Threshold-Driven Streaming Graph: Expansion and Rumor Spreading

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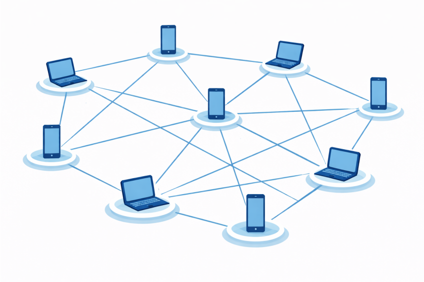
Application Scenario

- Dynamic model
- Bounded maximum degree
- Good Expansion

Suitable for: peer-to-peer, Bitcoin, opportunistic nets, …



Threshold-Driven Streaming Graph Model

$\mathcal{TSG}(n, d, c)$ is a dynamic-graph model $\{G_t\}_t = \{(V_t, E_t)\}_t,\ t \in \mathbb{N}$ that combines:

- **Streaming node-churn** protocol for V_t ([Becchetti et al. 2023)], [Cooper et al. (2008)]).
- **RAES** algorithm to update E_t ([Becchetti et al.(2020)]).

RAES(c, d)

Distributed, iterative protocol, designed to build an expander graph. Let $c, d \in \mathbb{N}$.

- At the beginning, every $v \in V$ sends d connection requests uniformly at random.
- Any v receiving some requests in the current round accepts all of them if doing so results in at most $c \cdot d$ total *incoming* edges. Otherwise, all the requests will be rejected.
- In the next round, every v will re-launch a new connection request for every edge that has been rejected previously.

Last step is iterated until every vertex has d outgoing edges.

OBS: Good expansion, bounded degree, but static setting.

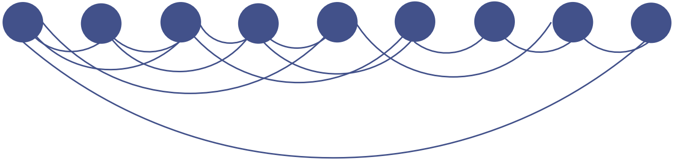
What kind of dynamic setting can we consider?

Streaming Node-Churn

Let $n \in \mathbb{N}$ and let $\{V_t : t \in \mathbb{N}\}$ be the deterministic process defined iteratively:

- $V_0 = \emptyset$.
- $\forall t \geq 1$ a new vertex joins the vertex set.
- At round $\forall t \geq n + 1$, the vertex that joined at time $t - n$ leaves the graph.

Example with $n = 9,\ d = 2,\ c = 1$



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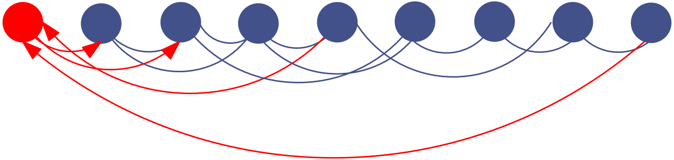

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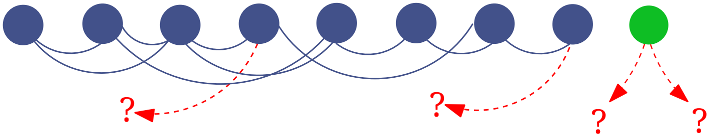

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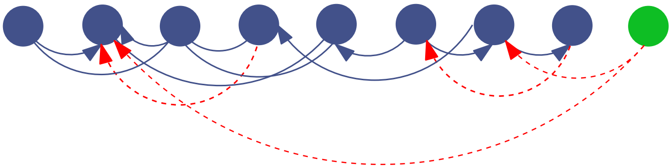

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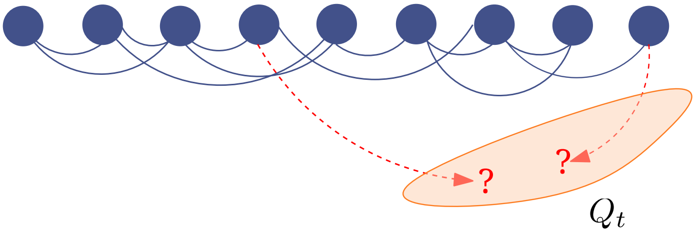


Preliminaries

Vertex Expansion of a Set $S \subseteq V$

$$\phi(S) = \frac{\#\{\text{Edges between } S \text{ and } S^c\}}{\#\{\text{Edges with at least an endpoint in } S\}} \tag{1}$$

Vertex Expansion of a graph G

$$\phi(G) = \min_{\substack{S \subseteq V \\ |S| \leq n/2}} \phi(S) \tag{2}$$

Main Results

Theorem: Expansion

In $\mathcal{TSG}(n, d, c)$, with d, c large enough, there exists a constant $\alpha \in (0, 1)$ such that:

- There exists $H_t \subseteq V_t$ with $|H_t| = n - O(\log n)$ such that the subgraph $G_t[H_t]$ has α expansion, with probability at least $1 - n^{-2}$.

