Unit Interval Selection in Random Order Streams

Cezar-Mihail Alexandru[1], Adithya Diddapur[2], Magnús M. Halldórsson[3], Christian Konrad[2], Kheeran K. Naidu[4]

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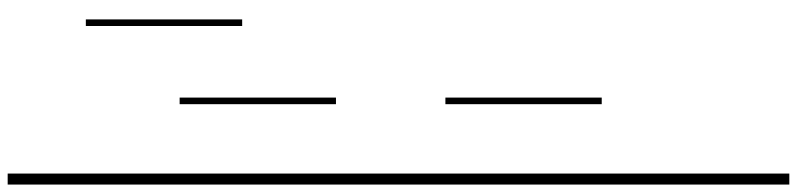

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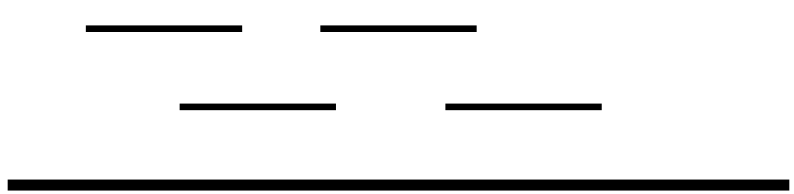

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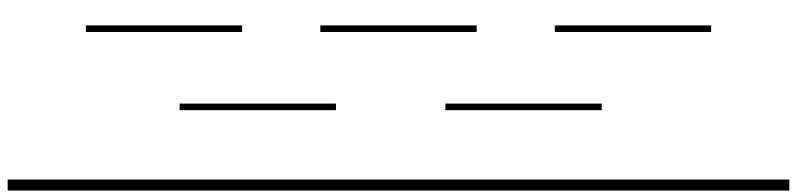

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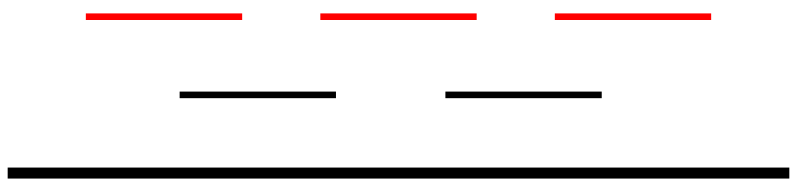

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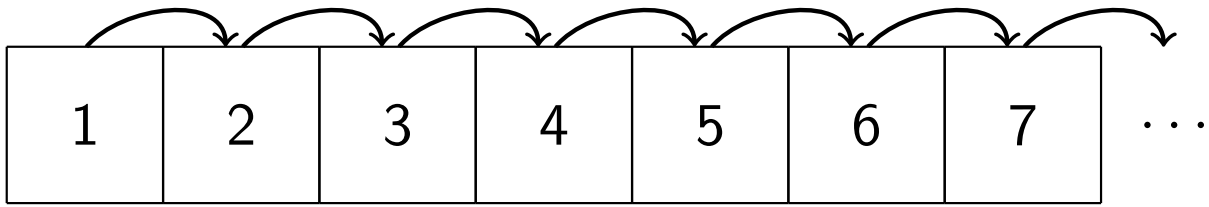
Goal

Given the input elements one at a time, compute an (approximate) solution using as **little space** as possible.



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- A 2/3-approximation is the right answer [EHR12, CPL15].

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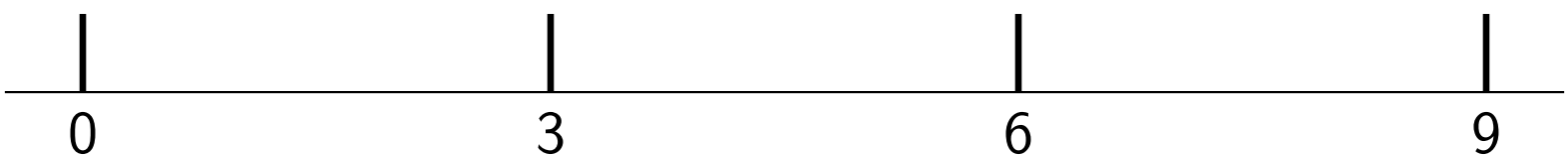

