Optimal Verification of a Minimum-Weight Basis in an Uncertainty Matroid

Haya Diwan, Lisa Hellerstein
New York University

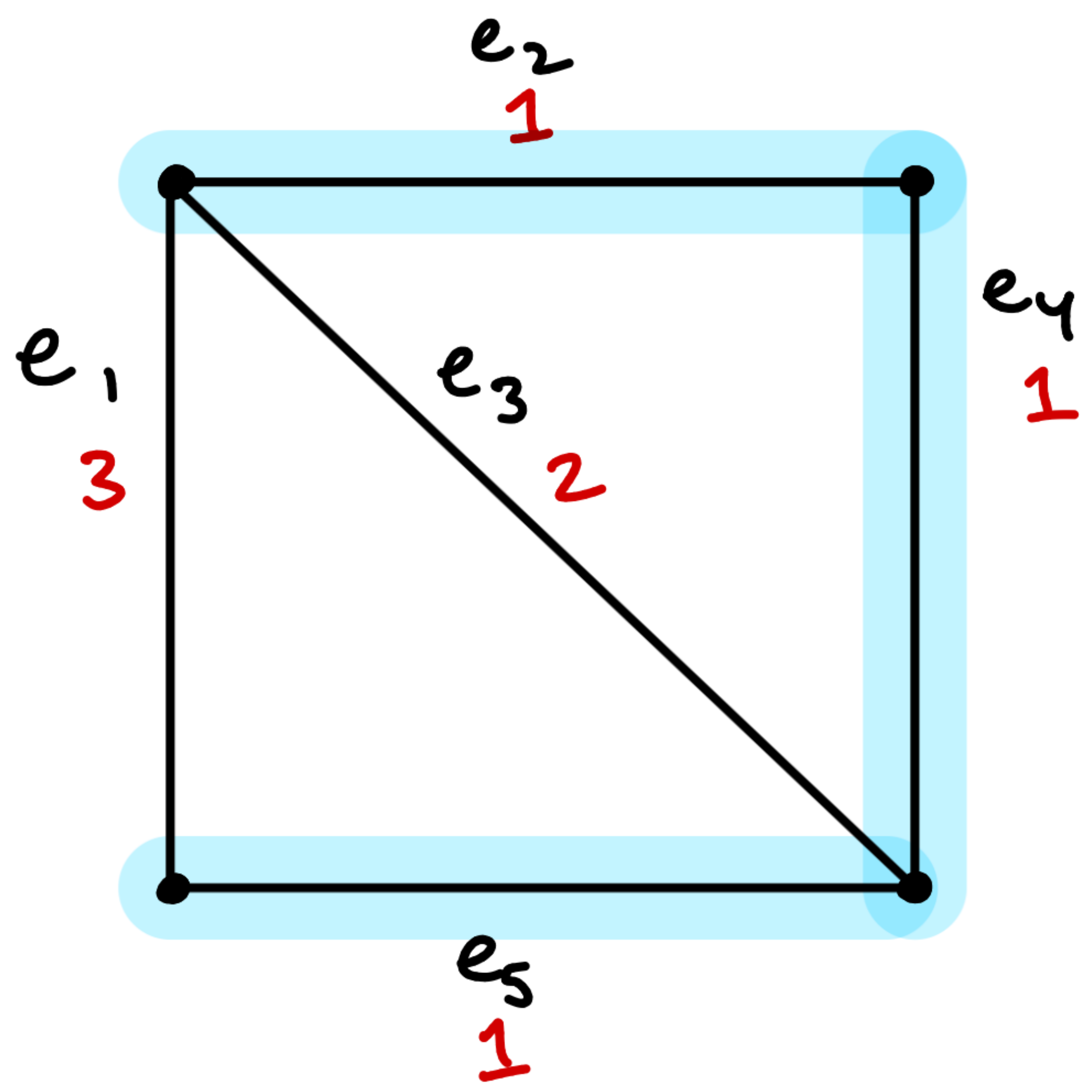
Nicole Megow
University of Bremen

Jens Schlöter
Centrum Wiskunde & Informatica (CWI)

Minimum Spanning Tree (MST) Problem

- Find an MST on the connected weighted graph $G = (V, E, w)$

- How can we easily verify that the highlighted edges form an MST?



