Planting and MCMC Sampling from the Potts model

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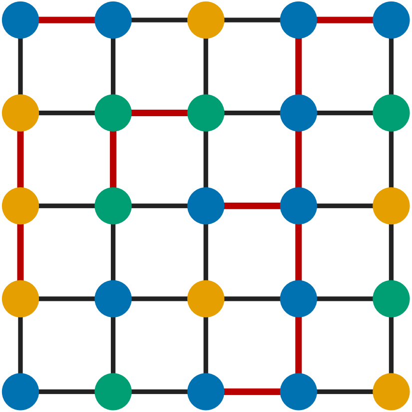
Potts Model

For a *graph* $G = (V, E)$ consider the set of *q-colourings* $\Omega = \{V \to [q]\}$

Probability of a colouring $\sigma \sim e^{\beta \, m(\sigma)}$

Inverse temperature parameter (β)

monochromatic edges under σ ($m(\sigma)$)



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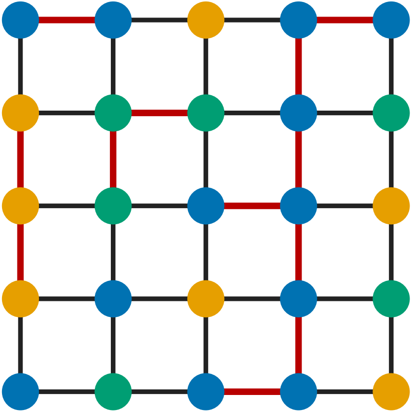

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A randomised algorithm with output distribution is (close to) π_G

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What about a “typical graphs”?

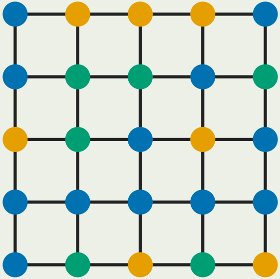
Random Cluster model

Markov chain-based algorithms working for all $\beta > 0$ on random regular graphs using representation via *random cluster model*

Random cluster model (RCM)

A distribution on *edge subsets*

Potts $\longrightarrow$ *RCM*: keep each edge with probability $1 - e^{-\beta}$



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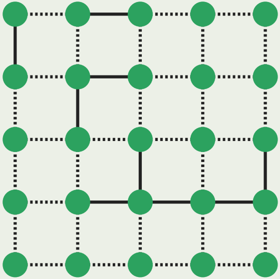

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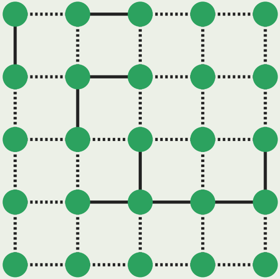

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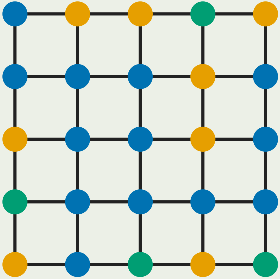

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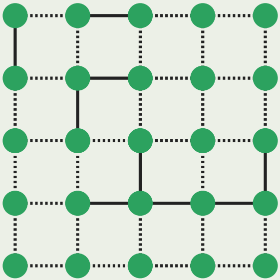
Probability of $F \subseteq E \sim$

$$q^{c(F)} (e^{\beta} - 1)^{|F|}$$

Number of components in (V, F)



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Sampling from RCM on random regular graphs

Glauber dynamics

States: $\Omega_G = \{F \subseteq E\}$

Transition $X_t \longrightarrow X_{t+1}$**:**

1. Pick a uniformly random edge $e \in E$
2. If e is a *cut-edge* in $X_t \cup \{e\}$ then
   1. With probability $\frac{1-e^{-\beta}}{1+e^{-\beta}(q-1)}$ set $X_{t+1} := X_t \cup \{e\}$
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Converges to the RC distribution

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How fast?

Sampling from RCM on random regular graphs

How many steps suffice for convergence?

- Blanca, Gheissari (2020): $O(n \log n)$ steps for low β and any q from any X_0
- Blanca, Gheissari (2023): $O(n^{1+o(1)})$ steps for high β and any q from any X_0
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Bottleneck between two typical sample types: worst-case exponential convergence for a window of β's

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Bottleneck between two typical sample types: worst-case exponential convergence for a window of β's

Can we extend the last result to all q?