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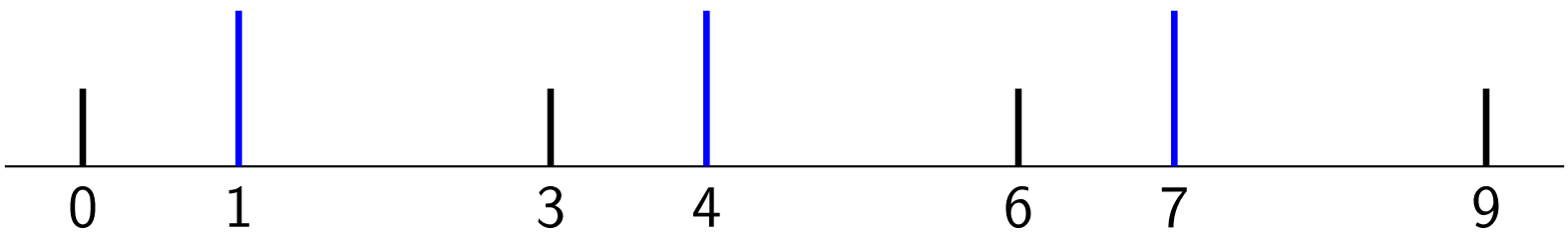

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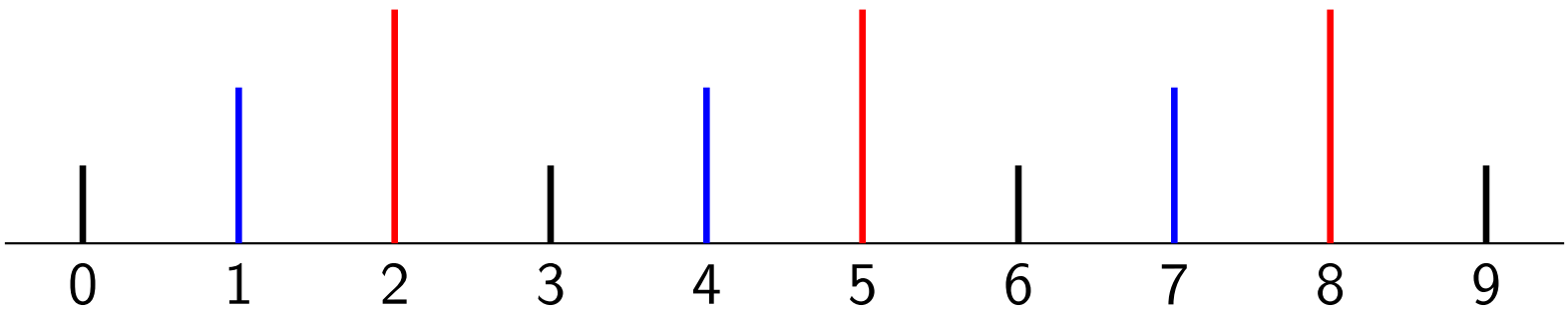

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- This is unlikely, but what if either the left or right-most arrives first?

