Polynomial Complementation of Nondeterministic Two-Way Finite Automata by 1-Limited Automata

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STACS 2026 — March 11th

The Sakoda & Sipser conjecture

n **states** NFA → 2^n **states** DFA

The Sakoda & Sipser conjecture

The Sakoda & Sipser conjecture

2FA: an automaton that can move its head back and forth

***n* states**
2NFA ⟶ 2DFA

The Sakoda & Sipser conjecture

2FA: an automaton that can move its head back and forth

n states
2NFA → ??? → 2DFA

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n **states**

2NFA —??? → 2DFA

$$L_n = \{u\, x\, \overleftarrow{u} \mid u \in \{a,b\}^n,\ x \in \{a,b\}^*\}$$

$\triangleright$	a	b	b	a	a	a	b	b	a	b	b	b	a	$\triangleleft$

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2FA: an automaton that can move its head back and forth

n **states**
2NFA ——???——→ 2DFA

fixed
$n = 3$

$$L_n = \{u\,x\,\overleftarrow{u} \mid u \in \{a,b\}^n,\ x \in \{a,b\}^*\}$$

▷	*a*	*b*	*b*	*a*	*a*	*a*	*b*	*b*	*a*	*b*	*b*	*b*	*a*	◁

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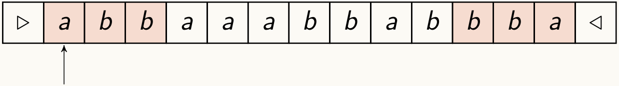

state
$i = 0$
$\sigma =$

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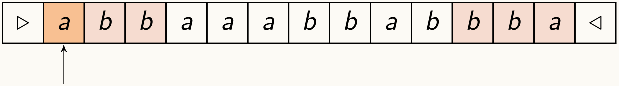

state

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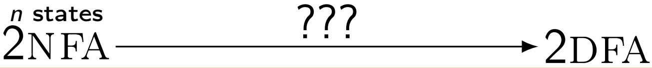

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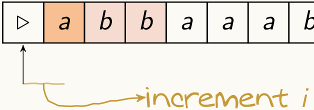

state
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$\sigma = a$

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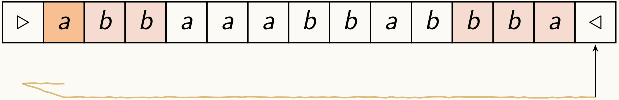

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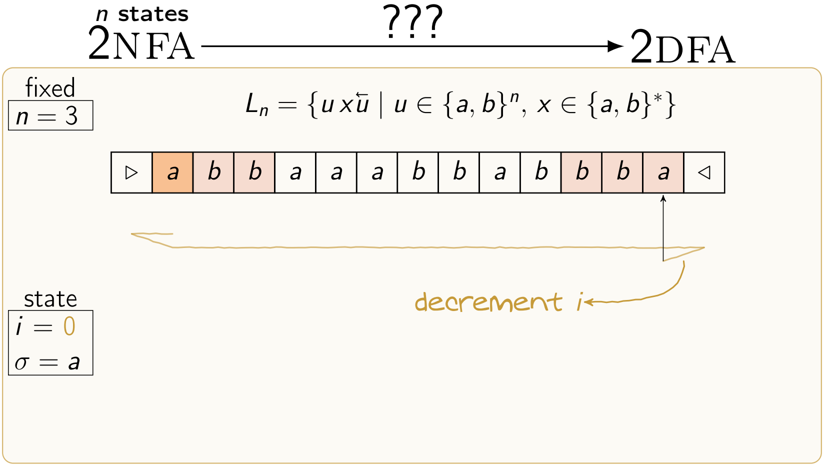

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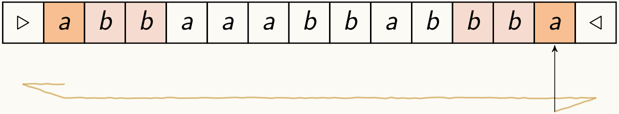

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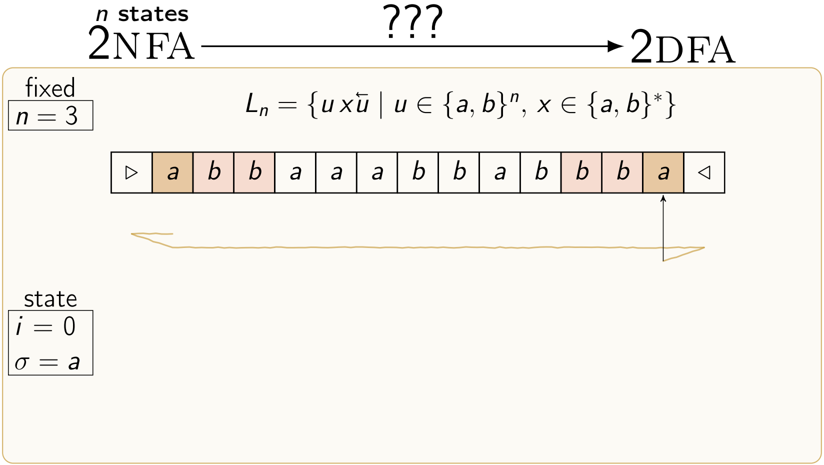

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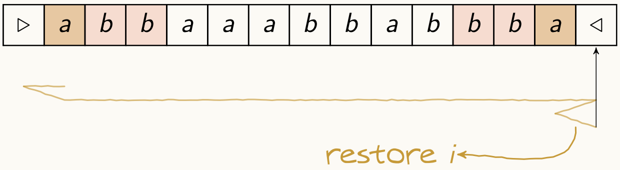

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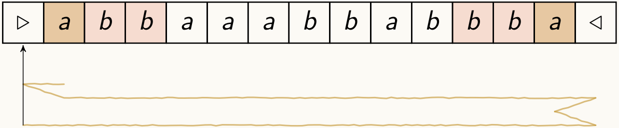

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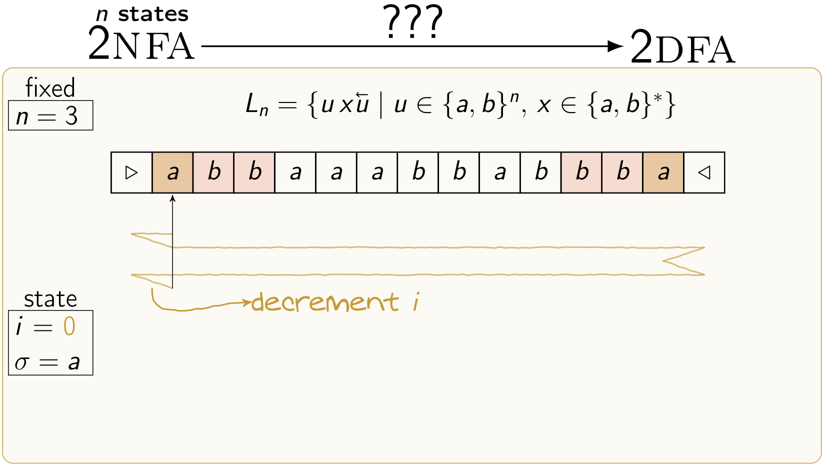

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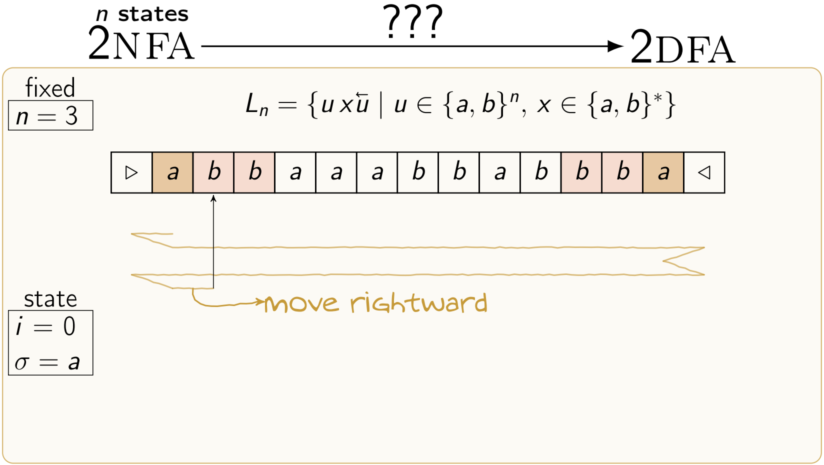

