Smaller Circuits for Bit Addition

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Contributions: One Slide Overview

Theory. Circuit size of bit addition:

$5n - 3m$	[Dadda, 65]
$4.5n - 2m$	[this paper]

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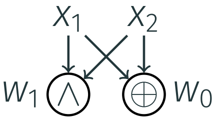
Practice. Generators: `Cirbo`. IWLS 2024 Contest:

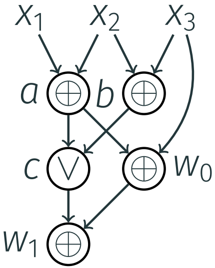
function	2023	team	our	
modulo8	1158	DeepMind	190	83%
sort15	114	TUW	73	35%

Computational Model: Boolean Circuits




```python
def half_adder(x1, x2):
    w0 = x1 ^ x2
    w1 = x1 * x2
    return w0, w1
```

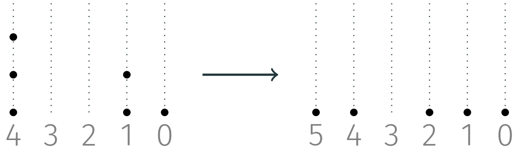
$$\mathrm{HA}(x_1, x_2) = (w_0, w_1)$$
$$x_1 + x_2 = w_0 + 2w_1$$




```python
def full_adder(x1, x2, x3):
    a = x1 ^ x2
    b = x2 ^ x3
    c = a | b
    w0 = a ^ x3
    w1 = c ^ w0
    return w0, w1
```

$$\mathrm{FA}(x_1, x_2, x_3) = (w_0, w_1)$$
$$x_1 + x_2 + x_3 = w_0 + 2w_1$$

Bit Addition and Dot Notation

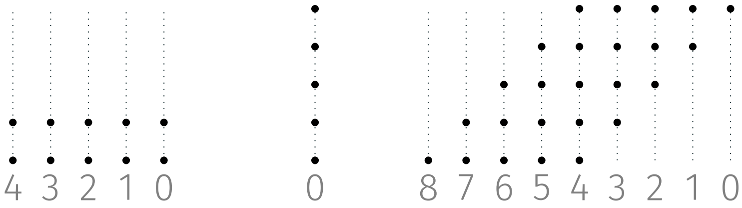


$$\mathrm{BA}_6^{0,1,1,4,4,4} \colon \{0,1\}^6 \to \{0,1\}^5$$

$$\mathrm{BA}_6^{0,1,1,4,4,4}(x_0, x_1, x_2, x_3, x_4, x_5) = (y_0, y_1, y_2, y_4, y_5)$$

$$x_0 + 2x_1 + 2x_2 + 16x_3 + 16x_4 + 16x_5 = y_0 + 2y_1 + 4y_2 + 16y_3 + 32y_5$$

Various Cases



ADD_5 SUM_5 $MULT_5$

Computing the Sum of Five Bits

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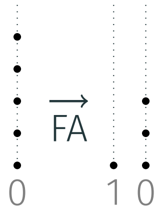

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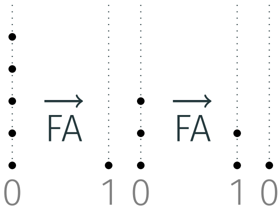

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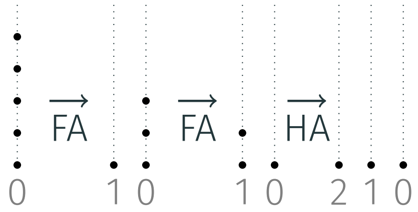

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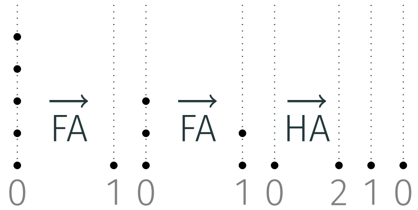

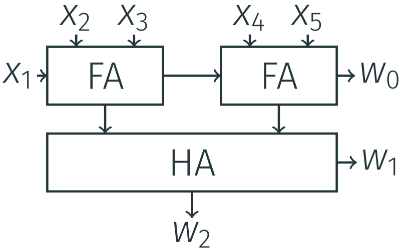

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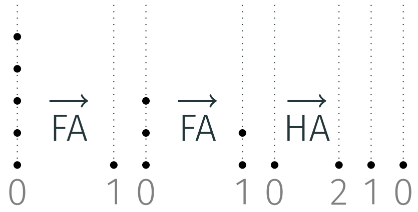