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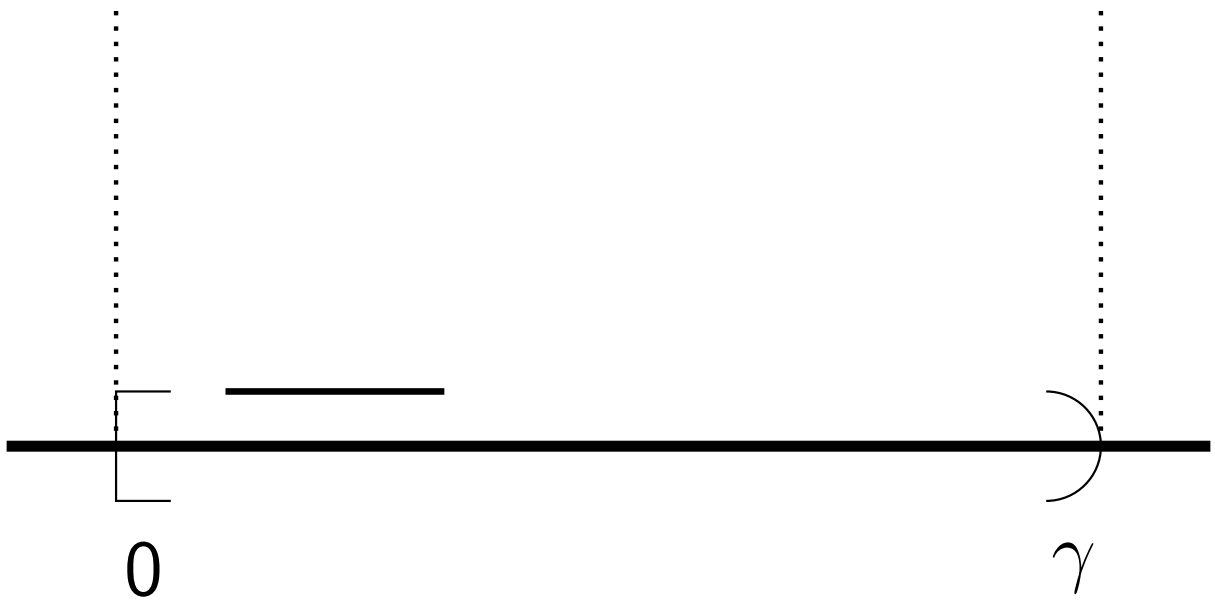

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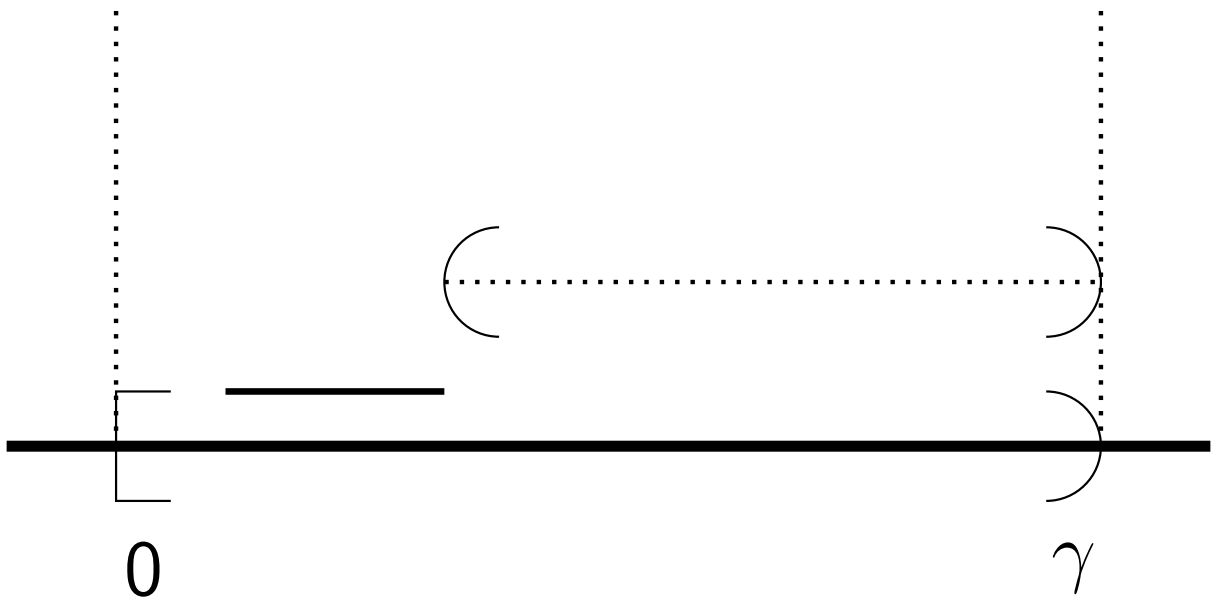