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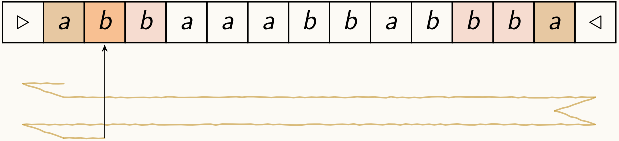

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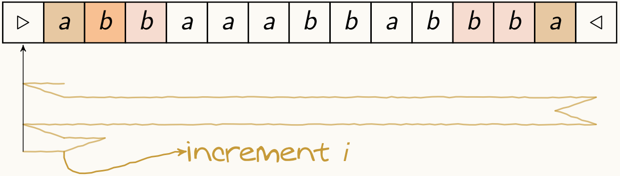

state
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n **states**

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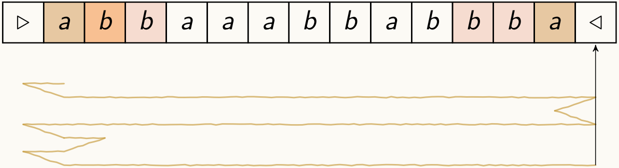

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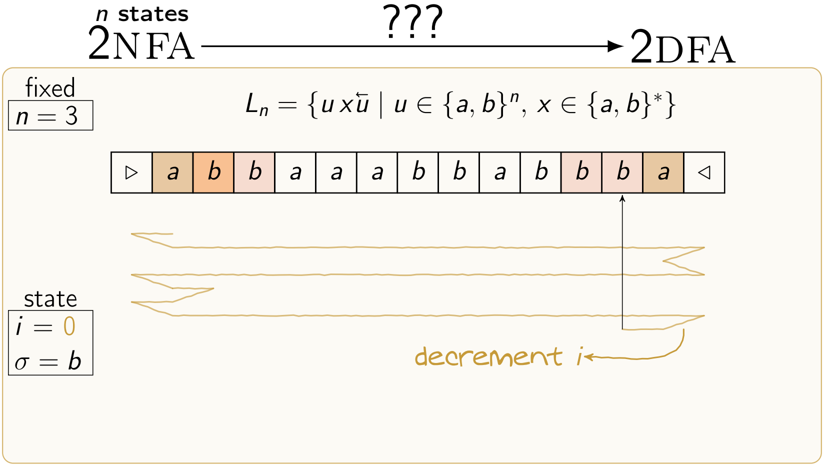

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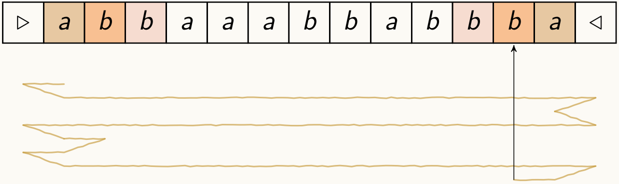

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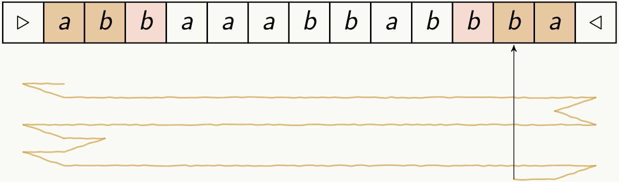

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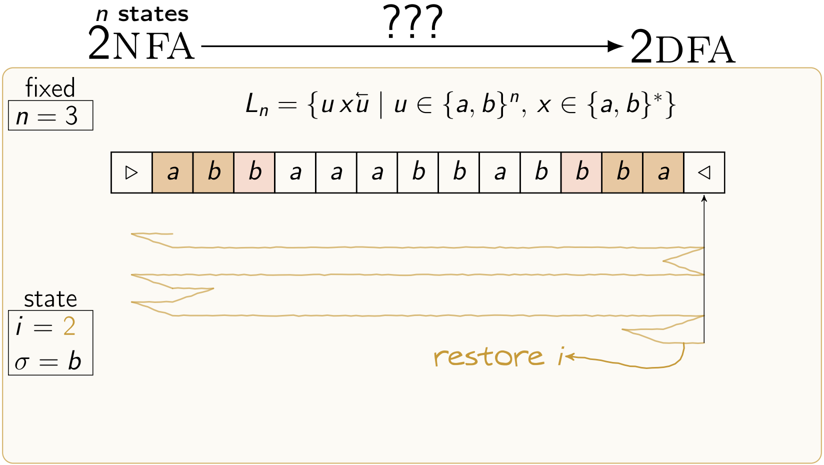

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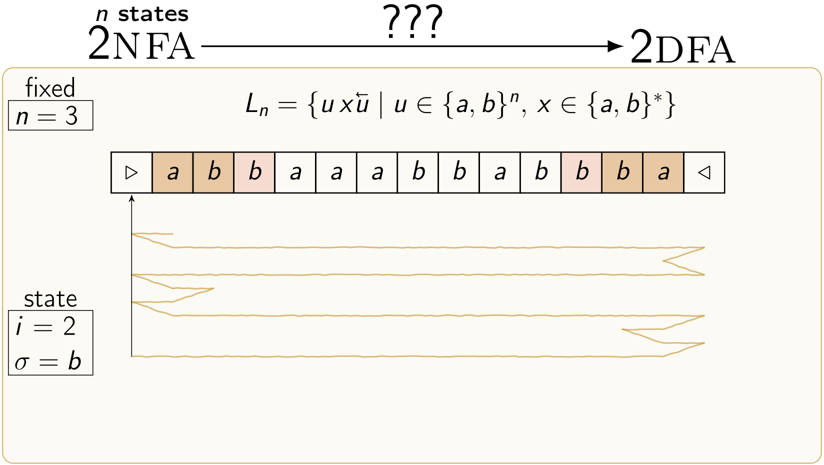

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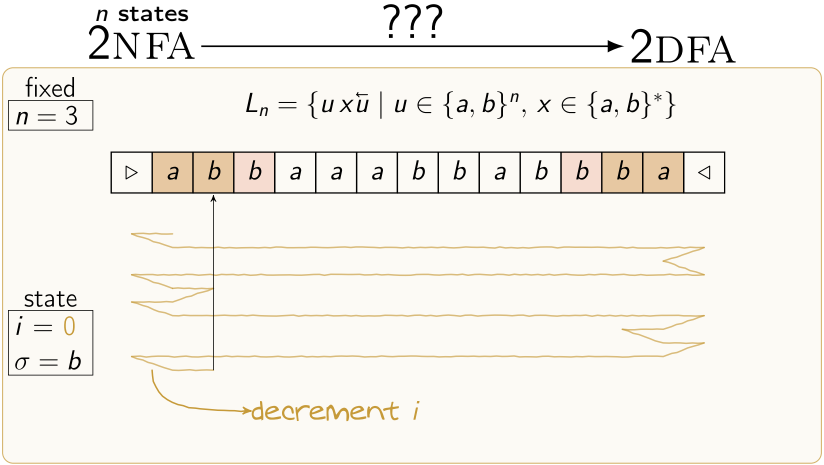

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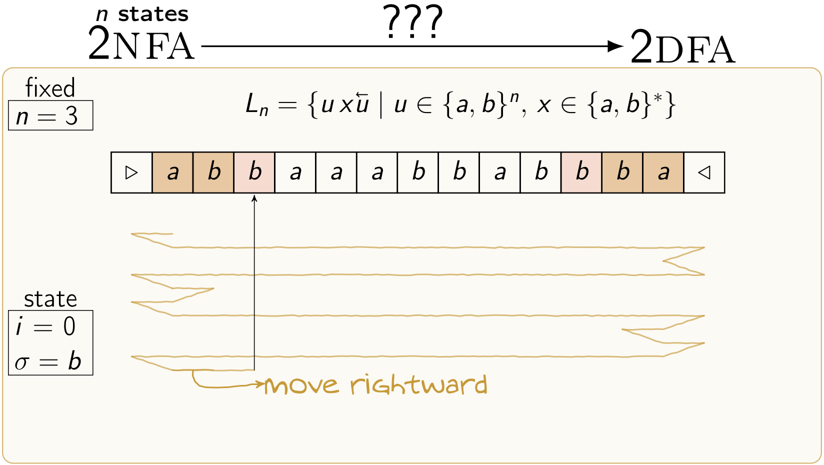