Our result

Theorem (Galanis, Goldberg, **S**)

Let $d, q \geq 3$ and $\beta > 0$. Then with high probability for a random d-regular graph G:

- *If $\beta < \beta_c$ then the Glauber dynamics with $X_0 := \emptyset$ converges in $O(n \log n)$ steps.*
- *If $\beta > \beta_c$ and $q \geq (5d)^5$, then the Glauber dynamics with $X_0 := E$ converges in $O(n \log n)$ steps.*

Where β_c is a critical temperature where a typical sample changes from disordered (few edges in, colour proportions are roughly equal) to ordered (most of edges in, most of vertices have one colour)

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We modify the proof techniques for high-q using a first-order method known as *planting*

Phases of the Potts model

Key part of the proof: *typical configuration* does not differ "too much" from X_t

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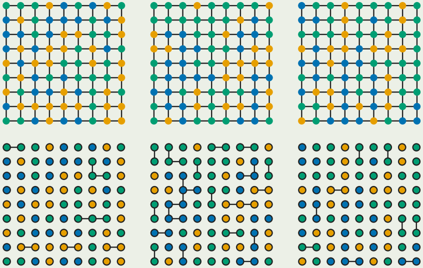
Key part of the proof: *typical configuration* does not differ "too much" from X_t
What is a *typical configuration*?

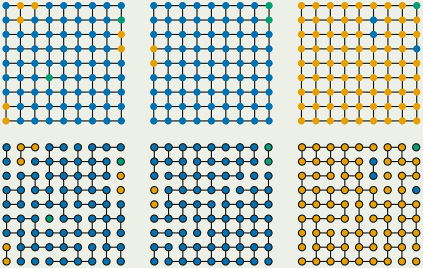
Disordered Phase ($\beta < \beta_c$)

Potts: all colours of roughly equal proportions
RCM: at most a small fraction of edges in a sample



Ordered Phase ($\beta > \beta_c$)

Potts: one dominant colour, the rest with ≈ equal proportions
RCM: vast majority of edges in a sample



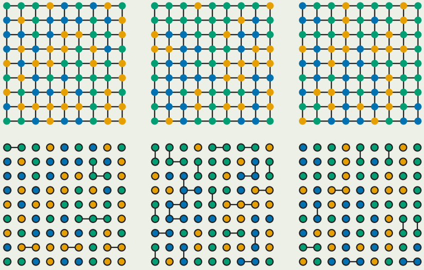
Planting of the Potts model

Disordered Phase ($\beta < \beta_c$)

Colours are symmetrical

$\#\,\textcircled{i} : \frac{n}{q} + o(n)$

$\#\,\textcircled{i}\!-\!\textcircled{j} \sim e^{\beta \cdot \mathbb{1}\{i=j\}}$



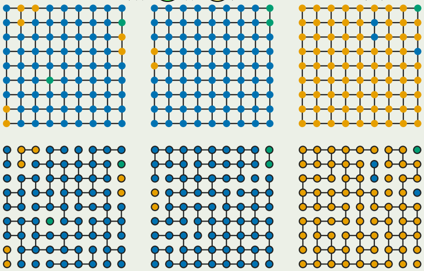
Ordered Phase ($\beta > \beta_c$)

Pick a u.a.r. dominant colour c

$\#\,\textcircled{i} \sim \frac{1-a}{q} + o(1)$ $\quad \#\,\textcircled{c} \sim a + o(1)$

$\#\,\textcircled{i}\!-\!\textcircled{j}$

$\sim e^{\beta \cdot \mathbb{1}\{i=j\}} (\#\textcircled{i}\#\textcircled{j})^{(d-1)/d} + o(1)$



Planting of the Potts model

Planting: we can generate a random graph and random sample at the same time

With high probability: (random graph, Potts sample) $\sim$ u.a.r. coloured graph with a given vertex and edge colour statistics

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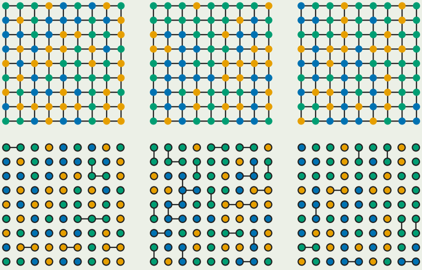

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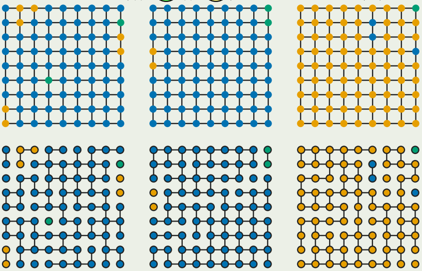