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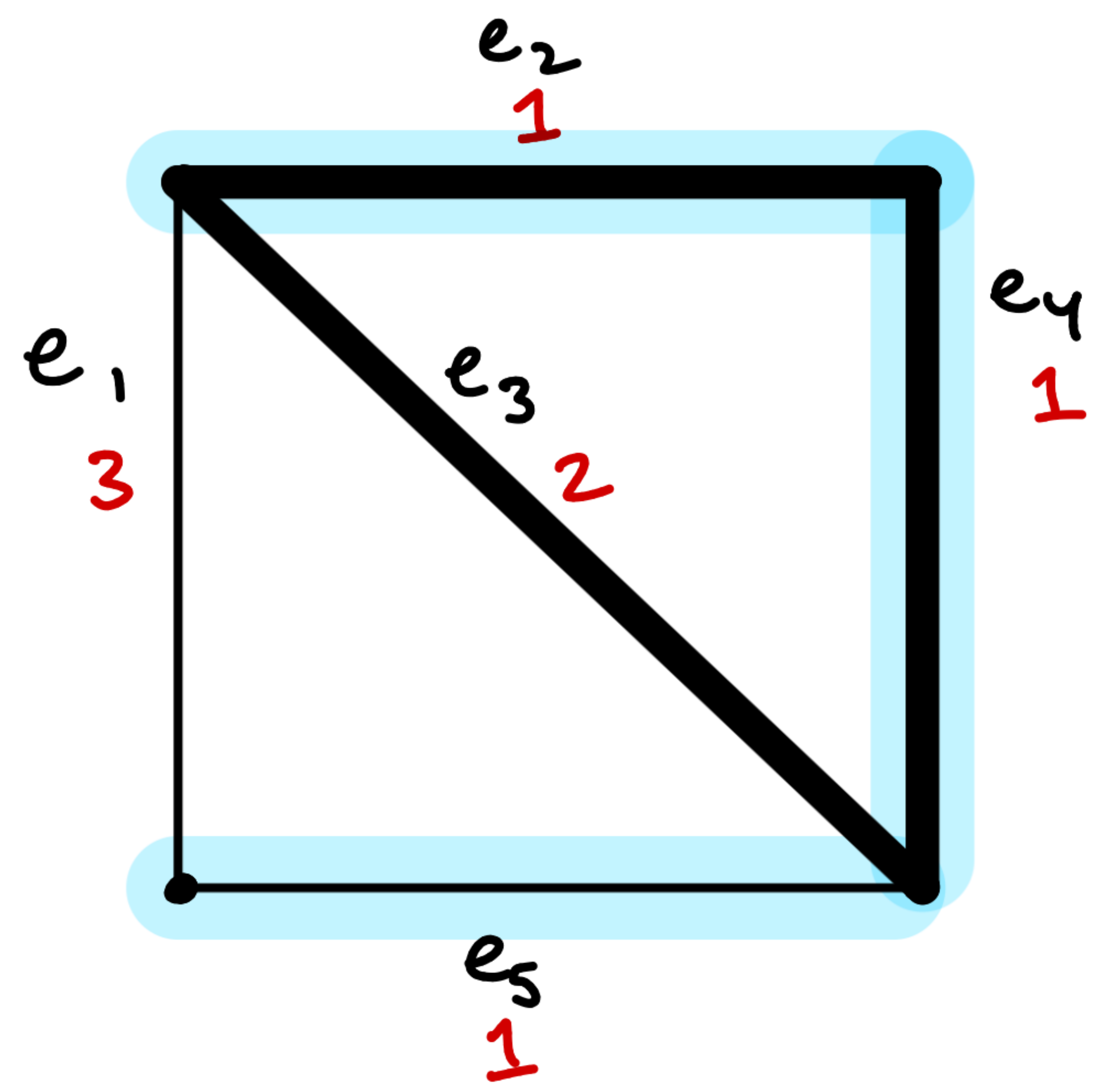
  - $e_3 = 2$ has maximum weight on cycle it forms with tree