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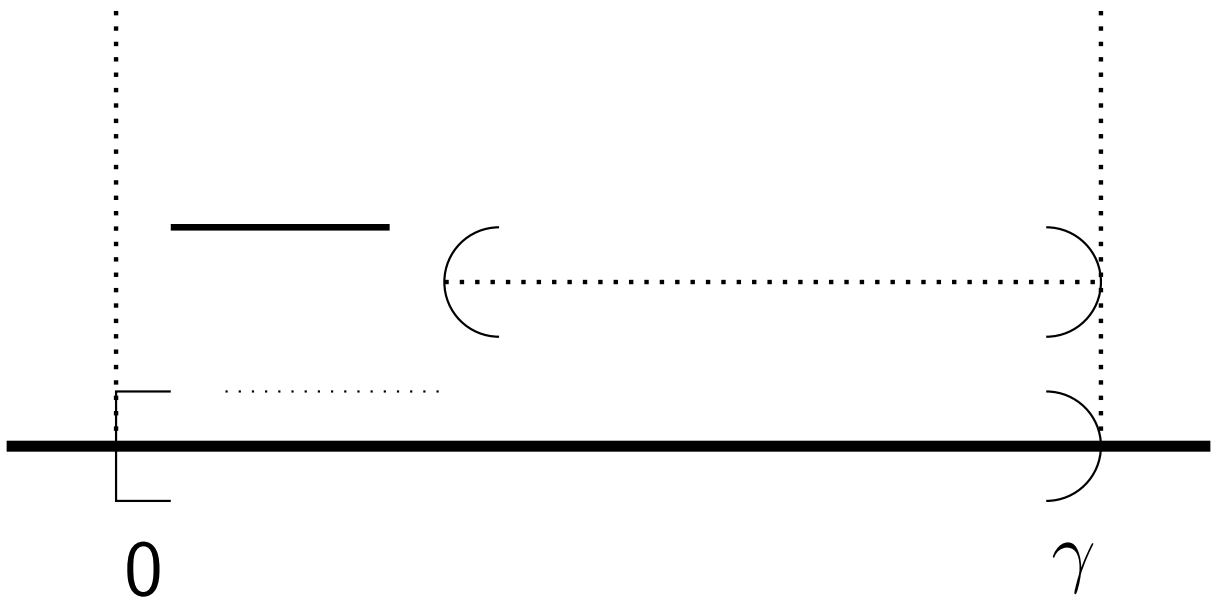

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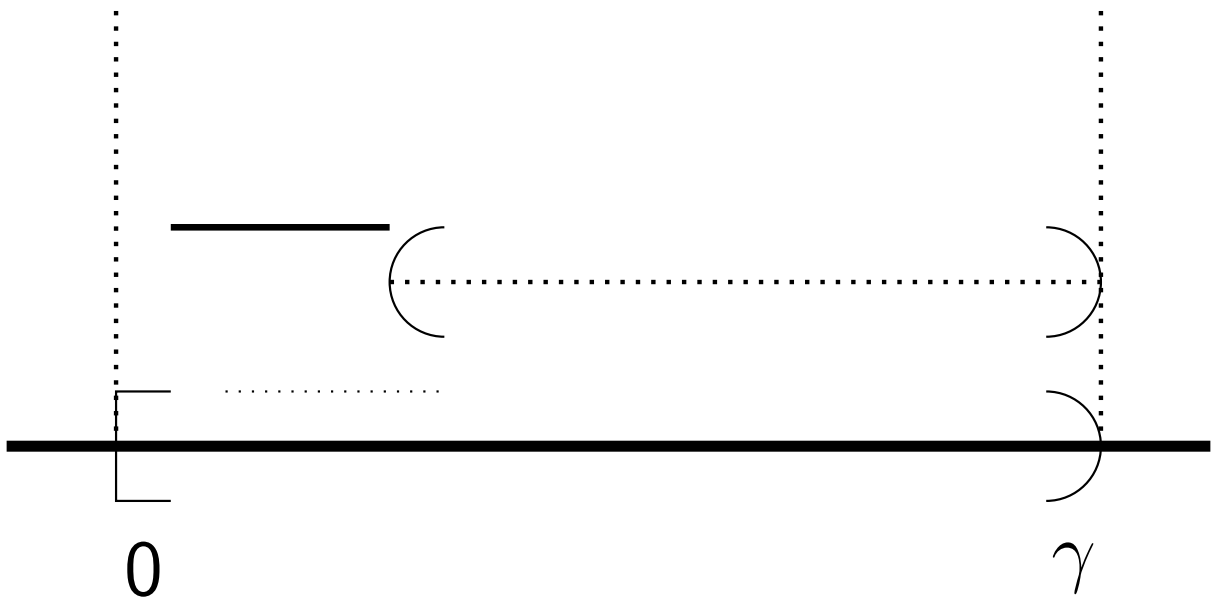