Key proof ingredient ($\beta < \beta_c$)

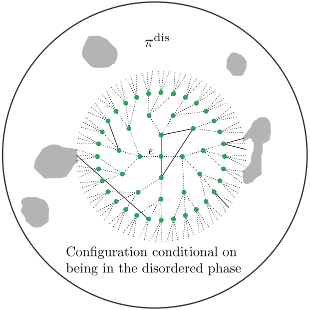
Weak Spatial Mixing within the disordered phase

Key proof ingredient: for every edge e show

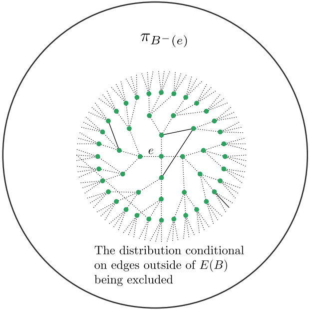
$$|\pi^{\text{dis}}(e \in \cdot) - \pi_{B^-(e)}(e \in \cdot)| = O(\tfrac{1}{n})$$

RC distribution conditional on being in the disordered phase (π^{dis})

RC distribution on $\approx \log n$-radius neighbourhood conditional on outside edges being out ($\pi_{B^-(e)}$)



Configuration conditional on being in the disordered phase



The distribution conditional on edges outside of $E(B)$ being excluded

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RC distribution conditional on being in the disordered phase

RC distribution on ≈ log n-radius neighbourhood conditional on outside edges being out

This is at most ≈ π^{dis}(two disjoint paths from the boundary to e)

Low if all but a constant number of boundary vertices are in different components

Bound using first-order methods by analysing the planted random graph

Key proof ingredient ($\beta > \beta_c$)

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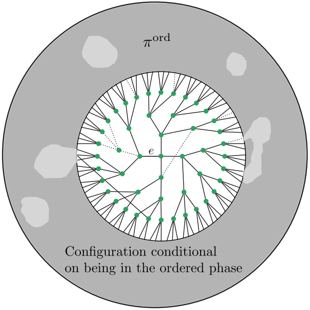
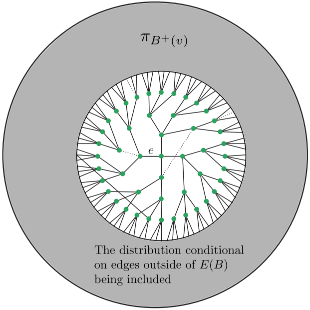
RC distribution on $\approx \log n$-radius neighbourhood conditional on outside edges being included ($\pi_{B^+(e)}$)





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Percolation of (almost) random regular graph $\implies$ largest component is very large

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Planting: with high probability, every length$\approx \log n$ path hits the giant

Open questions

- $\tilde{O}(n)$ sampling algorithm for Potts on random regular graphs for almost all parameter ranges
  - Can we improve q from $(5d)^5$ to 3?
    The bottleneck is in the size of the largest component of the typical RCM sample
  - Can we obtain algorithm for β_c?
    We need to be able to estimate the proportions of the phases
- Can we extend planting to random *average* bounded degree graphs?
  - Approach for random regular graphs does not result in correct degree statistics for the ordered phase

I am looking for a postdoc!

Additional Slides

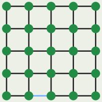
Proof Sketch ($p > p_c$)

Glauber from all-in — $\pi_G(\cdot \mid \cdot \text{ in ordered phase})$

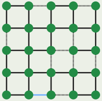
$$\|\pi_G(\cdot; q, p) - X_t\|_{TV} \le \underbrace{\|\pi_G(\cdot; q, p) - \pi^{\text{ord}}\|_{TV}}_{e^{-\Omega(n)}} + \underbrace{\|\pi^{\text{ord}} - X_t\|_{TV}}_{\textbf{Goal: } \text{show} \le \frac{1}{5}}$$

Two coupled copies of Glauber

X_t: starting from all-in



$\hat{X}_t$: starting from π^{ord}, restricted to ordered phase



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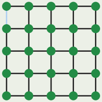

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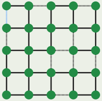

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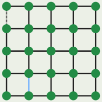

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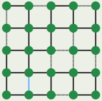