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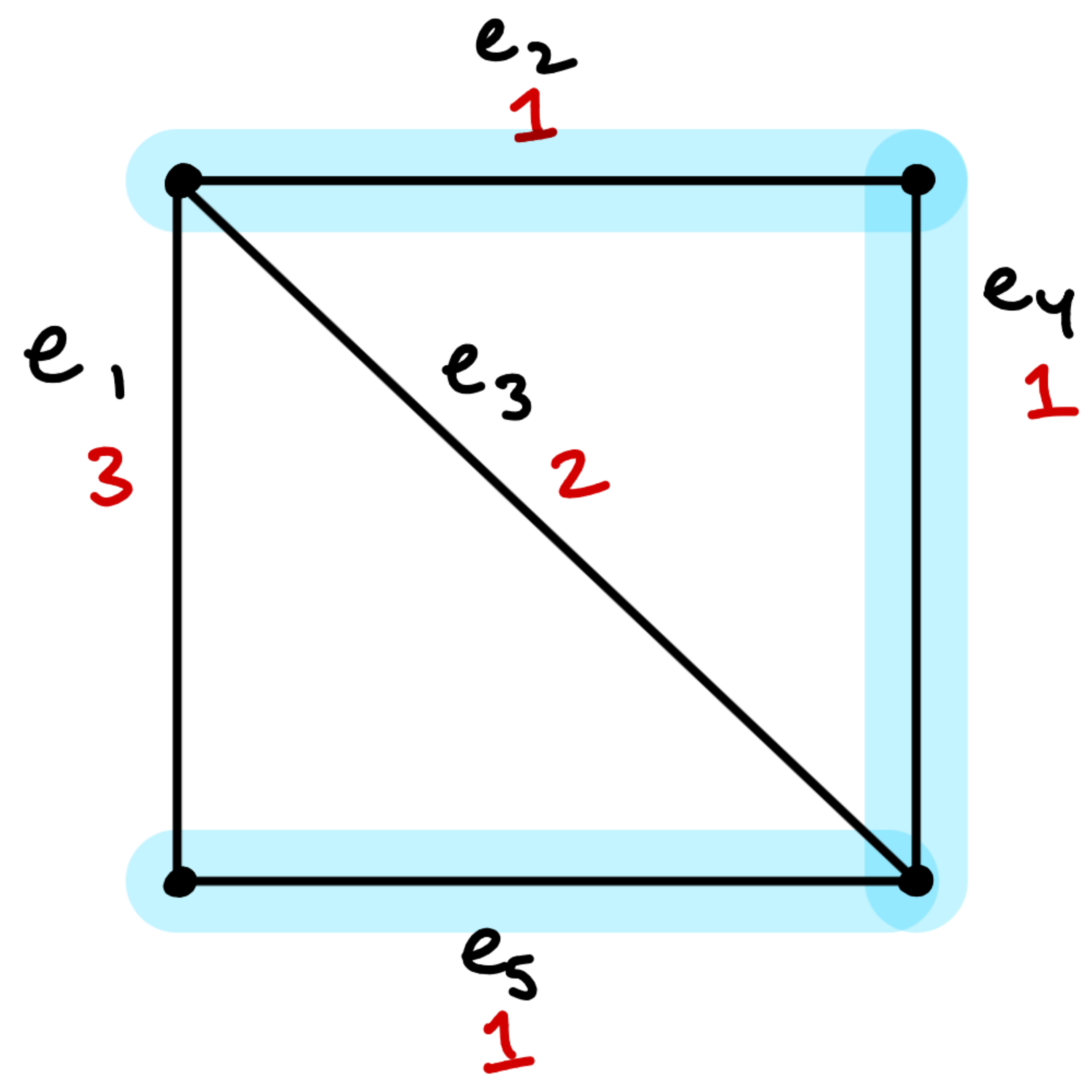

