Spectral Norm, Economical Sieve, and Linear Invariance Testing of Boolean functions

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Joint work with
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Contents

Definition: Property tester of functions

An algorithm (property tester) $\mathcal{A}$ $(\epsilon, \epsilon + \tau)$-tests $\mathcal{C}$, for a class of functions $f : \mathbb{F}_2^n \to \{-1, +1\}$, if given access to the truth table of a function f, whether

- f is "ϵ-close to $\mathcal{C}$"
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can be tested using $\mathcal{A}$ with success probability (called the *confidence*) $\geq 2/3$.

- The number of queries to the truth-table of f made by $\mathcal{A}$ is called the **query complexity** of $\mathcal{A}$.
- When $\epsilon = 0$, we call the property tester **non-tolerant**.
- When $\epsilon > 0$, we call the property tester **tolerant**.

Function Isomorphism

- $f : \mathbb{F}_2^n \to \{-1, +1\}$ and,

$$(f \circ \sigma)(x_1, \ldots, x_n) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

where $\sigma \in S_n$ (the group of all permutations of $[n]$).

One Fundamental Question

(f:known, g:unknown)

- Given query access to g,
- if there exists a permutation $\sigma \in S_n$ such that $f \circ \sigma = g$,
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- The **Hamming distance** between two functions $f_1 : \mathbb{F}_2^n \to \{-1, +1\}$ and $f_2 : \mathbb{F}_2^n \to \{-1, +1\}$ is defined as

$$\delta(f_1, f_2) = \Pr_x[f_1(x) \neq f_2(x)].$$

Linear Isomorphism

- For $A \in \mathbb{F}_2^{n \times n}$, let $f \circ A : \mathbb{F}_2^n \to \{-1, +1\}$ be the function $f \circ A(x) = f(Ax)$ for all $x \in \mathbb{F}_2^n$.

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- Two Boolean functions $f, g : \mathbb{F}_2^n \to \{-1, +1\}$ are **Linearly isomorphic** to each other if

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- **Question:** If f and g be linear transformation of each other?

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Linear Isomorphism Distance

The **Linear Isomorphism Distance** between two functions $f : \mathbb{F}_2^n \to \{-1, +1\}$ and $g : \mathbb{F}_2^n \to \{-1, +1\}$ is defined as

$$\mathrm{dist}_{\mathbb{F}_2^n}(f, g) = \min_{[A \in \mathbb{F}_2^{n \times n} : A \text{ is non-singular}]} \delta(f \circ A, g).$$

Testing Isomorphism of Boolean Functions

Let $f, g : \mathbb{F}_2^n \to \{-1, +1\}$ be Boolean functions.

- g is known. Given query access to f. For any $\epsilon \geq 0$ and $\tau > 0$, determine if
- $\exists$ linear transformation $A \in \mathbb{F}_2^{n \times n}$ such that $\delta(f \circ A, g) \leq \epsilon$, or
- $\forall$ linear transformation $A \in \mathbb{F}_2^{n \times n}$, $\delta(f \circ A, g) \geq \epsilon + \tau$.

In other words,

- $\mathrm{dist}_{\mathbb{F}_2^n}(f, g) \leq \epsilon$.
- $\mathrm{dist}_{\mathbb{F}_2^n}(f, g) \geq \epsilon + \tau$.

Why do we care?

Many important Boolean function classes are closed under function isomorphism:

- k-**juntas:** functions depending on at most k variables
- **Sparse functions:** functions having s many non-zero Fourier coefficients
- **Symmetric functions:** value depends only on Hamming weight

Why do we care?

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- **Sparse functions:** functions having s many non-zero Fourier coefficients
- **Symmetric functions:** value depends only on Hamming weight
- **Graph properties:** viewed as Boolean functions on adjacency matrices

Question: Can we efficiently detect structural equivalence under function isomorphism?