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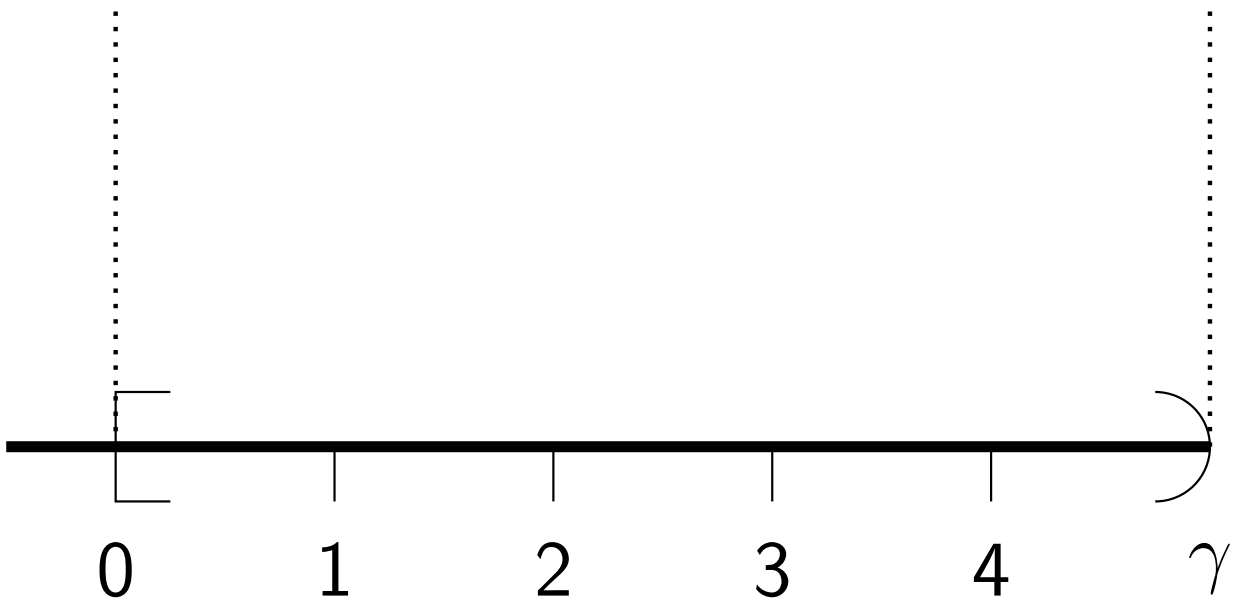

