Foremost, Fastest, Shortest: Temporal Graph Realization under Various Path Metrics

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Temporal graphs: motivations, history

Motivations:

- Graphs represent relationships between entities
- In many real-world applications, those relationships are time-related:
  - Public Transit network: there is a bus that goes directly from stop s to stop t only at time $\tau_1, \tau_2, \tau_3 \ldots$
  - Phone call network: Alice calls Bob from time τ to time τ'
  - Social network: Alice follows Bob at time τ
  - ...

Temporal graphs: motivations, history

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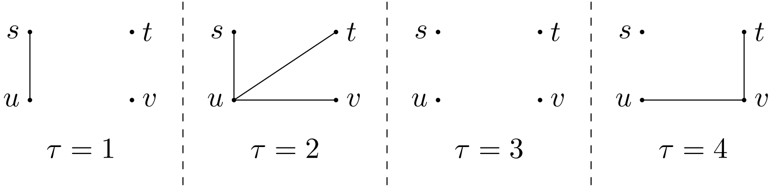
  - Phone call network: Alice calls Bob from time τ to time τ'
  - Social network: Alice follows Bob at time τ
  - $\ldots$

History:

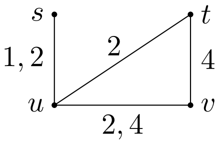
- Fragmented literature, plenty of models
- Time-dependent network (e.g.: road networks [Dreyfus 1969])
- Evolving graph (e.g.: ad-hoc networks [Xuan et al. 2003])
- Temporal network (e.g.: social network [Holme et al. 2011]), link stream [Latapy et al. 2017]
- Time-varying graph (ex: distributed computing [Casteigts et al. 2012]), temporal graph (ex: graph theory [Michail 2015])

Temporal graphs: models

Sequence of snapshots:



Labeled graph:



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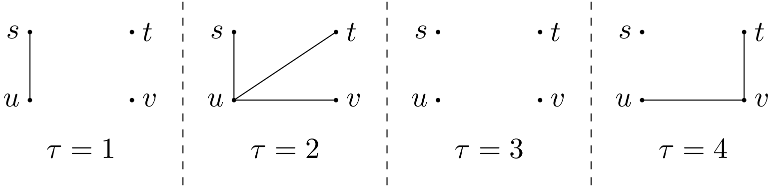

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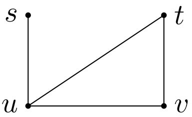
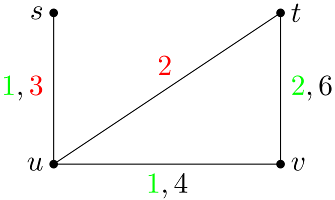


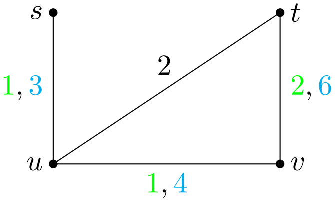
Underlying graph

Temporal paths: non strict and strict settings

Non-strict/Strict temporal path: A path and an non-decreasing/increasing sequence of times at which edges can be traversed one after the other

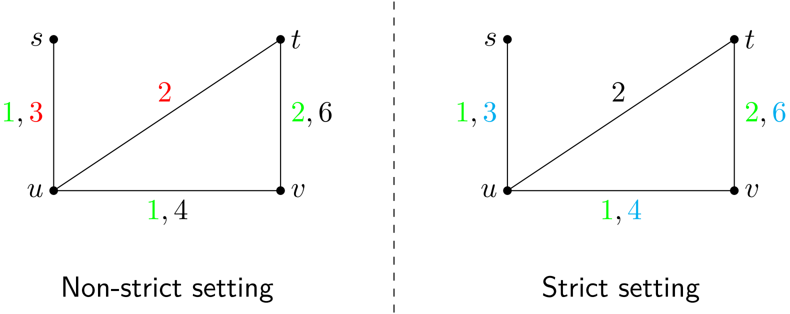


Non-strict setting



Strict setting

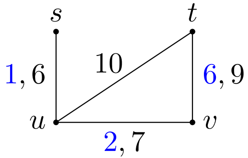
Non-strict/Strict temporal path: A path and an non-decreasing/increasing sequence of times at which edges can be traversed one after the other



All the following examples are in the strict setting

Temporal paths: foremost, fastest, shortest

Temporal path: A path and an increasing sequence of times at which edges can be traversed one after the other



Several notions of distance:

- Foremost temporal path: earliest arrival time
- Fastest temporal path: minimum duration
- Shortest temporal path: minimum number of edges

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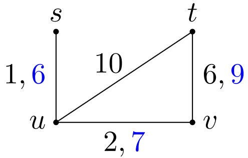

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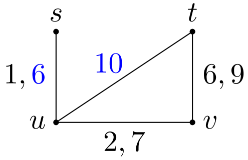

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Motivations and state of the art

Realization problems

Given a property P, is there a graph that satisfies the property?

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State of the art:

Periodic fastest temporal graph realization

Input: An $n \times n$ matrix D and a period Δ

Question: Is there a temporal graph with period Δ such that the minimum durations match the matrix?

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State of the art:

Periodic fastest temporal graph realization

Input: An $n \times n$ matrix D and a period Δ

Question: Is there a temporal graph with period Δ such that the minimum durations match the matrix?

- Railway networks, satellite networks
- NP-hard [Klobas et al. 2024]
- FPT by the feedback edge number of the underlying graph, W$[1]$-hard by the feedback vertex number [Klobas et al. 2024]
- Generalized to up to ℓ labels per period [Erlebach et al. 2024]

Questions

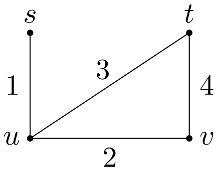
- Does the hardness remains if we lift the periodic constraint and allow an unlimited number of labels?
- What if the matrix D gives the earliest arrival times? the minimum number of edges of a temporal path?