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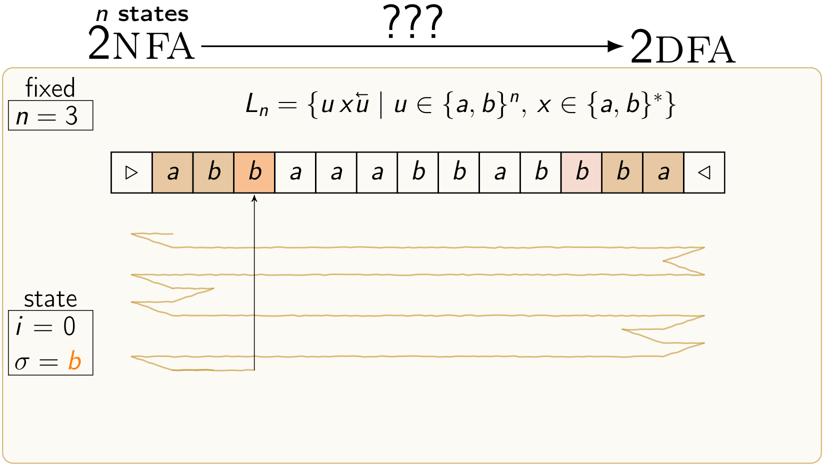

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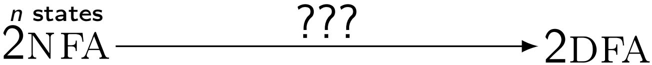

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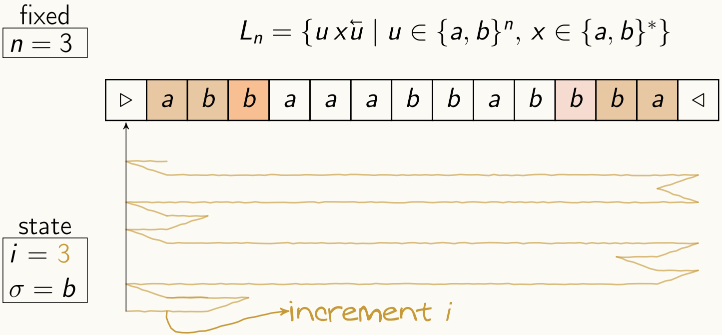

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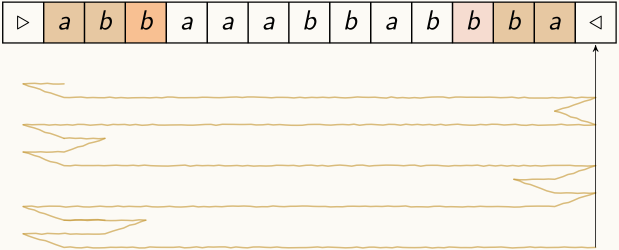

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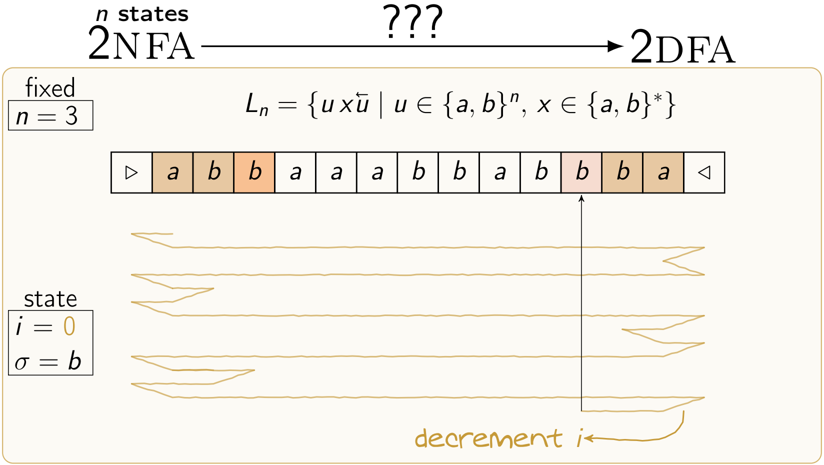

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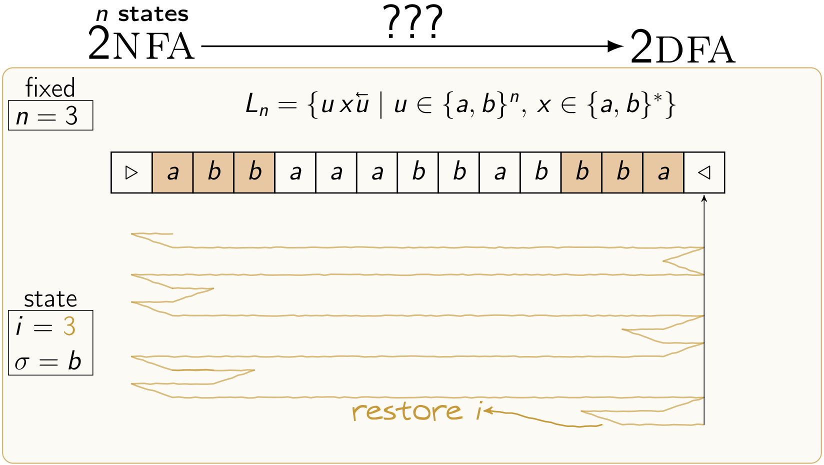

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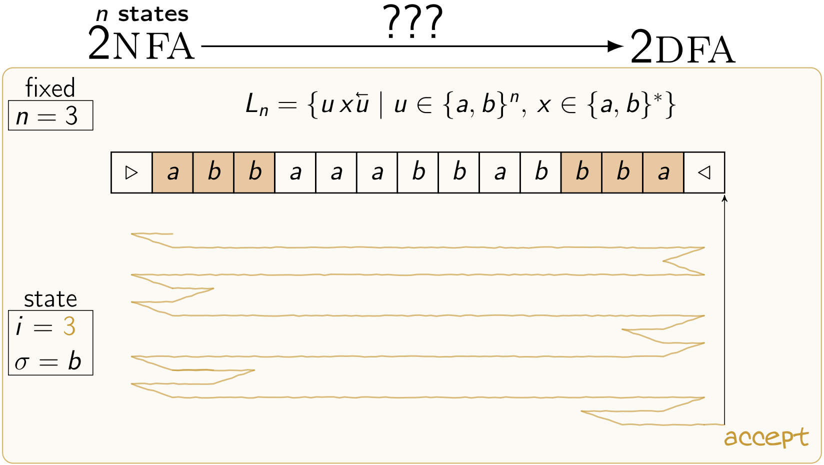

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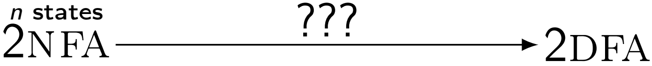

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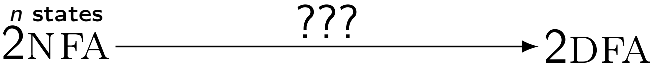

Conjecture (Sakoda & Sipser 1978)

The conversion of 2NFAs *into* 2DFAs *has super-polynomial cost*

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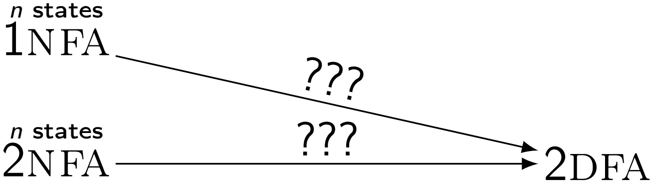

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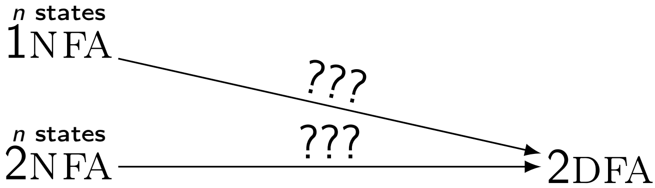

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- best known upper bound: exp ($\Longleftarrow$ 2NFA $\xrightarrow{\text{exp}}$ 1DFA)

