Rank 4 Maker-Maker Games are PSPACE-complete

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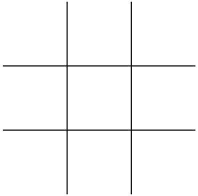
Positional Games (Hales et Jewett 1968)

Two players, **Left** and **Right**, play on a hypergraph $H = (V, E)$ ($E \subset 2^V$).

Left and **Right** take turns picking a previously unpicked vertex.

Hyperedges represent the goals.

"Achievement" $\rightarrow$ Hyperedges represent goals to achieve: winning sets.



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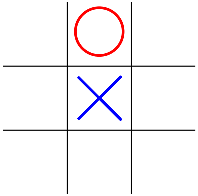

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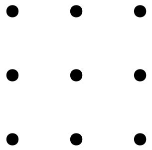

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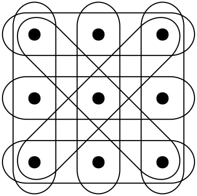


Maker-Maker Convention

Maker-Maker convention = Hyperedges are winning sets for both players (the first player to fill an hyperedge wins).

By strategy-stealing argument, only the first player to play, **Left**, can have a winning strategy.

Question : Does **Left** have a winning strategy ?

Theorem (Koepke 2025 (+ Biskov 2004))

6-*uniform Maker-Maker game is PSPACE-complete.*

Theorem (Trivial)

2-*uniform Maker-Maker game is in LOGSPACE.*

Maker-Breaker Convention

Maker-Breaker convention = Hypergedges are winning sets for **Left** , **Right** wins if **Left** does not win.

Theorem (Schaefer 1978)

11-*uniform Maker-Breaker game is PSPACE-complete.*

Theorem (Koepke 2025)

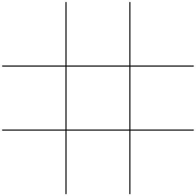
5-*uniform Maker-Breaker game is PSPACE-complete.*

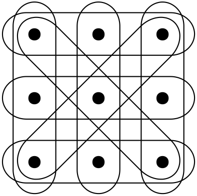
Theorem (Galliot, Gravier, Sivignon 2022)

3-*uniform Maker-Breaker game is polynomial.*

Intermediate position of Maker-Maker games

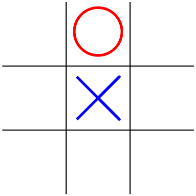
A Maker-Maker instance

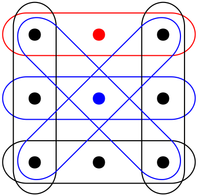




Intermediate position of Maker-Maker games

Not a Maker-Maker instance





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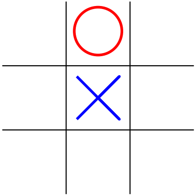

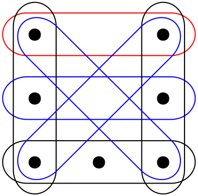

Two distinct colors of hyperedges, representing winning conditions for **Left** and **Right** (black hyperedges are both red and blue).

Achievement positional game (Galliot, Sénizergues 2025)

An **Achievement positional game** is a triplet $H = (V, E_L, E_R)$ such that

- (V, E_L) and (V, E_R) are hypergraphs.
- E_L (resp. E_R) are winning conditions for **Left** (resp. **Right**).
- Can be seen as a hyperedge-colored hypergraph with three colors (red, blue, and black = red and blue).

Special cases:

- $E_L = E_R \rightarrow$ Maker-Maker games
- $E_R = \emptyset \sim$ Maker-Breaker games

Previously known results

Question : Does **Left** has a winning strategy as the first player ?
Maximum respective sizes for elements of E_L (in blue) and E_R (in red).

	0, 1	2	3	4	5+
0,1	L	L	P	?	PSPACE-c
2	L	**P**	**NP-hard**	**NP-hard**	PSPACE-c
3+	L	**co-NP**-c	**PSPACE**-c	**PSPACE**-c	PSPACE-c

Theorem (Galliot, Sénizergues 2025)

Intermediate position of 4-uniform Maker-Maker games is PSPACE-complete (as a corollary of case (3,3)).