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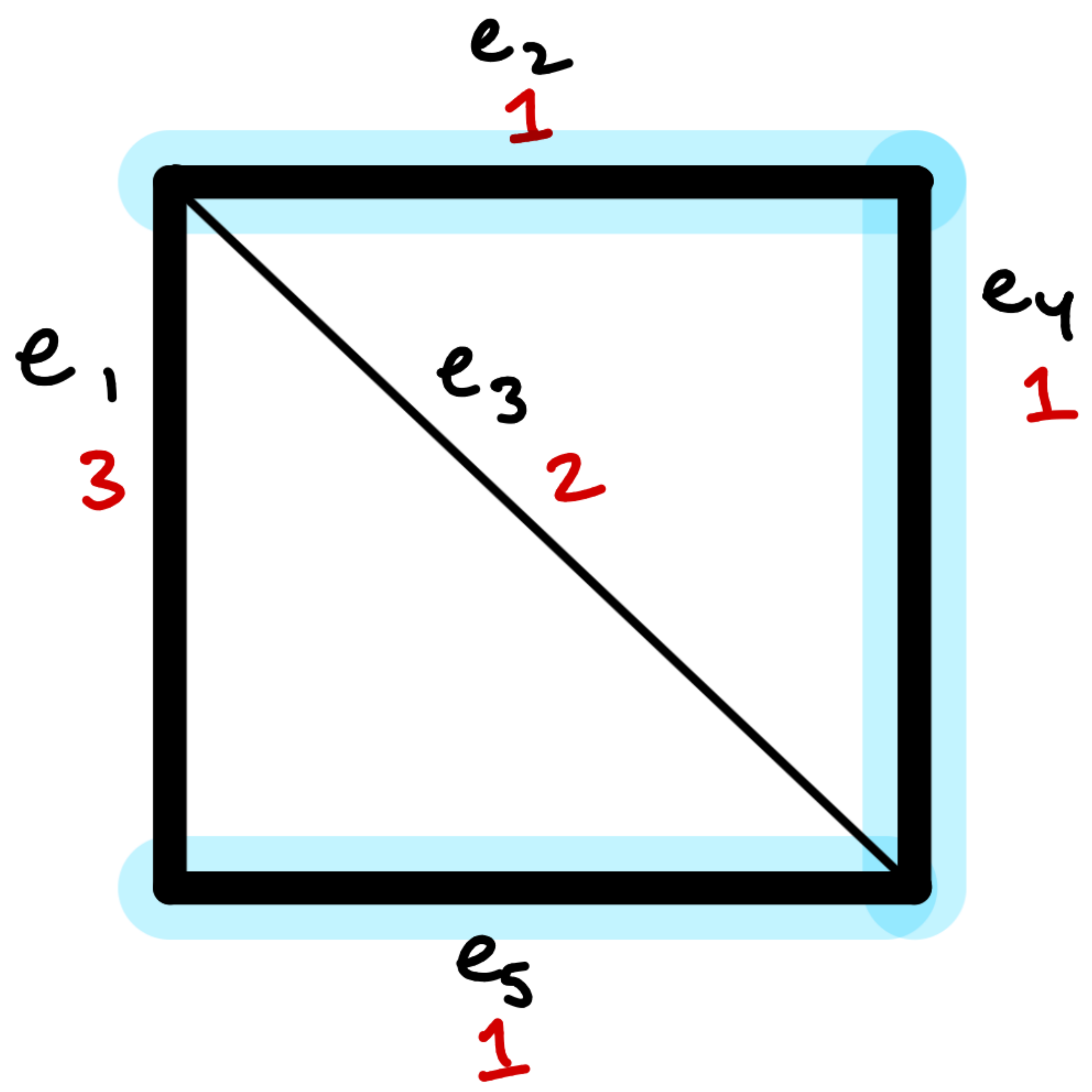
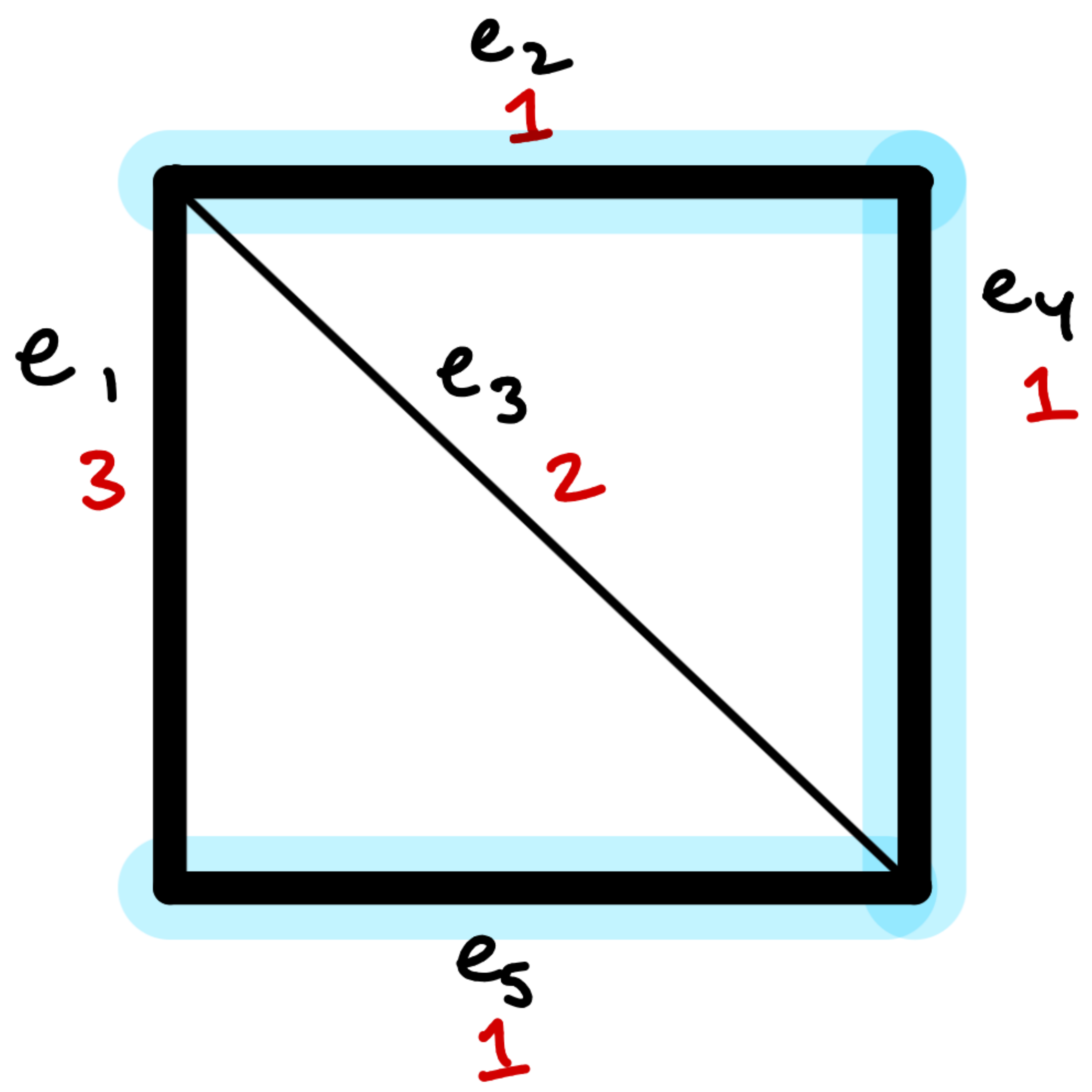
  - $e_1 = 3$ has maximum weight on cycle it forms with tree