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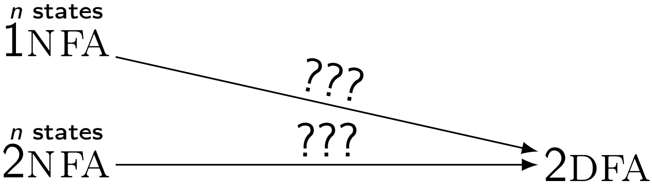

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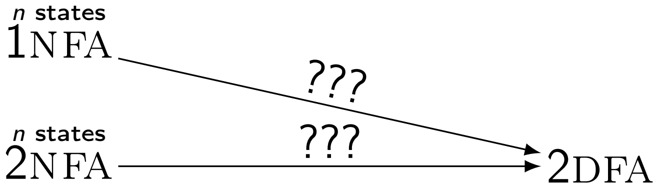

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- some common strategies to tackle the problem:
  - restrict/extend source/target   *e.g.*, 2NFA $\xrightarrow[\text{[Sipser'80]}]{\text{exp}}$ sweeping 2DFA

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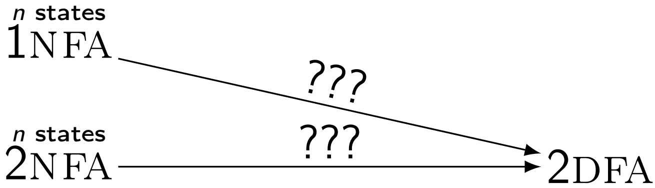

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- some common strategies to tackle the problem:
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  - consider the unary case: 1NFA $\xrightarrow[\text{[Chrobak'86]}]{\text{quadratic}}$ 2DFA $\xleftarrow[\text{[Geffert et al.'03]}]{<\text{subexp}}$ 2NFA

Cost of complementation

$$2\mathrm{NFA} \xrightarrow{\;?\;} \text{co-}2\mathrm{NFA}$$

for $L \subseteq \Sigma^*$ — for $\Sigma^* \setminus L$

Cost of complementation

∃ computation

?

not ∃ computation

2NFA → co-2NFA

for $L \subseteq \Sigma^*$

for $\Sigma^* \setminus L$

Cost of complementation

∃ computation	?	not ∃ computation
2NFA for $L \subseteq \Sigma^*$	$\longrightarrow$	co-2NFA for $\Sigma^* \setminus L$

Usually, complementation is cheap for deterministic devices
and expensive or unknown for nondeterministic ones

1DFA	2DFA	D1-LA	1NFA	2NFA	2NFA+cg $\sim$ 1-LA
trivial	linear	poly	exp	???	exp $\leq$? $\leq$ expexp

Cost of complementation

∃ computation — ? → not ∃ computation

2NFA for $L \subseteq \Sigma^*$ ⟶ co-2NFA for $\Sigma^* \setminus L$

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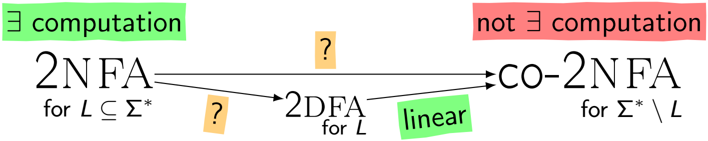

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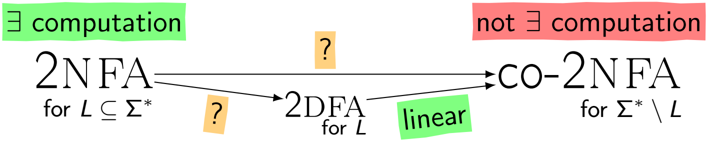

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1DFA	2DFA	D1-LA	1NFA	2NFA	2NFA+cg ∼ 1-LA
trivial	linear	poly	exp	???	exp ≤ ? ≤ expexp

Some known exceptions:

- NLogSpace = co-NLogSpace [Immerman'88, Szelepcsényi'87]
- complementing unary 2NFAs costs polynomial [Geffert *et al.*'07]
- idem for *outer-nondeterminsitic* 2NFAs [Geffert *et al.*'14]

Main result

$$2\mathrm{NFA} \xrightarrow{\ ?\ } \text{co-}2\mathrm{NFA}$$

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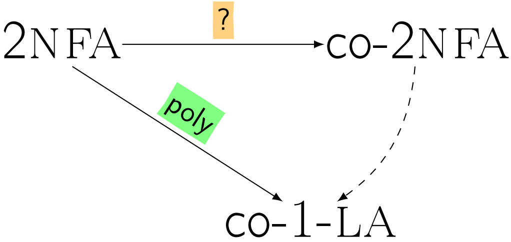

Theorem

Each 2NFA *can be complemented by a* 1-LA *of polynomial size*

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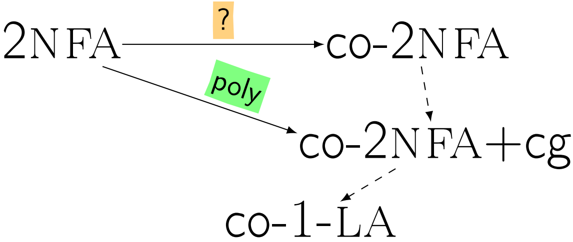

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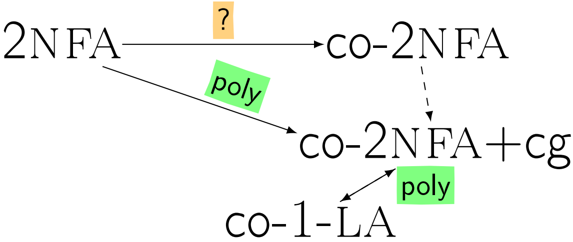

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1-limited automata & automata common-guess

2 extensions of 2NFAs with limited writing capacities

- 1-LAs: can rewrite each cell when visiting it **for the first time** only

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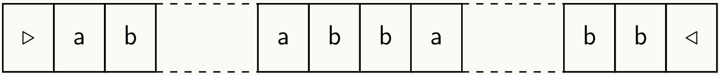

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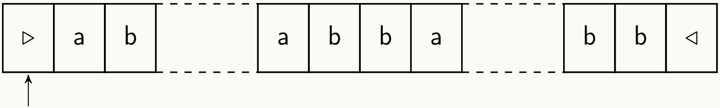

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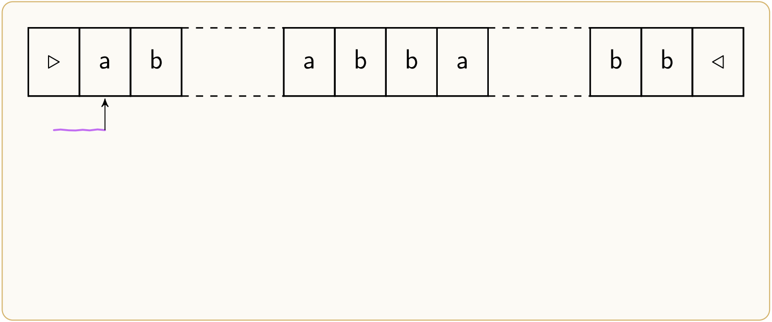

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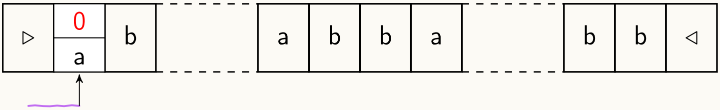

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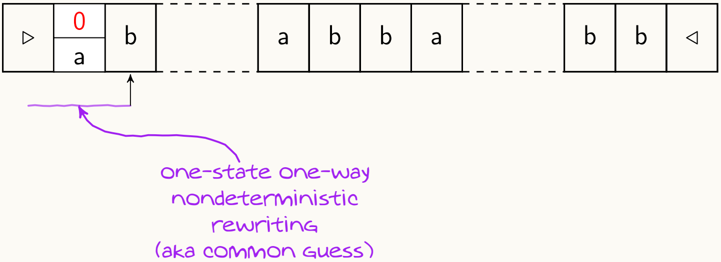

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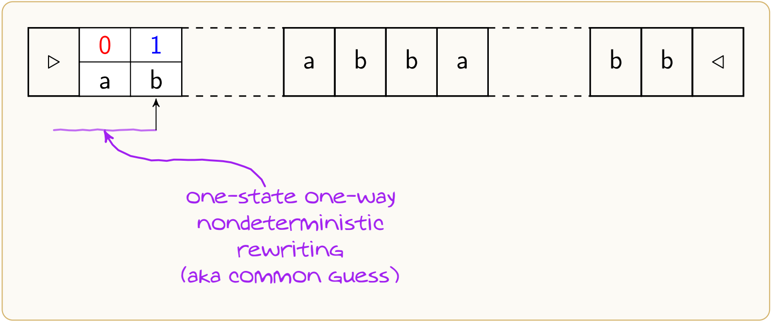