Rank 4 Maker-Maker games

Theorem (Galliot, Sénizergues 2026)

Deciding whether **Left** *has a winning strategy is PSPACE-complete even when hyperedges in E_L and E_R are of respective sizes 3 and 2 and the elements of E_R are pairwise disjoint.*

Corollary

Maker-Maker is PSPACE-complete even for hypergraph of rank 4.

About the corollary

Corollary

Maker-Maker is PSPACE-complete even for hypergraph of rank 4.

Sketch of the proof : If $H = (V, E_L, E_R)$ such as in the theorem hypothesis, $H' = (V \cup \{u, v\}, (u \times E_L) \cup (v \times E_R) \cup \{\{u, v\}\})$.

If **Left** plays u as its first move, then **Right** is forced to play v and the resulting game is H.

Else, **Right** plays u and either **Left** had played v as his first move or is forced to play v. As elements of E_R are pairwise disjoint, **Right** has a pairing strategy to prevent **Left** from winning.

Main theorem

Theorem (Galliot, Sénizergues 2025)

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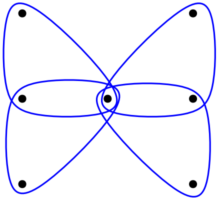
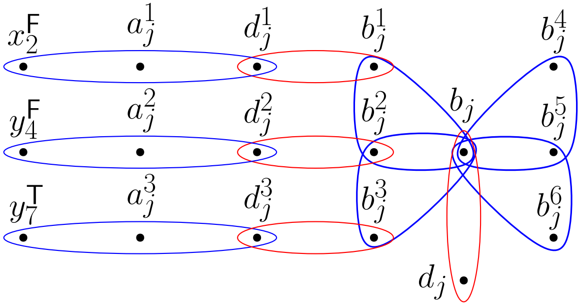
The proof is done by reduction from Quantified Boolean Formula.
Left acts as Falsifier while **Right** acts as Verifier.

The butterfly, a delayed menace



Clause gadget for $c_j = \neg x_2 \wedge \neg y_4 \wedge y_7$



A blue butterfly, with red *destruction edges* linked to it, connected to "valuation nodes".

If **Right** plays any one of those vertices, he can destroy the butterfly.

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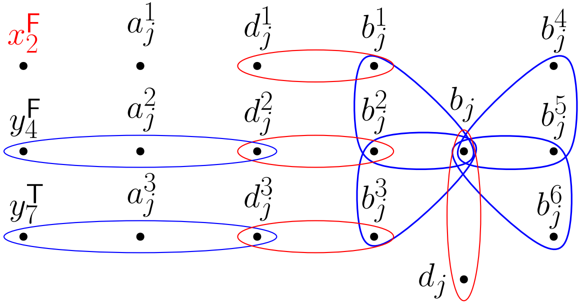

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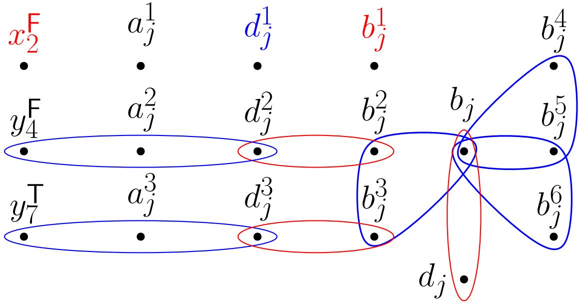

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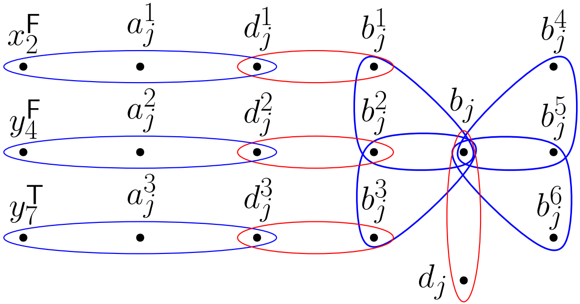

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If **Left** play the three of them, she can destroy the destruction edges and free the butterfly.