Minimum Spanning Tree (MST) Problem

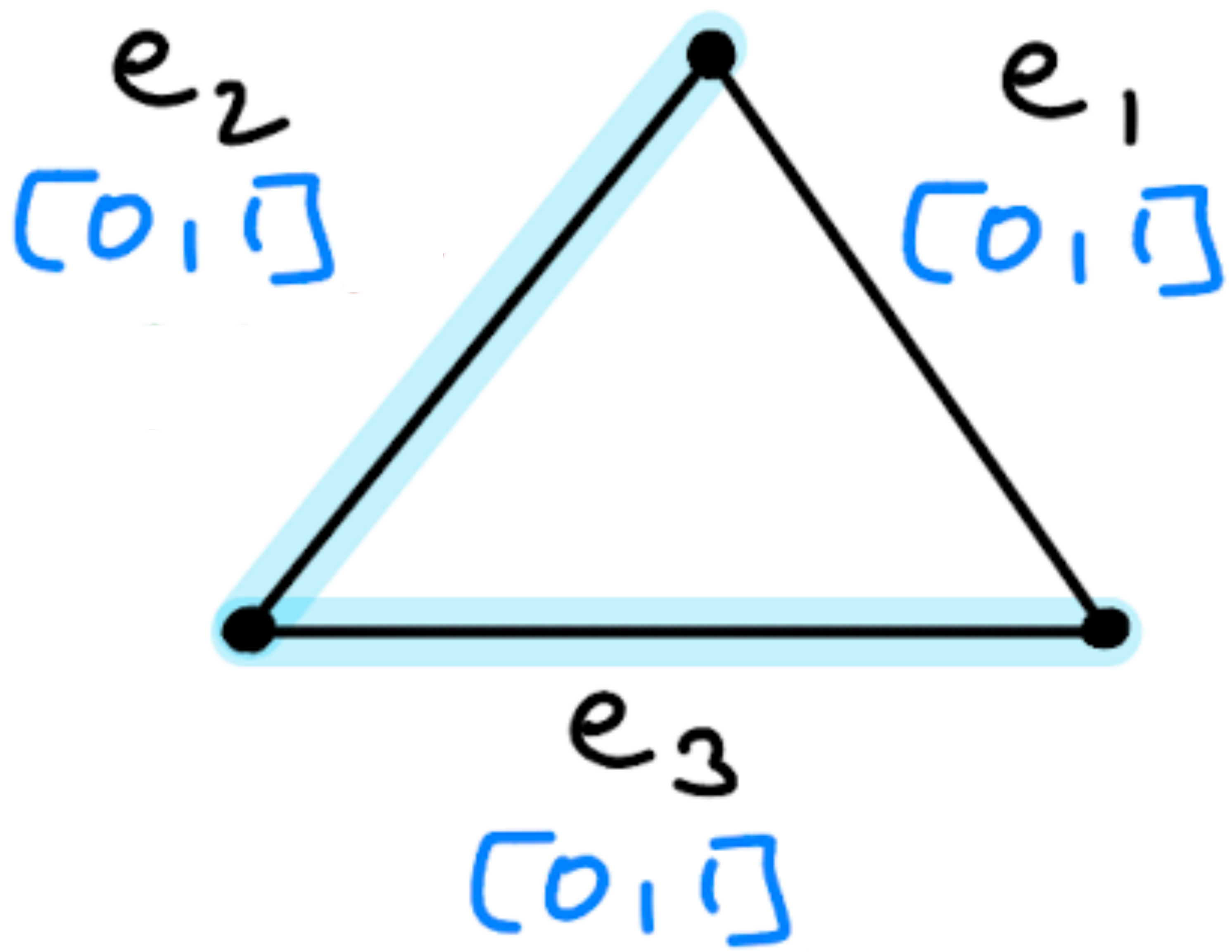
- Find an MST on the connected weighted graph $G = (V, E, w)$
- How can we easily verify that the highlighted edges form an MST?
  - $e_3 = 2$ has maximum weight on cycle it forms with tree
  - $e_1 = 3$ has maximum weight on cycle it forms with tree
  - Showed that each non-tree edge is a max-weight edge on the cycle it forms when added to the tree
    - Verified!