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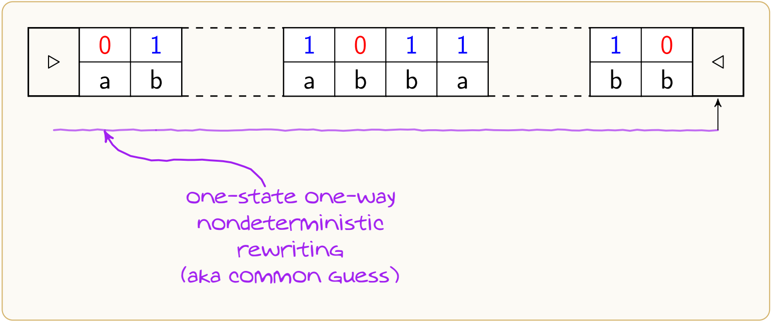

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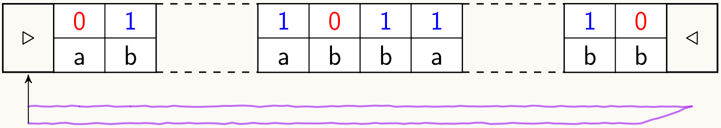

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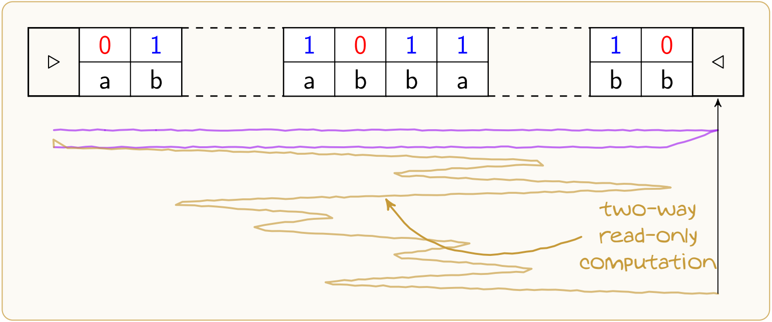

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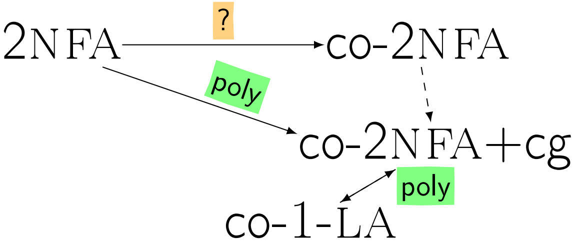
($\implies$ so do 2NFA+cgs*)*

Main result



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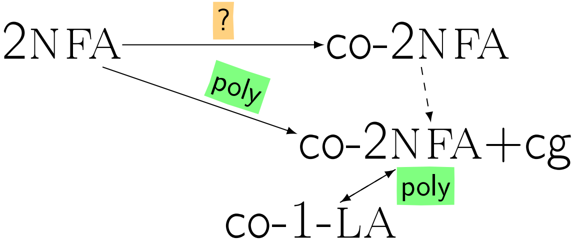

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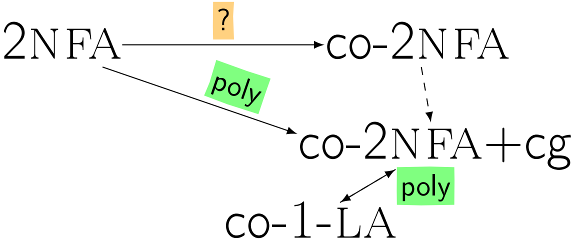

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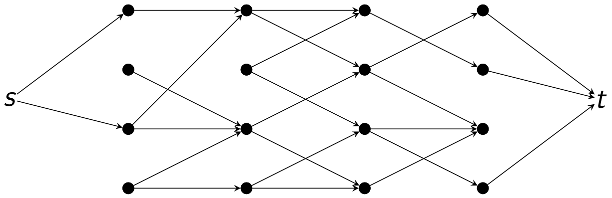

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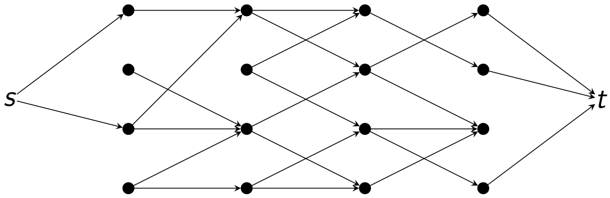

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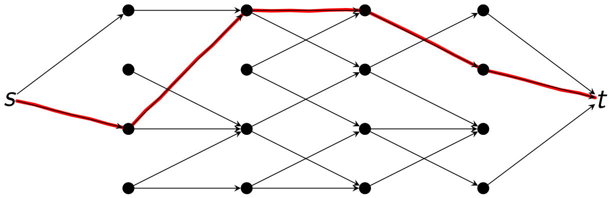

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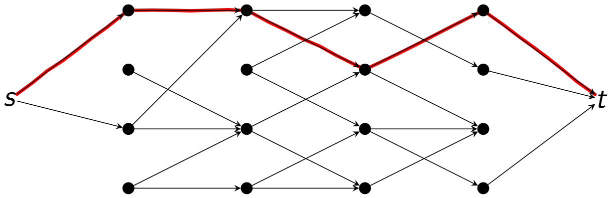

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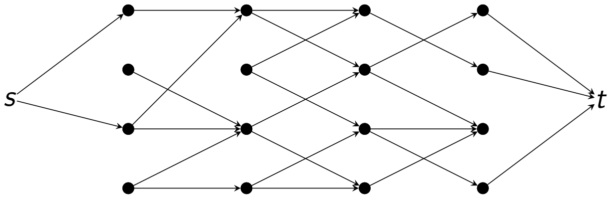

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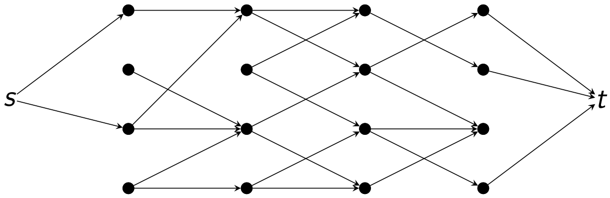

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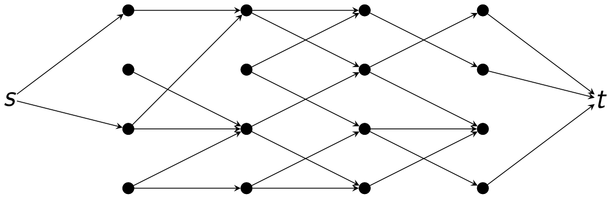

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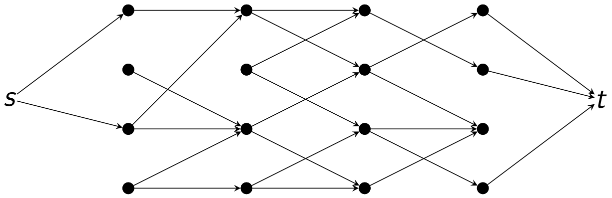

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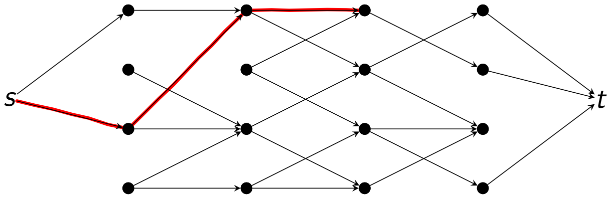

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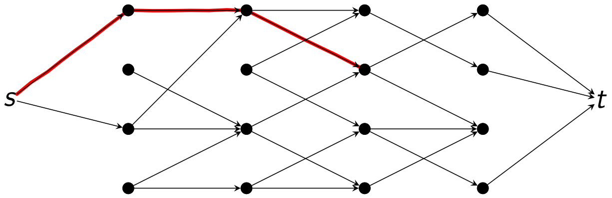

