On the *p*-adic Skolem Problem

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The Skolem Problem

Instance: An LRS u.
Problem: Is there $n \in \mathbb{N}$ such that $u_n = 0$?

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Next slides: what is $\mathbb{Z}_p$? What does it mean for $x \in \mathbb{Z}_p$ to be a zero of LRS u? Why do we care?

p-adic numbers – a very quick tour

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Facts:

- $\mathbb{Z}_p$ is a complete, compact ring – nice analytical properties!
- Keeps nice number theory structure - e.g. $x \bmod p^r$ makes sense for all $x \in \mathbb{Z}_p$.

p-adic numbers - examples

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Every $x \in \mathbb{Z}_p$ has a series expansion $x = \sum_{j=0}^{\infty} a_j p^j$ with $a_j \in \{0, 1, \ldots, p-1\}$.

- $2026 = 3 + 2 \cdot 7 + 6 \cdot 7^2 + 5 \cdot 7^3 \in \mathbb{Z}_7$
- $-1 = 1 + 1 \cdot 2 + 1 \cdot 2^2 + 1 \cdot 2^3 + \cdots \in \mathbb{Z}_2$
- $\frac{1}{2} = 2 + 1 \cdot 3 + 1 \cdot 3^2 + 1 \cdot 3^3 + \cdots \in \mathbb{Z}_3$
- $\sqrt{2} = 3 + 1 \cdot 7 + 2 \cdot 7^2 + 6 \cdot 7^3 + \cdots \in \mathbb{Z}_7$
- $\sqrt{-1} = 2 + 5 + 2 \cdot 5^2 + 1 \cdot 5^3 + \cdots \in \mathbb{Z}_5$

Why $\mathbb{Z}_p$?

Theorem (Skolem-Mahler-Lech, 1935-1957)

The set $Z(u) = \{n \in \mathbb{N} : u_n = 0\}$ of zeros of an LRS is a union of finitely many arithmetic progressions and a finite set.

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$$f_\ell : \mathbb{Z}_p \to \mathbb{Z}_p, \quad x \mapsto \sum_{k=0}^{\infty} b_{k,\ell} x^k$$

such that $f_\ell(n) = u_{Mn+\ell}$ for all $n \in \mathbb{N}$, for each $0 \leq \ell \leq M-1$.

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Each f_ℓ is analytic on compact domain $\rightarrow$ either identically zero or has finitely many zeros.

p-adic zeros of LRS

Key point: each subsequence $u_{Mn+\ell}$ is a function

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Definition (p-adic zero)

Call $x \in \mathbb{Z}_p$ a p-adic zero of LRS u if $f_\ell(x) = 0$ for some $0 \leq \ell \leq M - 1$.

The p-adic Skolem Problem

Theorem (Skolem-Mahler-Lech, 1935-1957)

The set of zeros of an LRS is a union of finitely many arithmetic progressions and a finite set.

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Theorem (Skolem-Mahler-Lech, 1935-1957) (true version)

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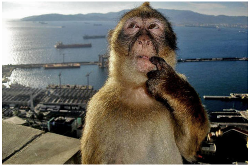
Correctness is unconditional!

Wait a minute...



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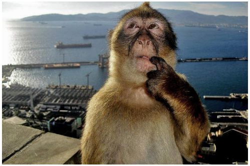

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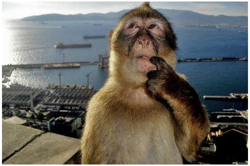

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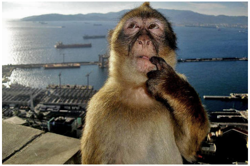

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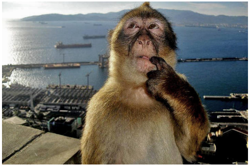

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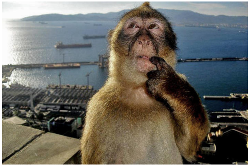

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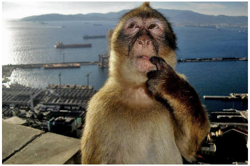

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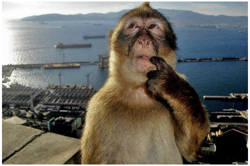

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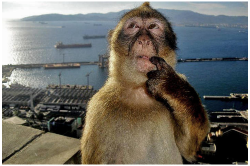

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But sometimes this is possible...

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Instance: Two LRS u, v.
Problem: Is there $n \in \mathbb{N}$ such that $u_n = v_n = 0$?

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$\rightarrow$ Use p-adic Skolem algorithm to find simultaneous p-adic zeros, then natural number zeros can be identified.