Problem Example

edge label

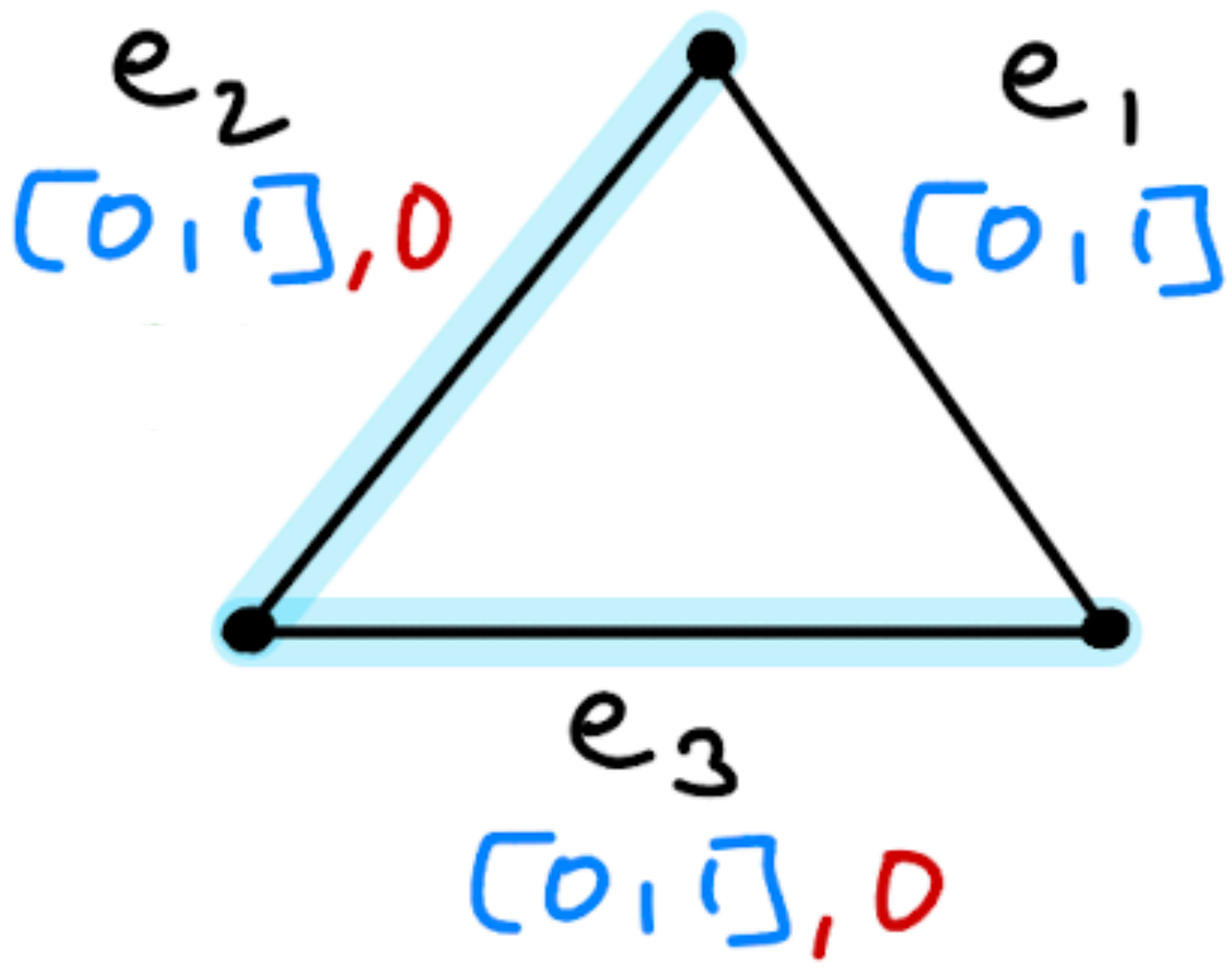
uncertainty area



Problem Example

edge label

uncertainty area, edge weight