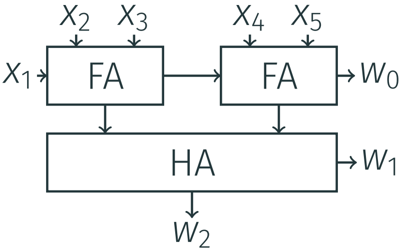

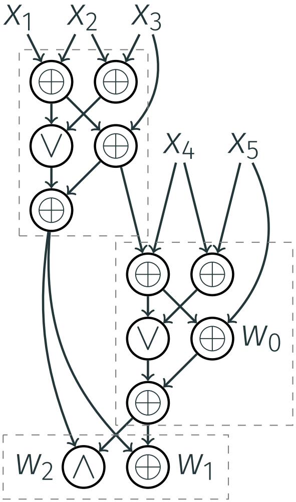

Synthesizing a Circuit for Bit Addition

- On each layer, apply FA's exhaustively, then apply HA if needed. The resulting size is at most $5(n-m)+2m=5n-3m$ [Dadda, 1965]

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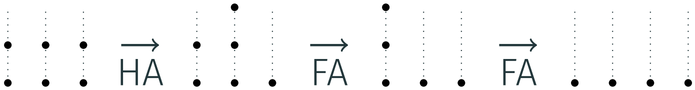

Synthesizing a Circuit for Bit Addition

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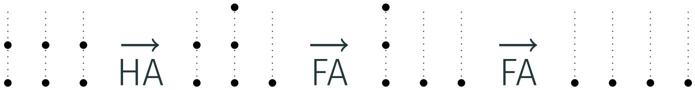
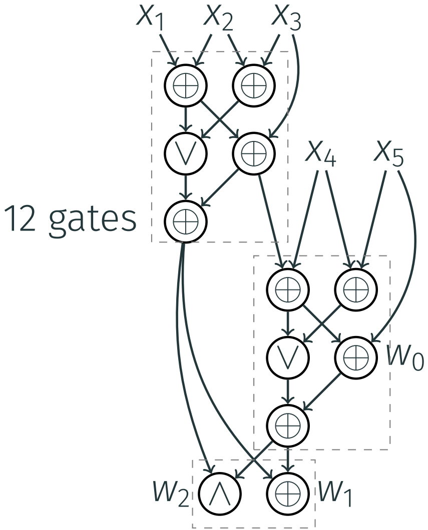


- For adding two n-bit integers, this gives a circuit of size $5n-3$ and it is provably optimal [Red'kin, 1981]

Surprise: $\text{size}(\text{SUM}_5) = 11$



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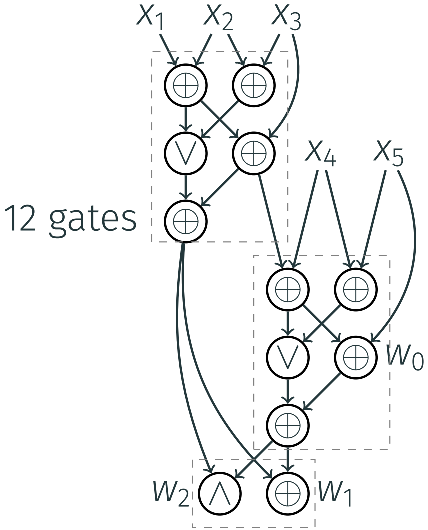
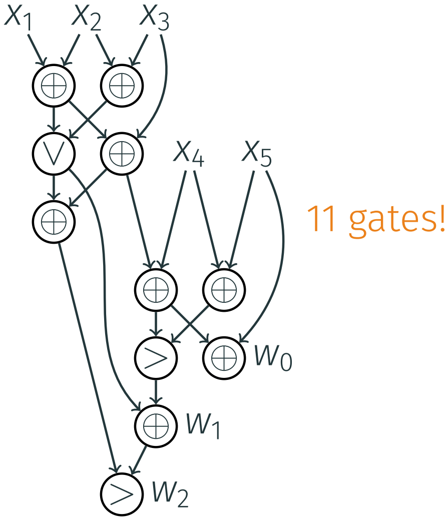
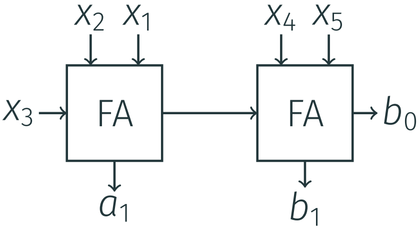
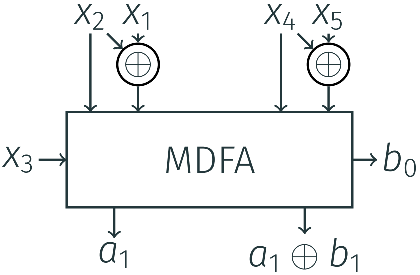
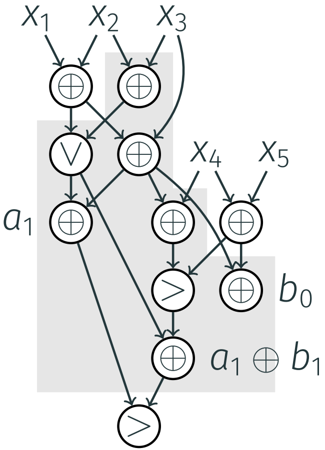