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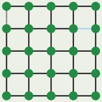

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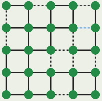

$$|X_t - \hat{X}_t|_{TV} \le \Pr(\hat{X}_t \ne X_t)$$

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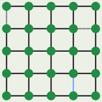
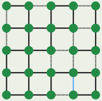
Two coupled copies of Glauber

X_t: starting from all-in



$\hat{X}_t$: starting from π^{ord},
restricted to ordered phase



$$|X_t - \hat{X}_t|_{TV} \le \Pr(\hat{X}_t \neq X_t)$$

Union bound:

$$\le \sum_{e \in E} \Pr(X_t, \hat{X}_t \text{ disagree on } e)$$

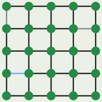
Proof Sketch ($p > p_c$)

Glauber from all-in — $\pi_G(\cdot \mid \cdot \text{ in ordered phase})$

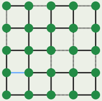
$$|\pi_G(\cdot; q, p) - X_t|_{TV} \le \underbrace{|\pi_G(\cdot; q, p) - \pi^{\text{ord}}|_{TV}}_{e^{-\Omega(n)}} + \underbrace{|\pi^{\text{ord}} - X_t|_{TV}}_{\textbf{Goal: } \text{show} \le \frac{1}{5}}$$

Two coupled copies of Glauber

X_t: starting from all-in



$\hat{X}_t$: starting from π^{ord}, restricted to ordered phase



$|X_t - \hat{X}_t|_{TV} \le \Pr(\hat{X}_t \ne X_t)$

Union bound:

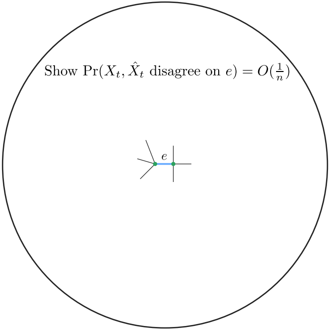
$$\le \sum_{e \in E} \Pr(X_t, \hat{X}_t \text{ disagree on } e)$$

Monotonicity: As long as $(\hat{X}_t)_{t \le T}$ has not ignored any updates: $\hat{X}_T \le X_T$

Unlikely to happen in polynomially many steps

Proof sketch ($p > p_c$): edge marginals

Glauber restricted to the ordered phase, from π^{ord}

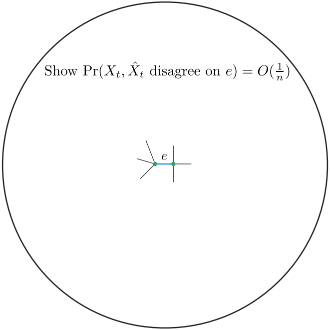


Goal: For any edge e, show that $\Pr(X_t, \hat{X}_t$ disagree on $e)$ is small enough

Glauber from all-in

Proof sketch ($p > p_c$): edge marginals

Glauber restricted to the ordered phase, from π^{ord}



Goal: For any edge e, show that $\Pr(X_t, \hat{X}_t$ disagree on $e)$ is small enough

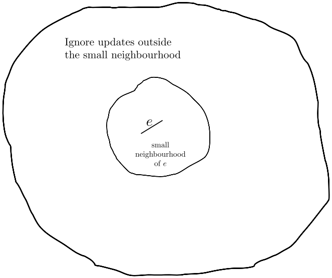
Glauber from all-in

Use *local behaviour* to extrapolate something *global*

Idea: can we show edges too far away have little influence on e?

Proof sketch ($p > p_c$): edge marginals

Glauber restricted to the ordered phase, from π^{ord}



Goal: For any edge e, show that

Pr(X_t , $\hat{X}_t$ disagree on e) is small enough

Glauber from all-in

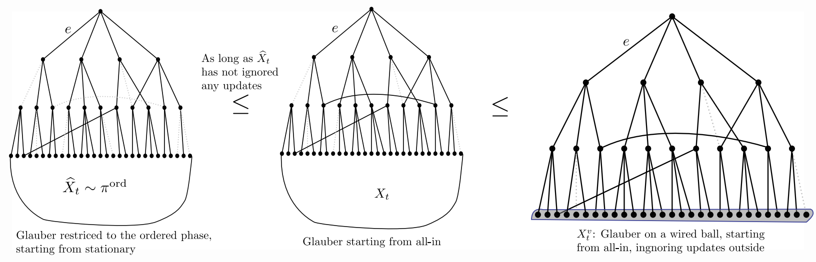
Use *local behaviour* to extrapolate something *global*

Idea: can we show edges too far away have little influence on e?