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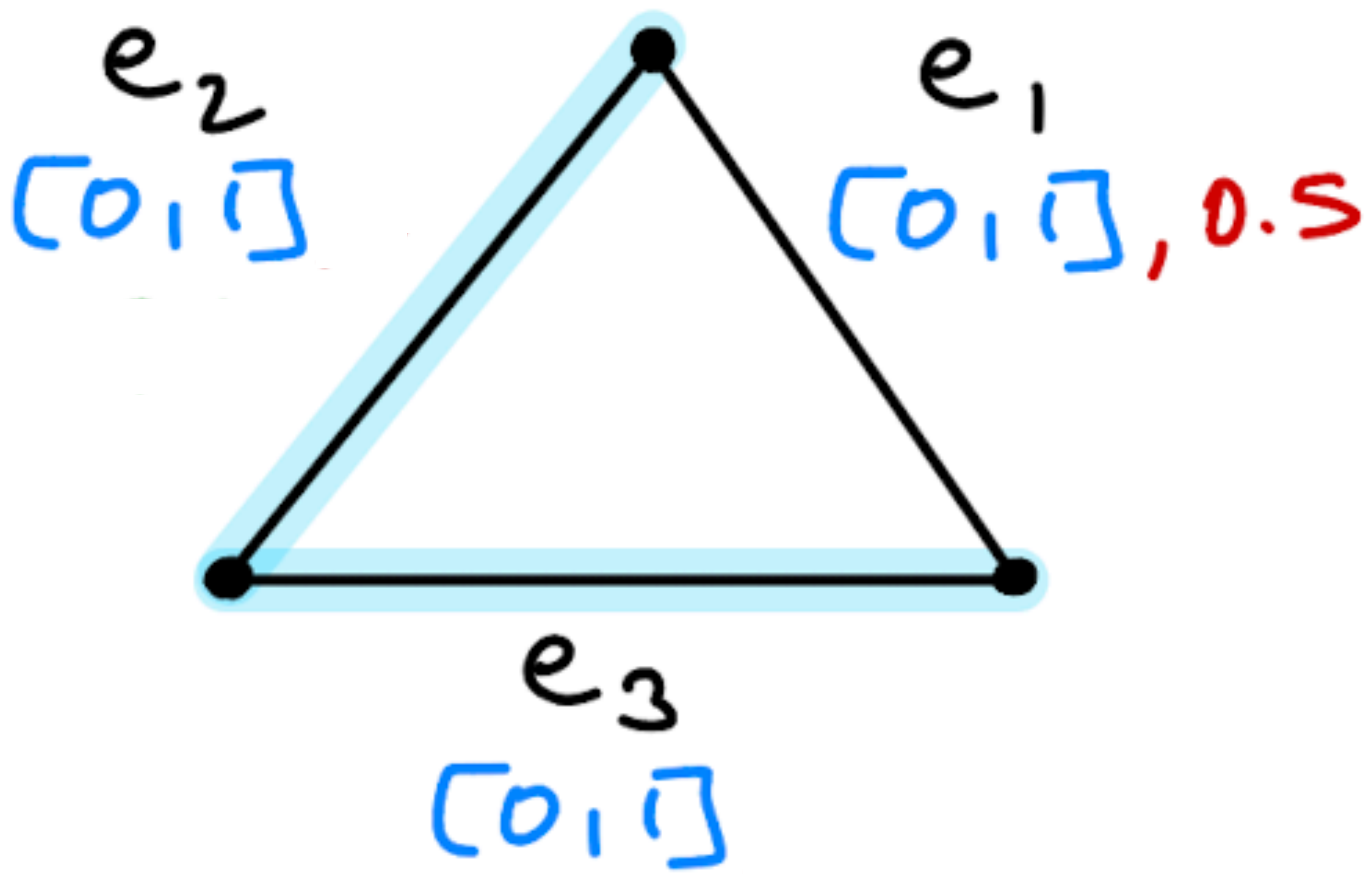
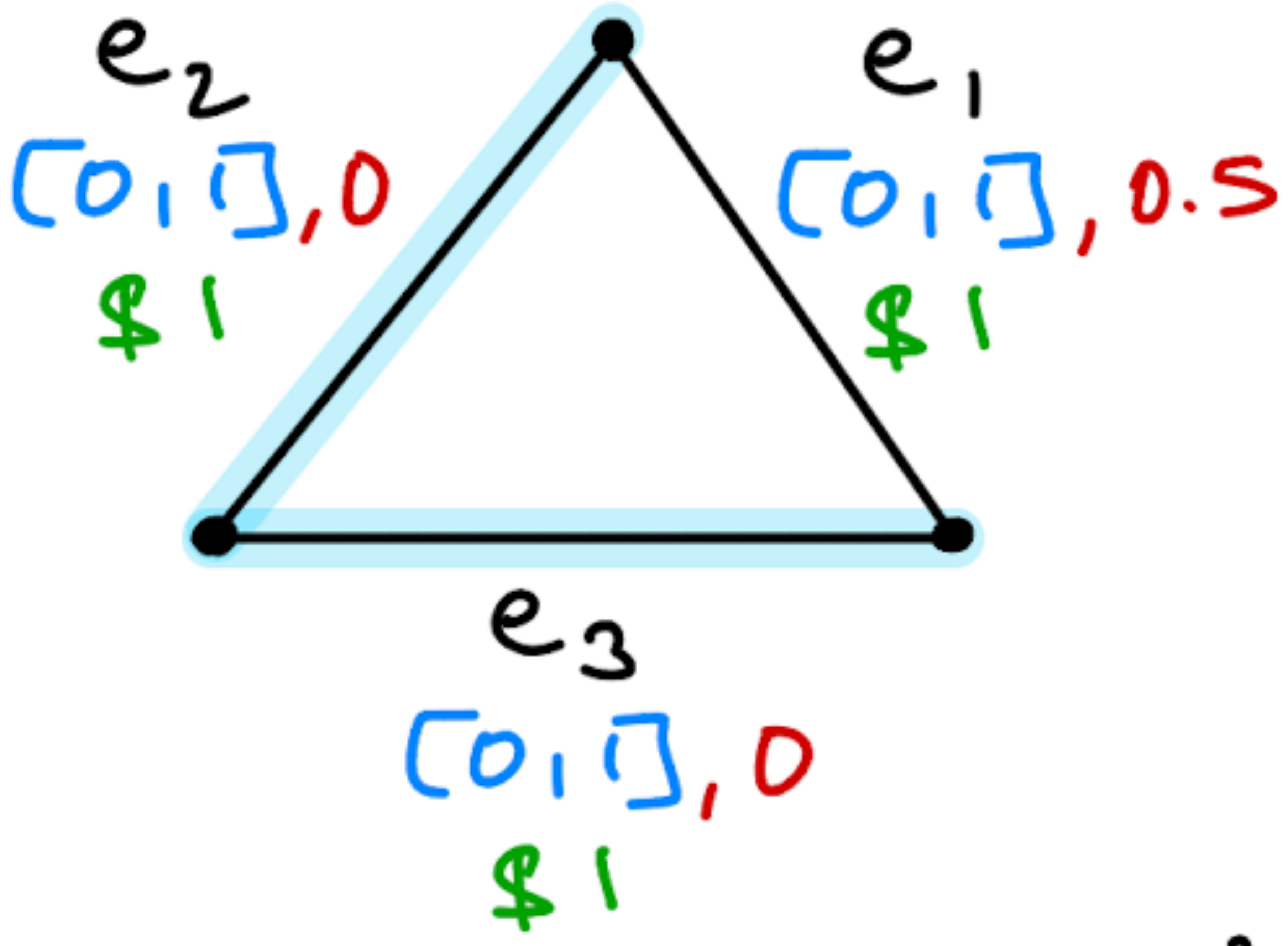