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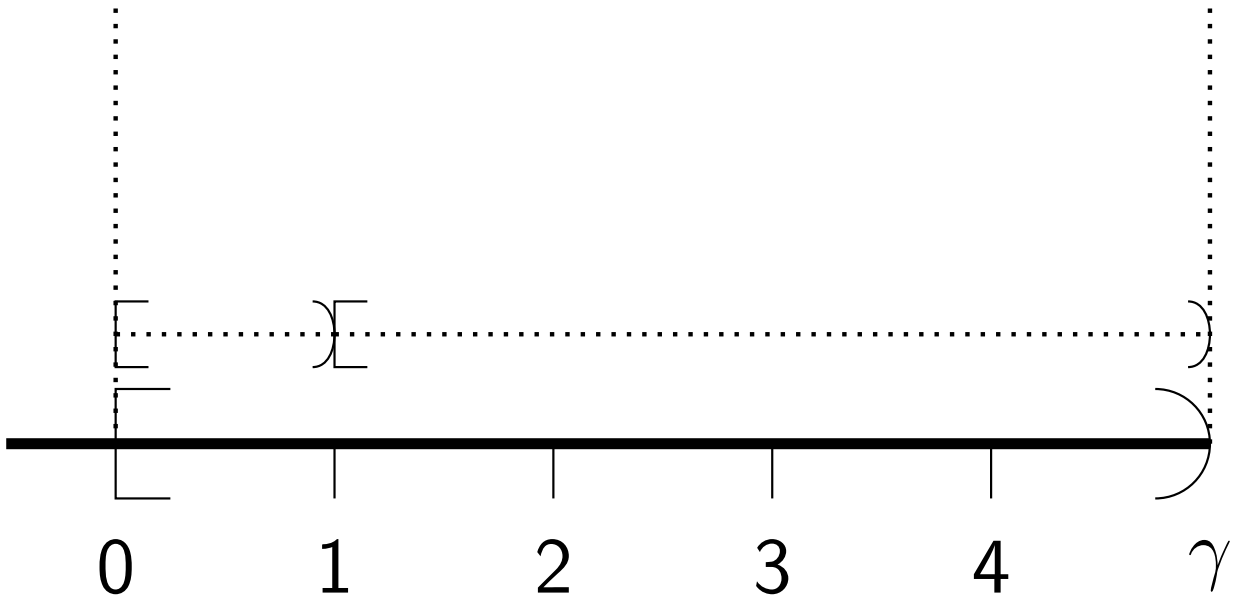

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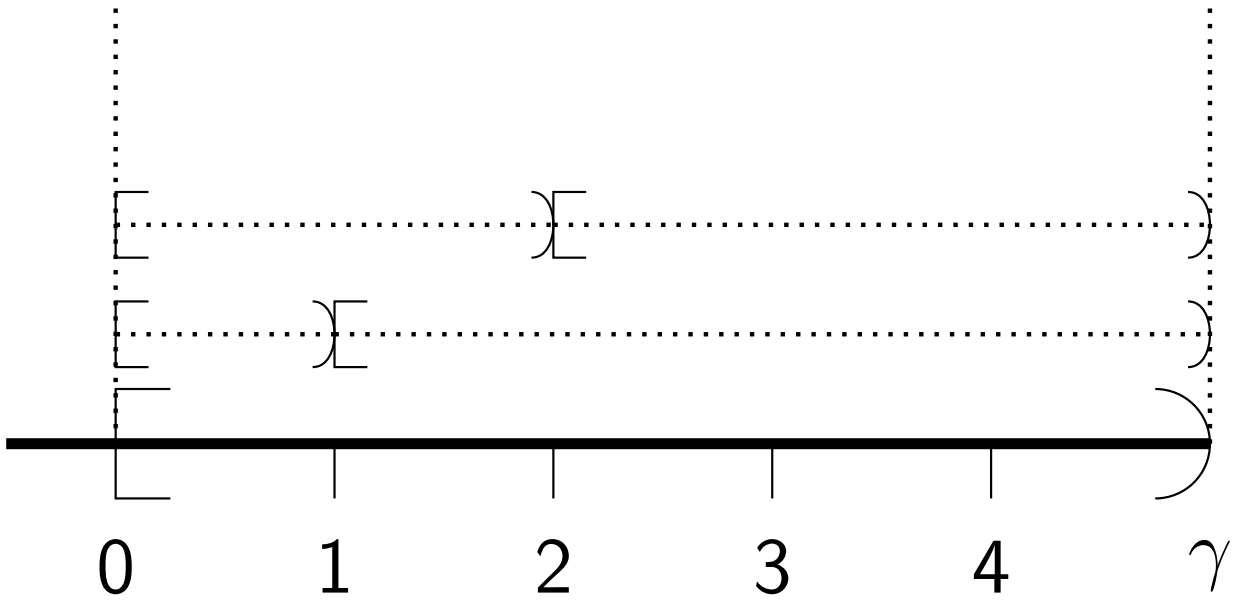