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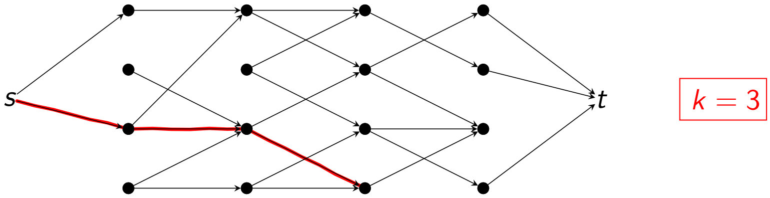

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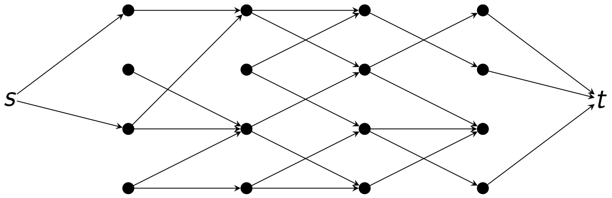

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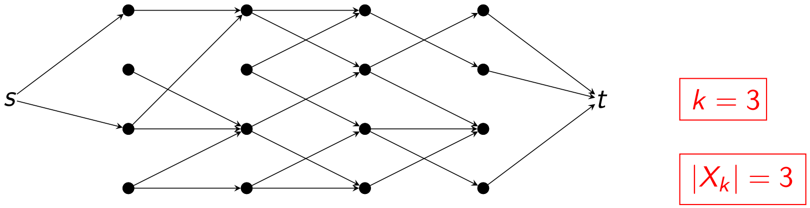

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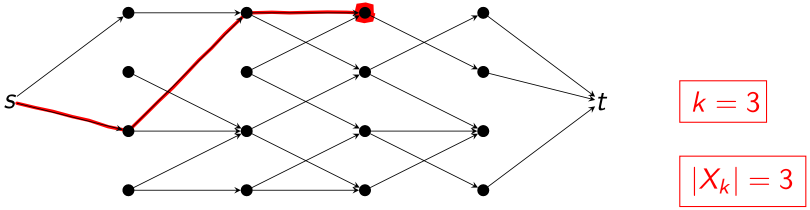

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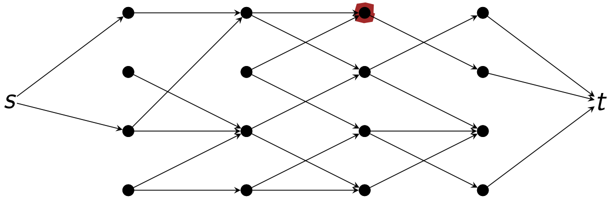

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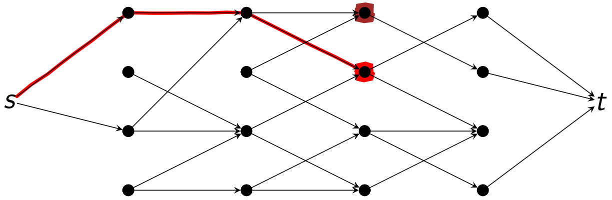

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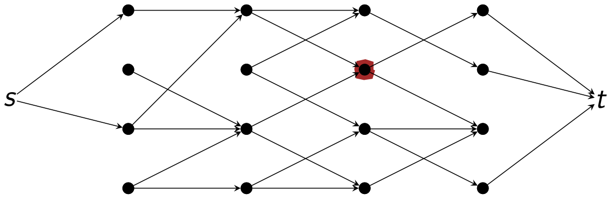

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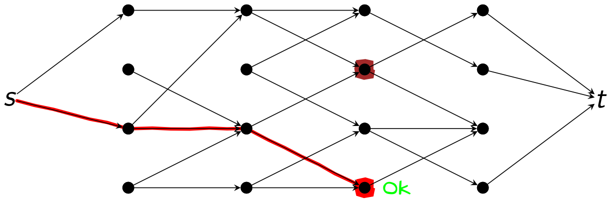

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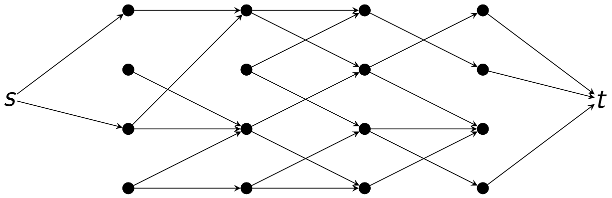

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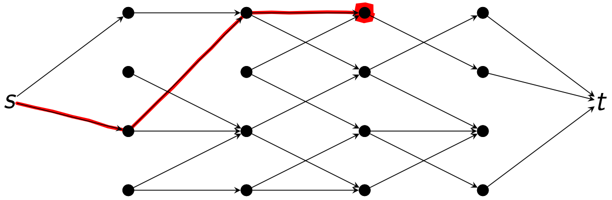

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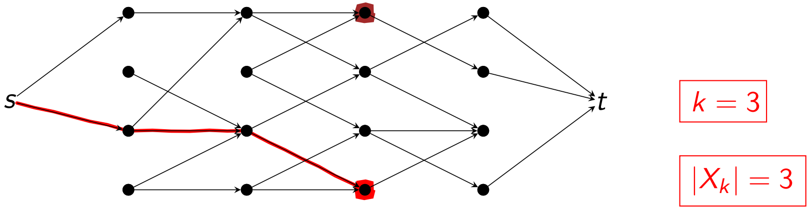

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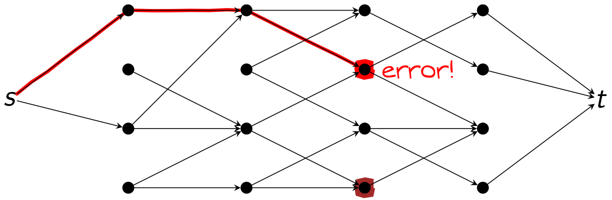
- $\exists$ path from s to t iff $t \in R_k$ for some $k < n$
- given k, using n-det and space $\log(n)$, we can find any $v \in R_k$



$k = 3$

$|X_k| = 3$

knowing $|R_k|$ we can enumerate R_k (assuming order)

Strategy: inductive counting

STCON problem

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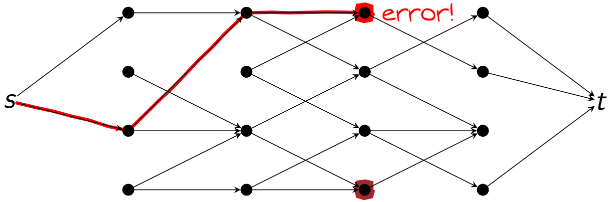

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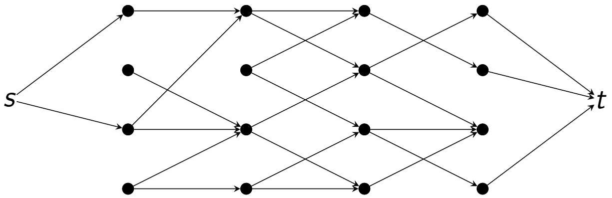

$|X_{k+1}| = 0$

$k = 3$

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knowing $|R_k|$ we can compute $|R_{k+1}|$

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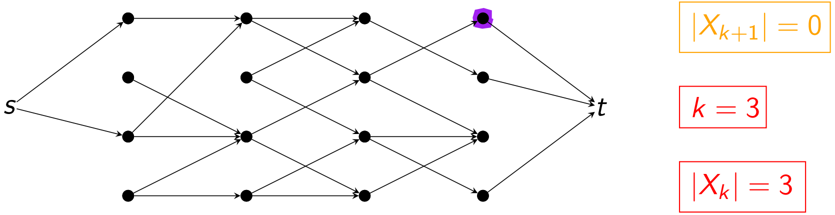

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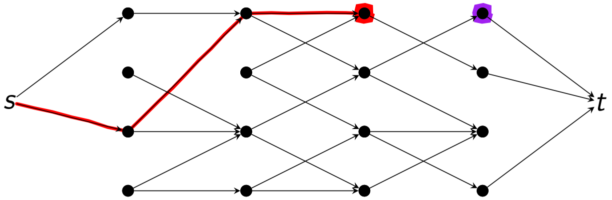