Compare to an *intermediate distribution*: a chain restricted to a *neighbourhood* around e, starting from all-in

Proof sketch ($p > p_c$): edge marginals

Pick $v \in V$ incident to e, $r \sim \frac{\log n}{d}$: consider a restricted chain, X_t^v, ignoring updates outside a the radius-r ball around v, starting from all-in



As long as $\hat{X}_t$ has not ignored any updates that would take it outside the phase, $\hat{X}_t \leq X_t \leq X_t^v$

Proof sketch ($p > p_c$): edge marginals

Conditioned on $\hat{X}_T$ having not ignored any updates, probability that X_T and $\hat{X}_T$ disagree on e is at most

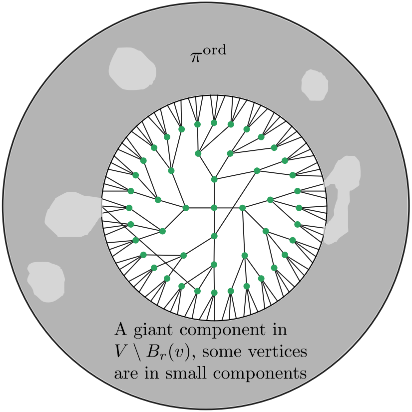
$$\left|\Pr(e \in \underbrace{X_T^v}_{\text{Glauber on the \textit{wired} ball}} \mid \hat{X}_T \text{ ignored no updates}) - \Pr(e \in \underbrace{\hat{X}_T}_{\text{Glauber restricted to the ordered phase}} \mid \hat{X}_T \text{ ignored no updates})\right|$$

$$\leq C \cdot \Pr(\hat{X}_T \text{ ignored no updates}) + |\Pr(e \in X_T^v) - \Pr(e \in \hat{X}_T)|$$

$$\leq C \cdot \overbrace{\Pr(\hat{X}_T \text{ ignored no updates})}^{\text{unlikely for } T \text{ a polynomial}}$$

$$+ \underbrace{|\Pr(e \in X_T^v) - \pi_{B^+(v)}(e \in \cdot)|}_{\textit{Local mixing on tree-like neighbourhoods}} + \underbrace{|\overbrace{\pi_{B^+(v)}(e \in \cdot)}^{\text{RC on the wired ball}} - \overbrace{\Pr(e \in \hat{X}_T)}^{\pi^{\mathrm{ord}}(e \in \cdot)}|}_{\textit{Weak spatial mixing within the ordered phase}}$$

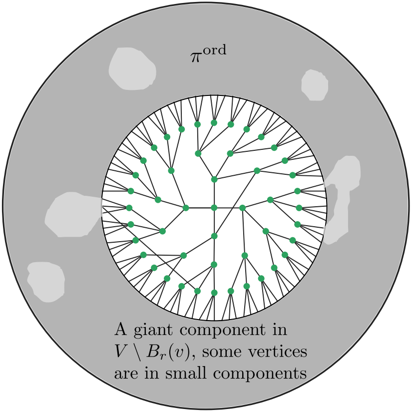
WSM within the ordered phase



If all vertices of $\partial B_r(v)$ are in the giant component, we are done

Otherwise, we need to reveal further

WSM within the ordered phase



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Question: what is a phenomenon that causes us to reveal further? How to characterize the event "*we reveal e before getting connected boundary*"?

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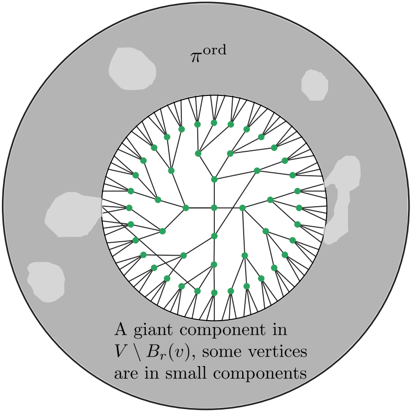

If all vertices of $\partial B_r(v)$ are in the giant component, we are done

Otherwise, we need to reveal further

Question: what is a phenomenon that causes us to reveal further? How to characterize the event "*we reveal e before getting connected boundary*"?

Answer: long path avoiding the giant of $(V, F \setminus \{e\})$