Related works

- Fischer et al. were the first to initiate this where A is a permutation matrix.
- Followed by a long line of work by Alon and Blais [RANDOM'2010], Blais and O'Donnell [CCC'10], Chakraborty et al. [CCC'12], and many more.
- Testing linear isomorphism of Boolean functions was studied by Wimmer and Yoshida [ICALP'13], followed by Grigorescu, Wimmer and Xie [RANDOM'13].

Wimmer and Yoshida [ICALP 2013]

- Wimmer and Yoshida [ICALP 2013] solved this using $O\left(\frac{m^{24}}{\tau^{24}}\right)$ many queries.
- Here m is the upper bound on the spectral norm of g. That is,

$$\|\widehat{g}\|_1 = \sum_{r \in \mathbb{Z}_2^n} |\widehat{g}(\chi_r)| \leq m.$$

- Solves Affine-ismorphism as well.

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- Solves Affine-ismorphism as well.
- Kushilevitz-Mansour learning algorithm in implicit set-up.

Contents

Our result: Upper Bound

Theorem [Datta, Ghosh, Kayal, Paraashar, Roy]

- The problem **"Testing Isomorphism of Boolean Functions"** can be solved using $O\left(\frac{m^4}{\tau^4}\right)$ many queries.
- Here m is the upper bound on the spectral norm of the known function g. That is,

$$\|\widehat{g}\|_1 = \sum_{r \in \mathbb{Z}_2^n} |\widehat{g}(\chi_r)| \leq m.$$

Our Result: Lower Bound

- Maiorana–McFarland functions!

Theorem [Datta, Ghosh, Kayal, Paraashar, Roy]

For any $m > 0$,

- There exists $h : \mathbb{F}_2^n \to \{-1, +1\}$ with spectral norm at most m,
- Every adaptive algorithm for the $(0, 1/4)$-Testing Isomorphism of Boolean Functions problem with respect to h requires $\Omega(m^2)$ queries to the unknown function.

- Wimmer and Yoshida [ICALP 2013] proved an $\Omega(m)$ lower bound for the $(0, \tau)$-'Testing Isomorphism of Boolean Functions' problem, for $\tau = \frac{1}{m}$.

Contents

Testing Isomorphism of Boolean Functions

Let $f, g : \mathbb{F}_2^n \to \{-1, +1\}$ be Boolean functions.

- g is known. Given query access to f. For any $\epsilon \geq 0$ and $\tau > 0$, determine if
- $\exists$ a non-singular matrix $A \in \mathbb{F}_2^{n \times n}$ such that $\delta(f \circ A, g) \leq \epsilon$, or
- $\forall$ non-singular matrices $A \in \mathbb{F}_2^{n \times n}$, $\delta(f \circ A, g) \geq \epsilon + \tau$.

The algorithm executes in two major steps:

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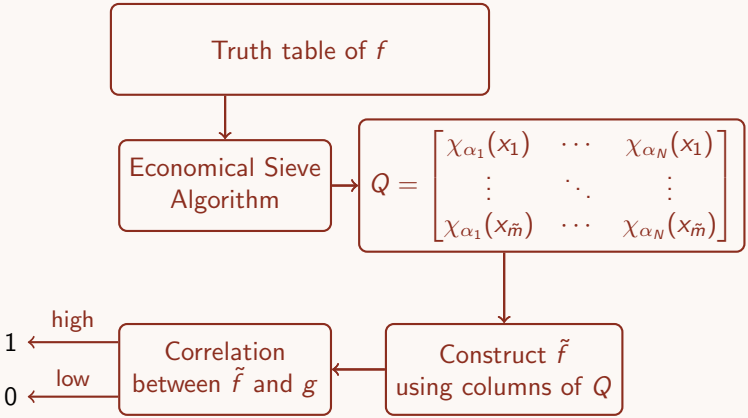
1. It learns the heavy Fourier co-efficients of the unknown function **implicitly**. **'Economical Sieve Algorithm'**.
2. Use the Economical Sieve Algorithm algorithm to create a function $\tilde{f}$ that approximates f. Check the Isomorphism **off-line**.