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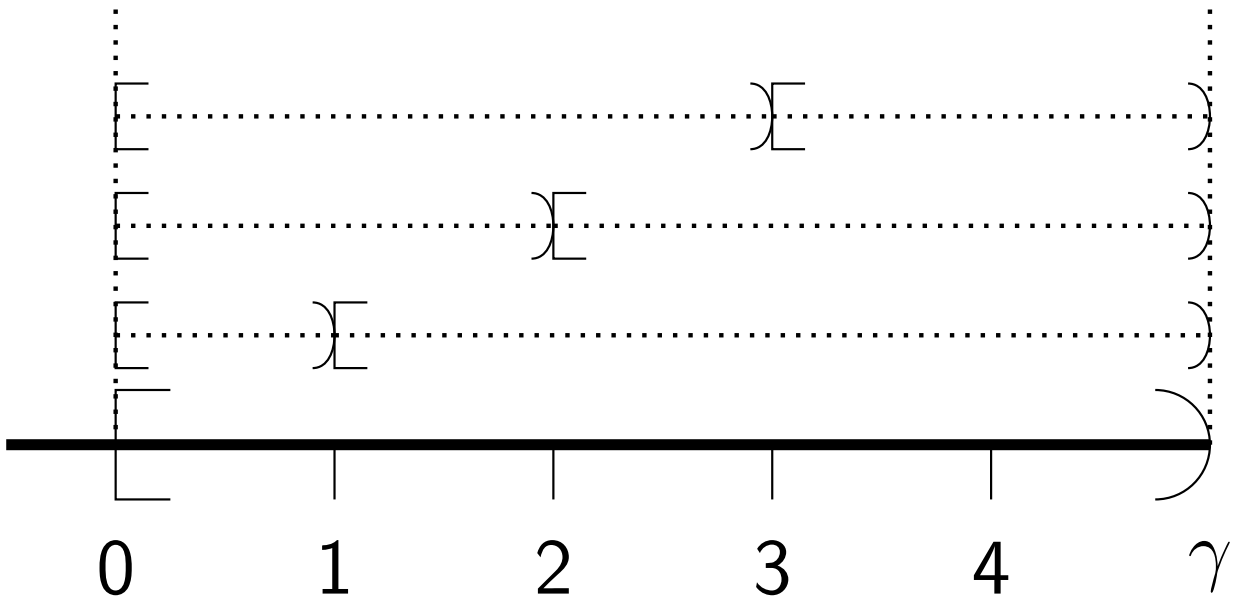

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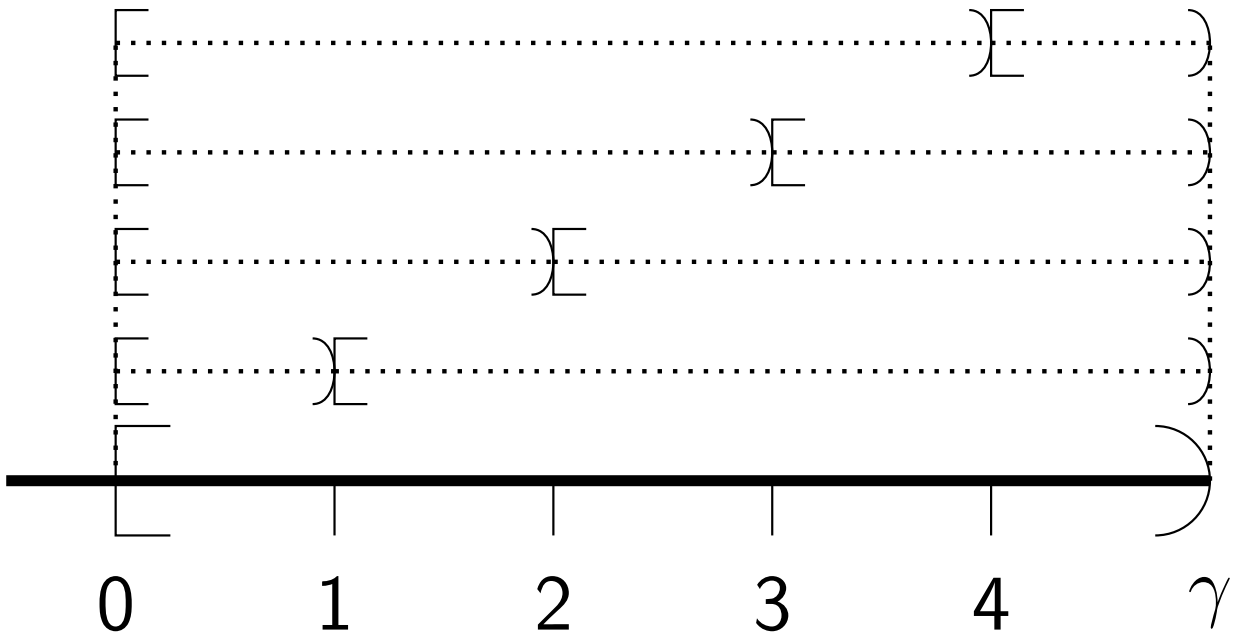