Remark: We need to study the distributions of the edges in E_t: how many of them are in Q_t? What is the probability of an edge $e \in E_t$ of being accepted from a node u?

Main Results

Corollary: Rumor Spreading

Let s be a *source* node joining the $\mathcal{TSG}(n, d, c)$ dynamic graph at some round $t_s \geq 2n$, with d, c large enough.

Then, after $T = O(\log n)$ rounds, the PUSH-PULL protocol informs at least $n - O(\log n)$ vertices in G_{T+t_s}, w.h.p..

Communication Cost

The overall number of requests each vertex makes during all its life has optimal (*constant*) expectation, w.h.p. .

The set of all the pending requests at time t is the **queue** Q_t.

Queue Lenght

$\exists \beta(d, c)$ such that, for every t

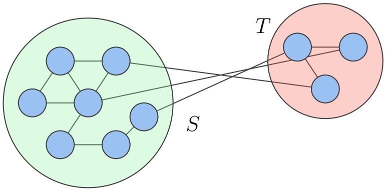
$$\mathbf{Pr}\left[|Q_t| \leq \beta \log n\right] \geq 1 - n^{-2}$$

Proof Sketch: Expansion of Small Sets

$$\forall t \geq 2n \quad \mathbf{Pr}\left[|E_t(S, S^c)| > \alpha \,\mathrm{vol}_t(S), \quad \forall S \subseteq V_t \text{ s.t. } |S| \leq \frac{n}{2000}\right] \geq 1 - n^{-2}$$

Let

$$E = \{\exists S \subseteq V_t,\ \exists T \subseteq V_t \setminus S \text{ s.t.}$$
$$|S| \leq \frac{n}{2000}$$
$$|T| = \alpha |S|$$
$$\forall (v, u) \in E_t, v \in S \implies u \in S \cup T\}$$



In particular, denoting as $X_t(r)$ the destination of and outgoing edge $r \in V_t \times \{1, \ldots, d\}$,

$\forall r \in S \times \{1, \ldots, d\},\ X_t(r) \in S \cup T$ or either $X_t(r) \in Q_t$.

Proof Sketch: Expansion of Small Sets

$$\mathbf{Pr}[E] \leq \sum_{\substack{S \subseteq V_t \\ |S| \leq \frac{n}{2000}}} \sum_{\substack{T \subseteq V_t \setminus S \\ |T| = \alpha|S|}} \mathbf{Pr}\left[\{\cap_{r \in R}\{X_t(r) \in S \cup T\}\}\right]$$

$$\leq \sum_{\substack{S \subseteq V_t \\ |S| \leq \frac{n}{2000}}} \sum_{\substack{T \subseteq V_t \setminus S \\ |T| = \alpha|S|}} \left(\frac{k|S \cup T|}{n-1}\right)^{|R|}$$

$$\leq \sum_{s \leq \frac{n}{2000}} \binom{n}{s}\binom{n-s}{\alpha s}\binom{ds}{ds - \beta \log n}\left(\frac{k(1+\alpha)s}{n-1}\right)^{ds - \beta \log n}$$

$$\lesssim n^{-2}$$

Where we also used that $|R| \geq d|S| - \beta \log n$ (since Q_t has logarithmic size w.h.p.).

$R \subseteq S \times \{1, \ldots, d\}$ is the set of accepted connections sent from S.

MAIN ISSUE: correlations between $\{X_t(r)\}_{r \in R}$ makes difficult to estimate $\mathbf{Pr}[\cap_{r \in R}\{X_t(r) \in S \cup T\}]$.

Technical lemma:

$\forall R \subseteq V_t \times \{1, \ldots, d\}$ and $A \subseteq V_t$

$\mathbf{Pr}[\cap_{r \in R}\{X_t(r) \in A\}] \leq \left(\frac{k|A|}{n-1}\right)^{|R|}$

Future Work

- Find optimal values of $c(d)$.
- Generalize the results to more realistic models (ex: Poisson).
- What if every vertex samples its neighbours from a restricted list, instead sampling over the whole network?

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Thank you for your attention!

Any questions?