The query complexity of the isomorphism testing algorithm solely depends on the Economical Sieve algorithm.

Algorithm Diagram



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- Input: a threshold θ, a set $M = \{x_1, \ldots, x_{\tilde{m}}\}$ of independently and uniformly chosen random points, and the unknown function f.

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- Input: a threshold θ, a set $M = \{x_1, \ldots, x_{\tilde{m}}\}$ of independently and uniformly chosen random points, and the unknown function f.
- It returns some values $\{\chi_{\alpha_j}(x_i), j \in [N]\}$, for all $x_i \in M$, where $S = \{\chi_{\alpha_1}, \ldots, \chi_{\alpha_N}\}$ is the set of *'heavy weight'* ($|\widehat{f}(\chi_{\alpha_i})| \geq \theta$) characters.
- So the set S is unknown.

- Fourier representation:

$$f(x) = \sum_{\alpha \in \mathbb{F}_2^n} \widehat{f}(\alpha)\chi_\alpha(x).$$

Random bucketing/Coset-Hashing

- $H =< \beta_1, \ldots, \beta_k >^{\perp}$, where $\beta_i \in \mathbb{F}_2^n$'s are chosen independently and uniformly.
- **Affine subspaces/cosets/buckets** $C(b) :=$ cosets of $\mathbb{F}_2^n/H$.
- Choose H in such a way that every Heavy Fourier coefficient end up in different buckets.

Let's first derive when f is sparse!

If f is sparse and we are re-using the samples following Wimmer and Yoshida [ICALP 2013], one can derive $O(\frac{m^4}{\tau^4})$ algorithm.

- A Boolean function f is *s-sparse* if it has s-many nonzero Fourier coefficients.
- The Fourier coefficients of f belong to different buckets with high probability on taking $O(s^2)$ many buckets.

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- **Goal:** Sign of the bucket is determined by sign of the heavy Fourier co-efficient.
- We do this via **Local-list correction** (ℓ_1 concentration): Precisely we bound the following:

$$\mathcal{P}_C(z) - \mathcal{P}_C^*(z)$$

where $\mathcal{P}_C f(z) := \sum_{\beta \in C} \widehat{f}(\beta)\, \chi_\beta(z)$.

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where $\mathcal{P}_C f(z) := \sum_{\beta \in C} \widehat{f}(\beta)\, \chi_\beta(z)$.

- How P_C relates sign?

$$P_C f(x) = \mathbb{E}_y[f(y)\chi_\alpha(y)\chi_\alpha(x)].$$

- Note that

$$P_C f(y_j) \cdot P_C f(y_j - x_i) = |\widehat{f}(\chi_\alpha)|^2\, \chi_\alpha(x_i).$$

- So $\operatorname{sign}(P_C f(y_j) \times P_C f(y_j - x_i)) = \chi_\alpha(x_i)$.

Key Differences from the Implicit Sieve

Implicit Sieve [Wimmer and Yoshida]

- Discards lightweight buckets in *two* rounds.
- First using wt_2, then using wt_4.
- Needed to estimate error for each bucket.

Economical Sieve

- Discards lightweight buckets in *one* round.
- Uses only wt_2 estimation.
- Uses ℓ_1-concentration.

Query complexity improvement:

- Reusing queries: $O\left(\frac{m^{24}}{\tau^{24}}\right) \to O\left(\frac{m^8}{\tau^8}\right)$.
- No wt_4 estimation: $O\left(\frac{m^8}{\tau^8}\right) \to O\left(\frac{m^4}{\tau^4}\right)$.

Contents

Conclusion!

- Our upper bound is $O(m^4)$ and lower bound is $\Omega(m^2)$. There's still a quadratic gap.
- Note that the lower bounds are non-tolerant. A lower-bound that is purely tolerant?

Thank You!