A Permanental Analog of the Rank-Nullity Theorem for Symmetric Matrices[1]

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STACS 2026

[1] Joint work with Priyanshu Pant, S. Chakrabartty.

Twins

Similar looks but contrasting nature

Determinant vs Permanent

School days revision

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Determinant of A is

$$\begin{aligned} \det(A) &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei - afh - bdi + bgf + cdh - ceg. \end{aligned}$$

Permanent of A is

$$aei + afh + bdi + bgf + cdh + ceg.$$

The determinant and permanent of an $n \times n$ matrix $M = (a_{ij})$ are defined as

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)},$$

$$\operatorname{per}(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

respectively, where S_n is the set of all permutations of $\{1, 2, \ldots, n\}$, the sign function $\operatorname{sgn}(\sigma)$ is 1, if σ is even and -1 if σ is odd, respectively.

Complexities

Determinant of any matrix can be computed in $\mathcal{O}(n^{2.37})$. Thanks to Gaussian elimination.

Computing the permanent is known to be $\#P$-complete even for a $(0,1)$-matrix.

Where they appear?

	Determinant	**Permanent**
	$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$	$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$
Quantum Physics	**Fermions**	**Bosons**
Uses	Linear Algebra, Volume/Geometry/ Everywhere	Combinatorics / Statistical mechanics
Counting in Graphs	**Spanning trees**	**Perfect Matchings**
Computationally	**Easy**	**Hard (?)**

So we have

- **Determinant** $\in$ **P** [KV05] & **Permanent** $\in \#$**P** [Val79].
- One sign change but a huge complexity gap.
- Why removing the sign factor makes the complexity jump from **P** to #**P**?

Why is determinant faster?

For any $n \times n$ matrix A (after reordering rows if needed),

$$A = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \ell_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}}_{U}.$$

$$\det(A) = \det(LU) = \det(L)\det(U)$$

$$\det(A) = \prod_{i=1}^{n} u_{ii}.$$

This fails for the permanent:

$$\mathrm{per}(AB) \neq \mathrm{per}(A)\,\mathrm{per}(B) \text{ in general.}$$

One key algebraic property: multiplicativity

Determinant

Multiplicative:

$$\det(AB) = \det(A)\det(B)$$

Permanent

Not multiplicative:

$$\mathrm{per}(AB) \neq \mathrm{per}(A)\,\mathrm{per}(B) \text{ in general.}$$

So elimination-style factorizations do *not* yield a shortcut for $\mathrm{per}(\cdot)$.

$\Rightarrow$ This single identity is why LU helps det, but not per.

What survives in both worlds? Studying analogs

Determinant

- **Characteristic polynomial:**

$$\chi(A) = \det(xI - A)$$

- **Roots** = eigenvalues $\lambda_1, \ldots, \lambda_n$.

Permanent

- **Permanental polynomial**[1]:

$$\pi(A) = \operatorname{per}(xI - A)$$

- **Per-roots:** roots of $\pi(A) = 0$.

[1] Permanental polynomials of graphs: [Tur68; MRW81; KTG81].

More analogs: rank and nullity

Determinant

- **Nullity** (if $A^{\mathsf{T}} = A$):
  $\eta(A) =$ multiplicity of 0 as an eigenvalue.
- **Rank**:
  largest k such that some $k \times k$ minor has det $\neq 0$.

Permanent

- **Permanental nullity**[1]:
  $\eta_{\text{per}}(A) =$ multiplicity of 0 as a root of $\pi(A)$.
- **Permanental rank**[2]:
  largest k such that some $k \times k$ submatrix has per $\neq 0$.

[1] Per-nullity: [WZ15].
[2] Per-rank: [Yu99].

A natural question

Determinant

- **Rank–Nullity theorem:**

$$\rho(A) + \eta(A) = n.$$

Permanent

- **Do we also have:**

$$\rho_{\mathrm{per}}(A) + \eta_{\mathrm{per}}(A) = n \text{ ?}$$

Our work

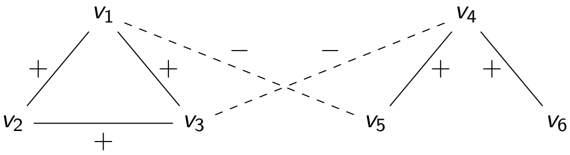
A permanental analog of rank–nullity

Let A be an $n \times n$ matrix. Then

$$\rho_{\text{per}}(A) + \eta_{\text{per}}(A) \geq n.$$

For the following classes equality holds:

1. Symmetric nonnegative matrices
2. Positive semidefinite matrices
3. Adjacency matrices of balanced signed graphs



How did we prove it?

(High level idea for nonnegative symmetric matrices)

Let $\mathrm{per}(xI - A) = \sum_{i=0}^{n} b_i x^{n-i}$ then $b_k = (-1)^k \sum_{|S|=k} \mathrm{per}(A[S,S])$ [Min78]

$\Downarrow$

If $\rho_{\mathrm{per}}(A) = k \Rightarrow b_i = 0$ for $i > k \Rightarrow \rho_{\mathrm{per}}(A) + \eta_{\mathrm{per}}(A) \geq n$

$\Downarrow$

Goal: show $b_k \neq 0 \Rightarrow \eta_{\mathrm{per}}(A) = n - k$

$\Downarrow$

We prove if $A = A^{\mathrm{T}}$ and $\rho_{\mathrm{per}}(A) = k \Rightarrow$ a k-vertex directed cycle cover on G associated with A

$\Downarrow$

$\exists\, S,\ |S| = k$ with $\mathrm{per}(A[S,S]) \neq 0$

$\Downarrow$

If $A \geq 0$: $b_k = (-1)^k \sum_{|S|=k} \mathrm{per}(A[S,S]) > 0 \Rightarrow b_k \neq 0$

How did we prove it?