Modified Double Full Adder







Theory: New Upper and Lower Bounds

Upper bounds: $\text{size}(\mathrm{BA}_n^S) \leq 4.5n - 2m$

This generalizes $\text{size}(\mathrm{SUM}_n) \leq 4.5n$ [Demenkov et al. 2010] and improves $\text{size}(\mathrm{BA}_n^S) \leq 5n - 3m$ [Dadda 1965]

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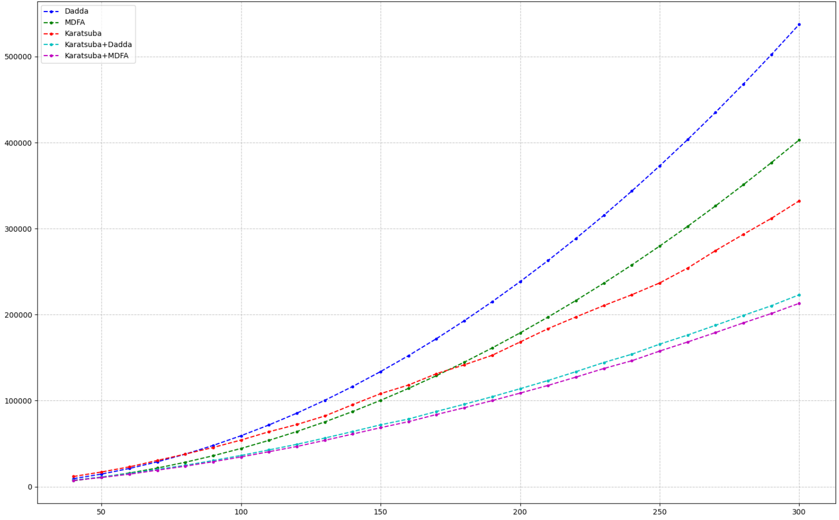
Lower bounds: $\text{size}(\text{INC}_n) = 2n \pm c$

Hence, $\text{size}(\text{BA}_n^S) \not\leq 5n - 3.1m$ and $\text{size}(\text{BA}_n^S) \not\leq 4.5n - 2.6m$

Practice: IWLS 2024 Contest on Logic Synthesis

function	2023	team	our	
modulo8	1158	DeepMind	190	83.59%
square12	1284	DeepMind	324	74.77%
div8	306	DeepMind	142	53.59%
sqrt16	226	DeepMind	136	39.82%
sort15	114	TUW	73	35.96%
maj15	68	TUW	48	29.41%

Practice: Generators for $MULT_n$



Contributions and Open Problems

Contributions

- **Upper bounds:** $\mathsf{size}(\mathrm{BA}_n^s) \leq 4.5n - 2m$
- **Lower bounds:** $\mathsf{size}(\mathrm{INC}_n) = 2n \pm c$
- **Software:** `Cirbo` package

Open problems

- Close the gap $2.5n \leq \mathsf{size}(\mathrm{SUM}_n) \leq 4.5n$!
- Improve $4.5n - 2m$ further!