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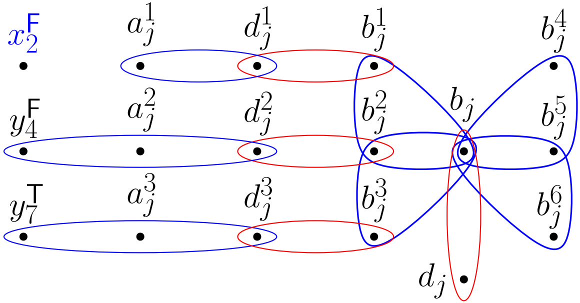

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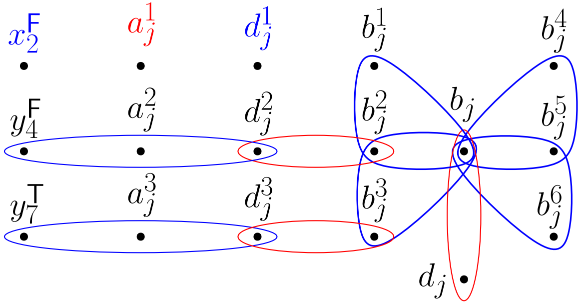

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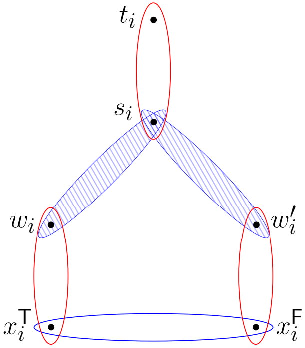
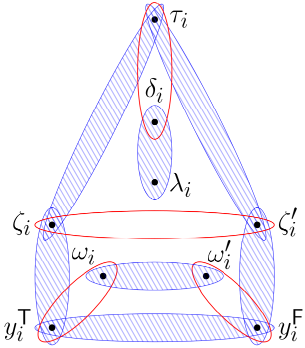
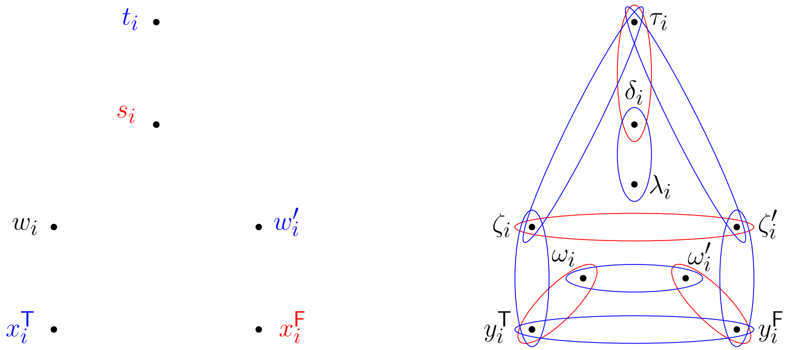
Variable gadget for round i





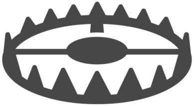
Each dashed edge represents two size 3 edges containing respectively x_i^T and x_i^F. Once **Left** plays either of those, the sequence is forced.

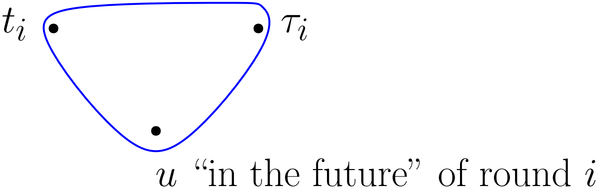
Variable gadget for round i



Then **Right** must choose between y_i^T and y_i^F, and the rest of the sequence is forced

It's a trap!





Every vertex adjacent to red edges is booby trapped by **Left**! If it is not supposed to be played as of round i, $\{t_i, \tau_i, u\}$ is a blue *trap edge*.

Trap edges ensure that **Right** cannot deviate from expected play pattern.

Conclusion

Theorem (Galliot, Sénizergues 2026)

Maker-Maker is PSPACE-complete even for hypergraph of rank 4.

The first proof about Maker-Maker games that is not from trivial reduction of Maker-Breaker games!

What's left about Maker-Maker:

- Our construction is rank 4 but not 4-uniform,
- Rank 3 is still open.

Thank you for your attention