Foremost TGR

Input: An $n \times n$ matrix D with values in $\mathbb{N} \cup \{\infty\}$
Question: Is there a temporal graph G such that the earliest arrival times match the matrix?

$$\begin{array}{c} \\ s \\ t \\ u \\ v \end{array} \begin{array}{cccc} s & t & u & v \\ \end{array}$$

$$\begin{array}{c|cccc} & s & t & u & v \\ \hline s & 0 & 3 & 1 & 2 \\ t & \infty & 0 & 3 & 4 \\ u & 1 & 3 & 0 & 2 \\ v & \infty & 3 & 2 & 0 \end{array} \longrightarrow$$



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NO

Our results

	Periodic	Non periodic
Foremost TGR	Polytime	Polytime
Shortest TGR	Polytime	NP-hard
Fastest TGR	NP-hard	NP-hard

Foremost TGR

Theorem

Foremost TGR can be solved in $\mathcal{O}(n^3 \log n)$ time and $\mathcal{O}(n^2)$ space.

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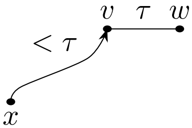
Edge compatibility

A temporal edge $(\{v, w\}, \tau)$ is edge-compatible with the matrix D if it satisfies: $EdgeCompat(D, \{v, w\}, \tau) := \forall x \in [n], D_{xv} < \tau \Rightarrow D_{xw} \leq \tau$ and $D_{xw} < \tau \Rightarrow D_{xv} \leq \tau$.



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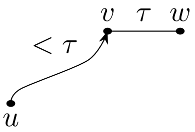
Algorithm: For each entry $\tau = D_{uw}$, check that there exists v such that:

- $D_{uv} < \tau$,
- $(\{v, w\}, \tau)$ is edge-compatible.



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- Produces a labeling with at most n labels per edge and at most n^2 labels in total: this is somehow tight

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- Valid for strict and non-strict settings, periodic graphs (slight change to the definition of edge-compatibility)

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Foremost TGR can be solved in $\mathcal{O}(n^3 \log n)$ time and $\mathcal{O}(n^2)$ space.

Remarks:

- Produces a labeling with at most n labels per edge and at most n^2 labels in total: this is somehow tight
- Valid for strict and non-strict settings, periodic graphs (slight change to the definition of edge-compatibility)
- Valid with a prescribed graph (if non-strict setting $\Rightarrow \mathcal{O}(n^2 m)$ with m the number of edges of the prescribed graph)

Foremost TGR: limitations of the tractability

Theorem

Foremost TGR is NP-hard when allowing at most one label per edge.

FOREMOST TGR: limitations of the tractability

Theorem

FOREMOST TGR is NP-hard when allowing at most one label per edge.

RANGED-FOREMOST TGR

Input: An $n \times n$ matrix D where each entry represents a range $[\ell, r]$
Question: Is there a temporal graph G such that the earliest arrival times match the matrix?

Theorem

- RANGED-FOREMOST TGR is NP-hard.

Foremost TGR: limitations of the tractability

Theorem

Foremost TGR is NP-hard when allowing at most one label per edge.

Ranged-Foremost TGR

Input: An $n \times n$ matrix D where each entry represents a range $[\ell, r]$
Question: Is there a temporal graph G such that the earliest arrival times match the matrix?

Theorem

- Ranged-Foremost TGR is NP-hard.
- Ranged-Foremost TGR can be solved in $\mathcal{O}(k^2 3^k n^4)$ time and $\mathcal{O}(2^k + n^2)$ space, where k is the number of undetermined entries of D.

General idea: Process by increasing time and guess which undetermined entries can be realized at the current time.

Shortest and Fastest TGR

Theorem

Shortest TGR is NP-hard.

Remarks:

- Valid for strict and non-strict settings
- Polynomial-time solvable in periodic graphs (check if D is the distance matrix of the underlying graph)

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Shortest TGR is NP-hard.

Remarks:

- Valid for strict and non-strict settings
- Polynomial-time solvable in periodic graphs (check if D is the distance matrix of the underlying graph)

Theorem

Fastest TGR is NP-hard and W[1]-hard when parameterized by the vertex cover number of the underlying graph plus the largest entry of the matrix.

Remark: Valid for strict and non-strict settings, periodic graphs

Conclusion and future work

Summary:

	Periodic	Non periodic
Foremost TGR	Polytime	Polytime
Shortest TGR	Polytime	NP-hard
Fastest TGR	NP-hard	NP-hard

Future work:

- Does Shortest TGR become polynomial-time solvable if we restrict the respective labeling?
- Are there structural parameters for which we can solve Shortest TGR in FPT-time? (e.g. treewidth)
- Are there constant factor approximations for Fastest TGR and Shortest TGR? (Under the measurement of fulfilling as many entries as possible)

Conclusion and future work

Summary:

	Periodic	Non periodic
FOREMOST TGR	Polytime	Polytime
SHORTEST TGR	Polytime	NP-hard
FASTEST TGR	NP-hard	NP-hard

Future work:

- Does SHORTEST TGR become polynomial-time solvable if we restrict the respective labeling?
- Are there structural parameters for which we can solve SHORTEST TGR in FPT-time? (e.g. treewidth)
- Are there constant factor approximations for FASTEST TGR and SHORTEST TGR? (Under the measurement of fulfilling as many entries as possible)

Thank you for listening!

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