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5. $\gamma = 5000$ yields a 0.7401-approximation.

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[RY20] Solving INDEX_t requires a message of size $\Omega(t)$.

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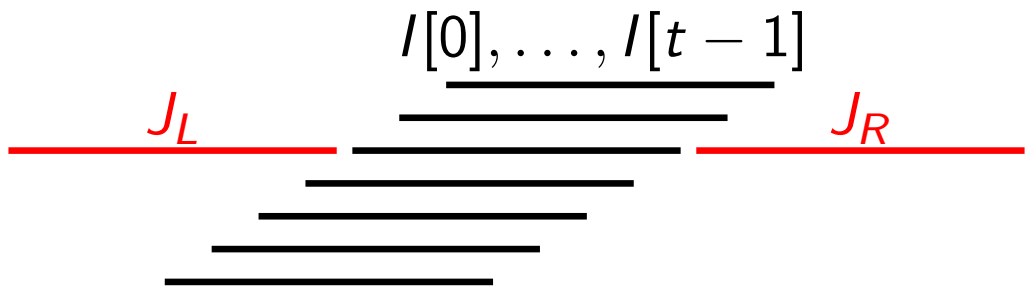

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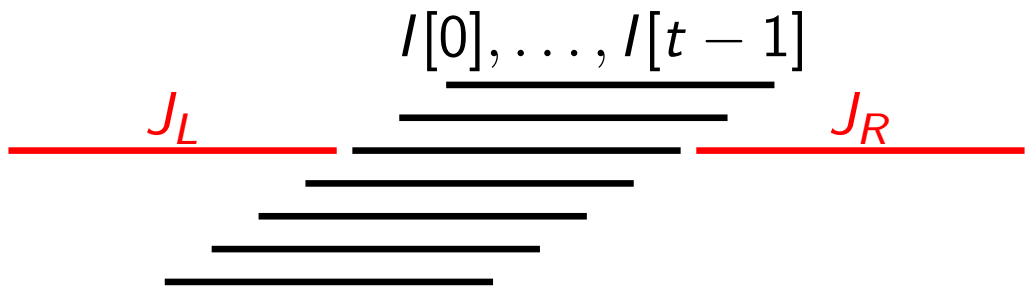

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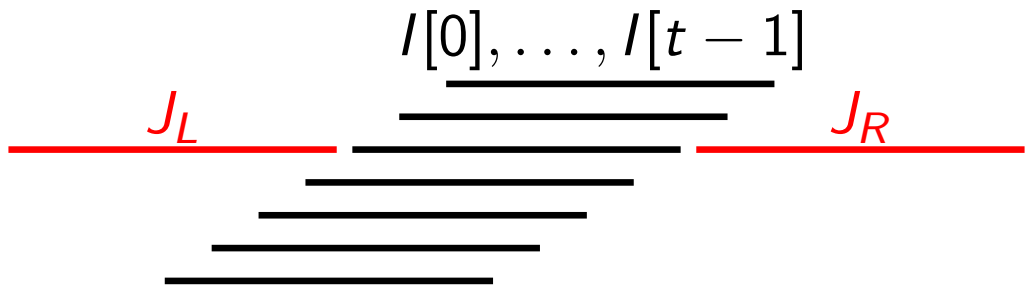

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- This event occurs with probability $2/3$.

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- Can we further close this gap between 0.7401 and $8/9 \sim 0.889$?
- Can we beat the adversarial-order bounds known for variable-length intervals?

References I

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