(Positive semidefinite matrices)

If $\rho_{\text{per}}(A) = k$, then some $k \times k$ submatrix has

$$\text{per} \neq 0.$$

For PSD matrices, we show this implies

$$\exists\, S \subseteq [n],\ |S| = k, \quad \text{per}(A[S,S]) \neq 0.$$

Also, every principal submatrix is PSD, so[2]

$$\text{per}(A[S,S]) \geq 0 \qquad \text{for all } S.$$

Hence

$$b_k = (-1)^k \sum_{|S|=k} \text{per}(A[S,S]) \neq 0,$$

and therefore

$$\rho_{\text{per}}(A) + \eta_{\text{per}}(A) = n.$$

[2] [Pak23]

How did we prove it?

(High level idea for balanced signed graphs)

Let A be the signed adjacency matrix of a balanced signed graph.
We can write[3],

$$A = DBD,$$

where B is the adjacency matrix of the underlying unsigned graph, and D is a suitable diagonal matrix entries ± 1. We show

$$\rho_{\text{per}}(A) = \rho_{\text{per}}(B).$$

Since B is nonnegative matrix

$$\rho_{\text{per}}(B) + \eta_{\text{per}}(B) = n \implies \rho_{\text{per}}(A) + \eta_{\text{per}}(A) = n.$$

[3] [Har53]

Characterization

Let $A \in \{0, \pm 1\}^{n \times n}$ be symmetric and let $k = \rho_{\mathrm{per}}(A)$. Then

$$\rho_{\mathrm{per}}(A) + \eta_{\mathrm{per}}(A) = n \quad \Longleftrightarrow \quad E_k \neq O_k,$$

where E_i and O_i denote the positive- and negative-weight i-vertex cycle-cover contributions, respectively; namely,

$$E_i := (-1)^i \sum_{\substack{S \subseteq [n] \\ |S| = i}} |\{C \in \mathcal{L}(S) : w(C) = +1\}|, \qquad O_i := (-1)^i \sum_{\substack{S \subseteq [n] \\ |S| = i}} |\{C \in \mathcal{L}(S) : w(C) = -1\}|.$$

In words, equality holds exactly when the positive and negative k-vertex cycle-cover contributions do not cancel.

Computational takeaway:

Decision problem:

$$\textbf{Per-Rank}: \text{ Given } (A,k), \text{ decide whether } \rho_{\text{per}}(A) \geq k.$$

In general: PER-RANK is NP-hard.

But on some structured classes (nonnegative /PSD/ balanced signed adjacency),

Per-Rank is **poly-time**

Thank You

Questions?

References I

[Har53] Frank Harary. "On the notion of balance of a signed graph.". In: *Michigan Mathematical Journal* 2 (1953), pp. 143–146. URL: https://doi.org/10.1307/MMJ%2F1028989917.

[KTG81] D Kasum, N Trinajstić, and I Gutman. "Chemical graph theory. III. On the permanental polynomial". In: *Croatica Chemica Acta* 54.3 (1981), pp. 321–328. URL: https://hrcak.srce.hr/194320.

[KV05] Erich Kaltofen and Gilles Villard. "On the complexity of computing determinants". In: *Computational Complexity* 13.3 (2005), pp. 91–130. ISSN: 1420-8954. URL: https://doi.org/10.1007/s00037-004-0185-3.

[Min78] Henryk Minc. *Permanents*. Addison-Wesley, 1978.

References II

[MRW81] Russell Merris, Kenneth R Rebman, and William Watkins. “Permanental polynomials of graphs”. In: *Linear Algebra and Its Applications* 38 (1981), pp. 273–288. URL: https://doi.org/10.1016/0024-3795(81)90026-4.

[Pak23] Vehbi Emrah Paksoy. “A permanent inequality for positive semidefinite matrices”. In: *Electronic Journal of Linear Algebra* 39 (2023), p. 71. URL: https://doi.org/10.13001/ela.2023.7701.

[Tur68] James Turner. “Generalized matrix functions and the graph isomorphism problem”. In: *SIAM Journal on Applied Mathematics* 16.3 (1968), pp. 520–526. URL: https://doi.org/10.1137/0116041.

[Val79] Leslie G Valiant. “The complexity of computing the permanent”. In: *Theoretical computer science* 8.2 (1979), pp. 189–201. URL: https://doi.org/10.1016/0304-3975(79)90044-6.

References III

[WZ15] Tingzeng Wu and Heping Zhang. "Per-spectral characterizations of graphs with extremal per-nullity". In: *Linear Algebra and its Applications* 484 (2015), pp. 13–26. URL: http://dx.doi.org/10.1016/j.laa.2015.06.018.

[Yu99] Yang Yu. "The permanent rank of a matrix". In: *J. Comb. Theory, Ser. A* 85.2 (1999), pp. 237–242. URL: https://doi.org/10.1006/jcta.1998.2904.