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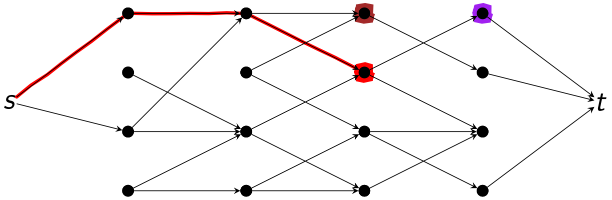

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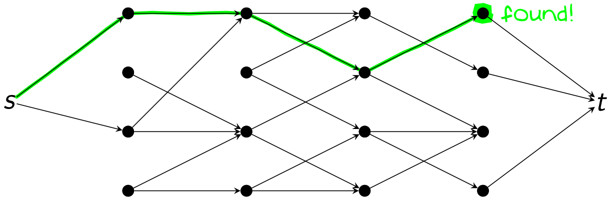

$|X_{k+1}| = 1$

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$|X_k| = 3$

knowing $|R_k|$ we can compute $|R_{k+1}|$

Strategy: inductive counting

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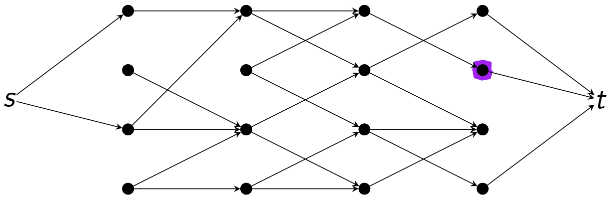

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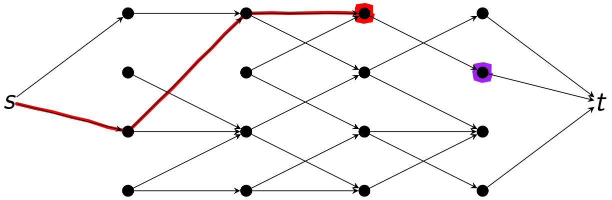

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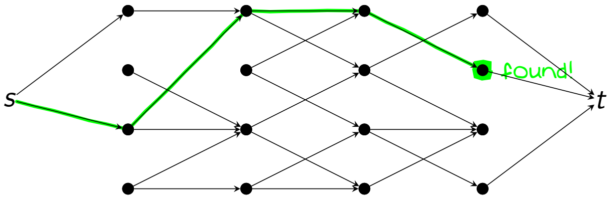

$|X_{k+1}| = 2$

$k = 3$

$|X_k| = 3$

knowing $|R_k|$ we can compute $|R_{k+1}|$

Strategy: inductive counting

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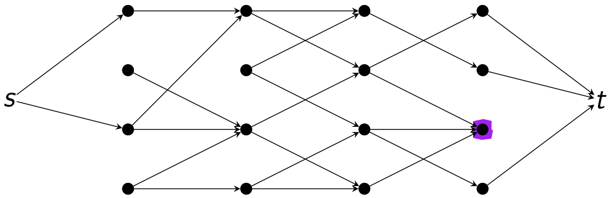

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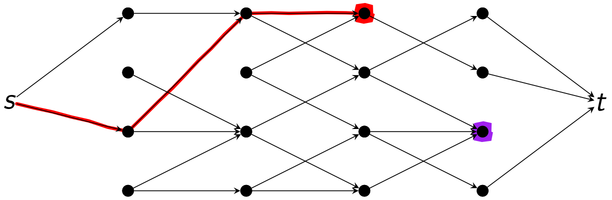

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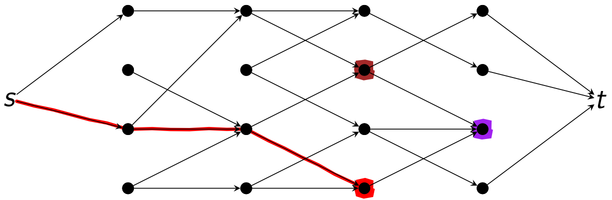

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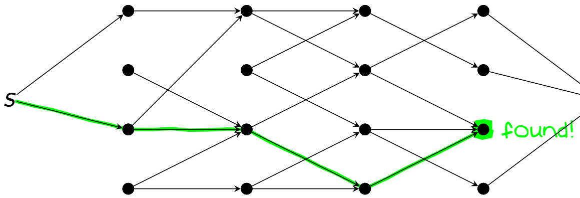

$|X_{k+1}| = 3$

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knowing $|R_k|$ we can compute $|R_{k+1}|$

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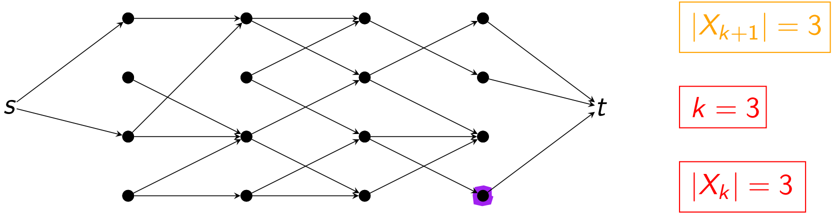

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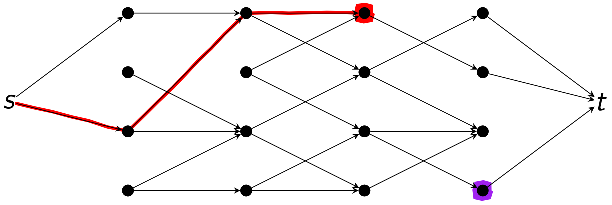

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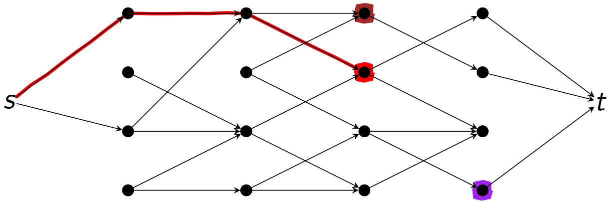

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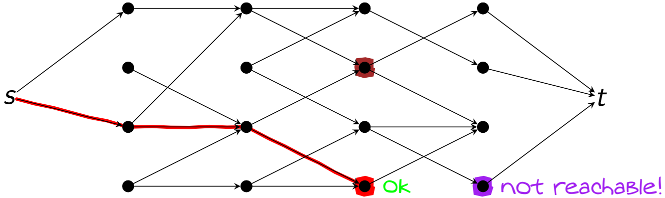

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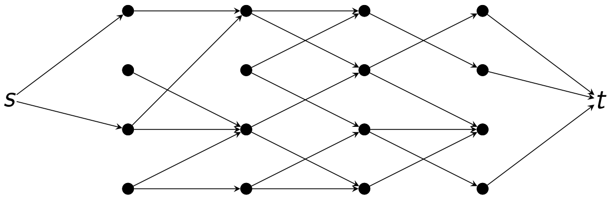

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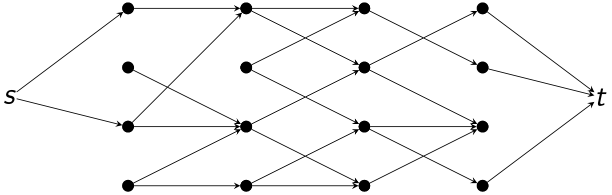

$|X_{k+1}| = 0$

$k = 4$

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inductively compute $|R_k|$ for $k < n$;

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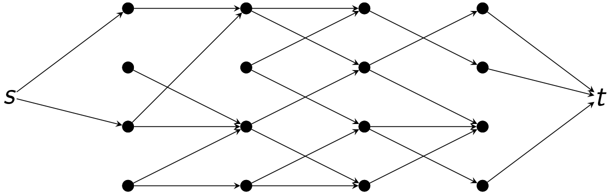

$|X_{k+1}| =$

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inductively compute $|R_k|$ for $k < n$; accept if t is reached, reject otherwise

Strategy: inductive counting

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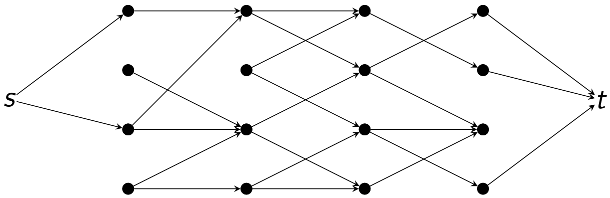

$|X_{k+1}| =$

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inductively compute $|R_k|$ for $k < n$; accept if t is reached, reject otherwise

space: store k, $|X_k|$, $|X_{k+1}|$ and 3 vertices

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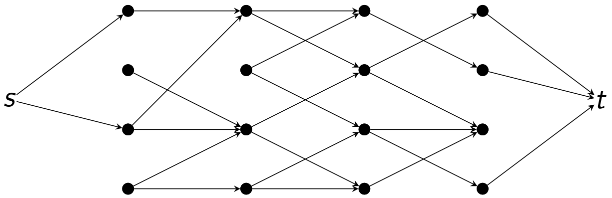

$|X_{k+1}| =$

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$|X_k| =$

inductively compute $|R_k|$ for $k < n$; accept if t is reached, reject otherwise

space: store k, $|X_k|$, $|X_{k+1}|$ and 3 vertices $\implies \mathcal{O}(\log(n))$

Complementing 1NFAs

Fix a 1NFA A with n states, and an input word w:

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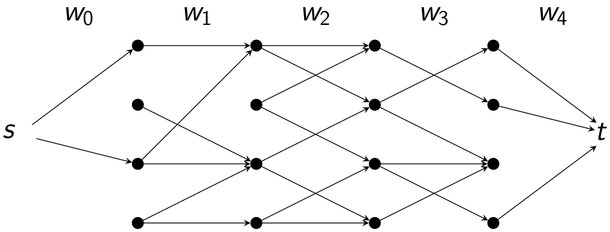

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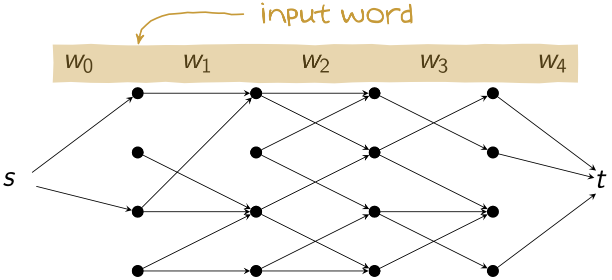

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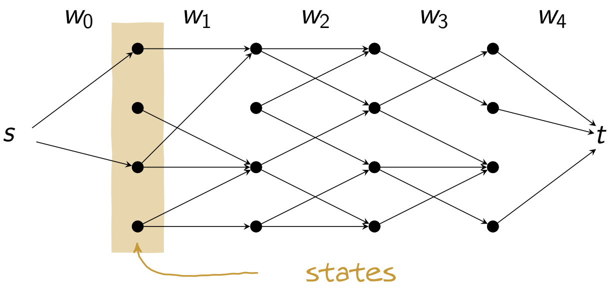

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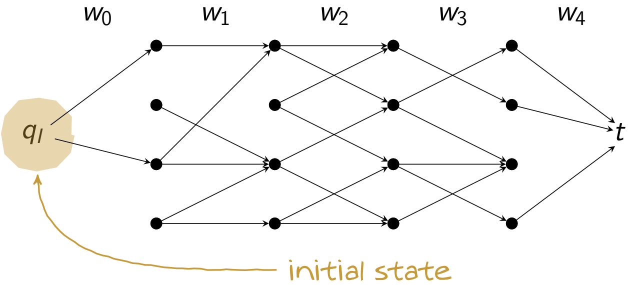

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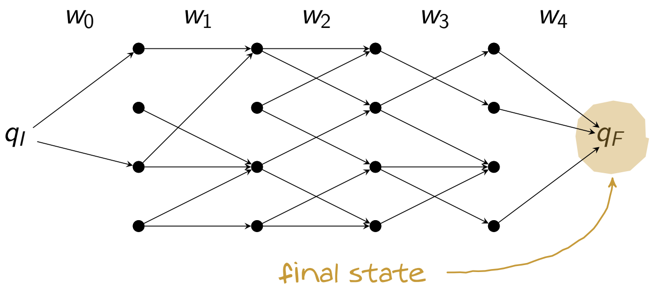

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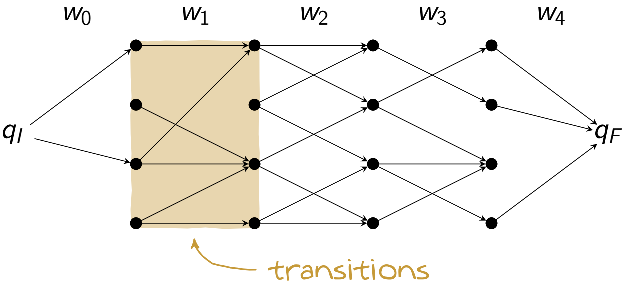

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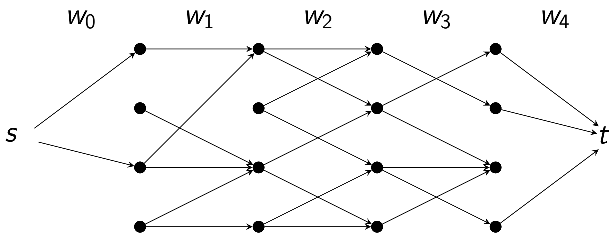

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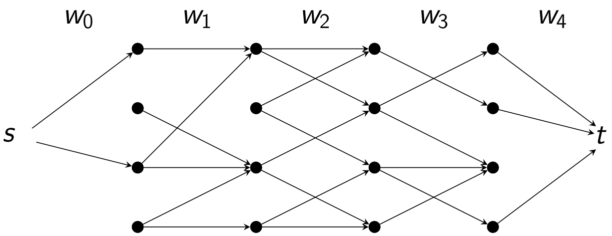

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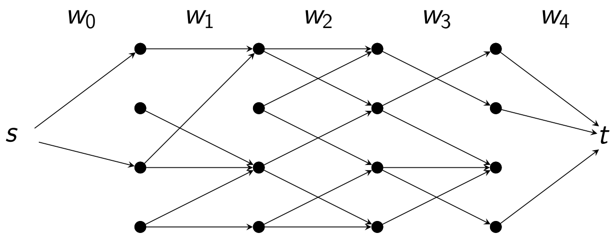

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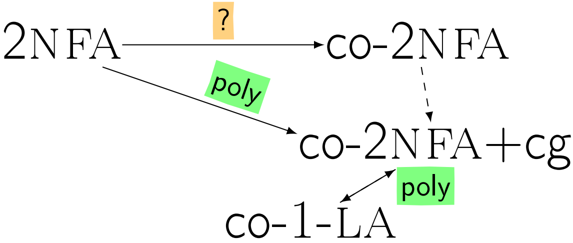
solution: annotate the tape so that it is sufficient
to rerun $\leq k$ steps of A from a config that is not far

Main result



Theorem

Each 2NFA *can be complemented by a* 1-LA *of polynomial size*

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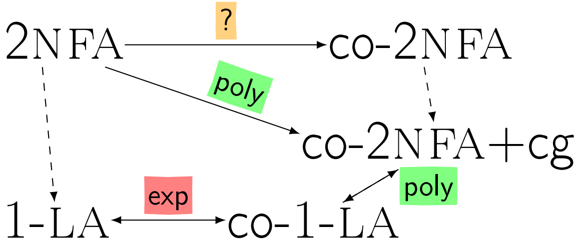

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Corollary

*Each 1-*LA *can be complemented by a 1-*LA *of exponential size (tight)*

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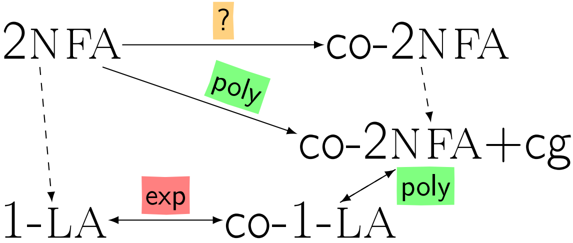

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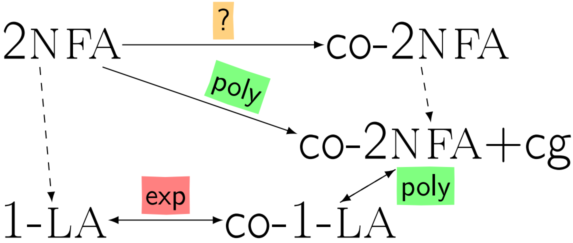

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thank you!