Problem Example

edge label

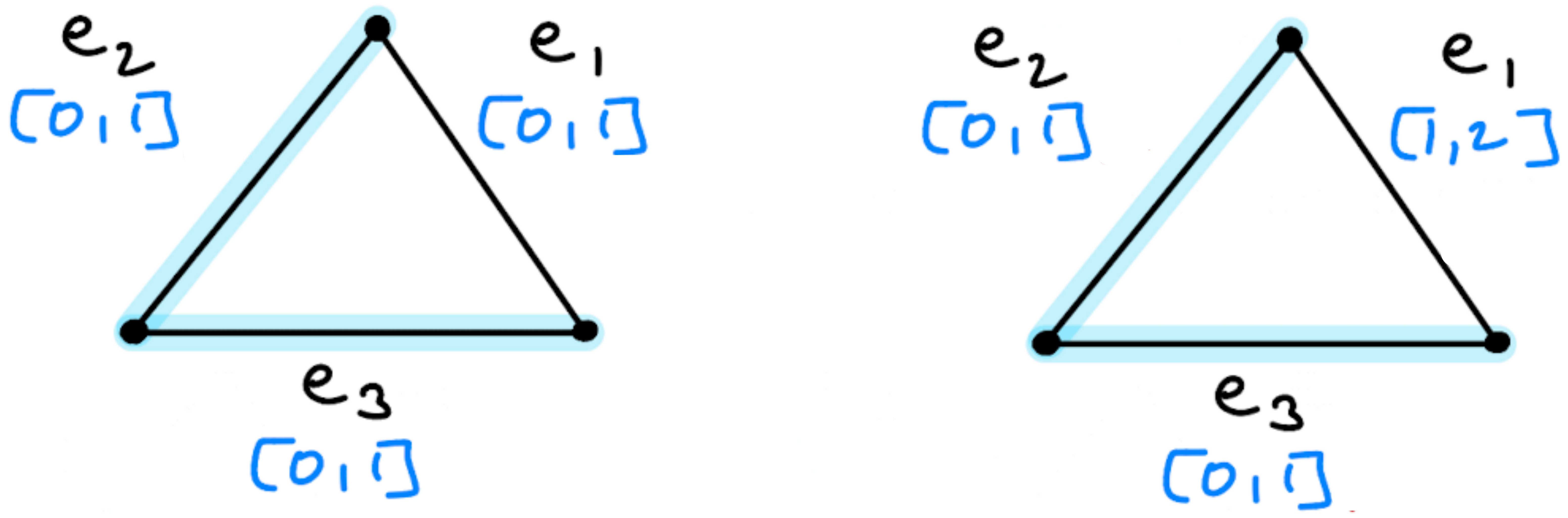
uncertainty area, edge weight

Cost to reveal weight



certificate : $\{e_2 = 0, e_3 = 0\}$

More Examples



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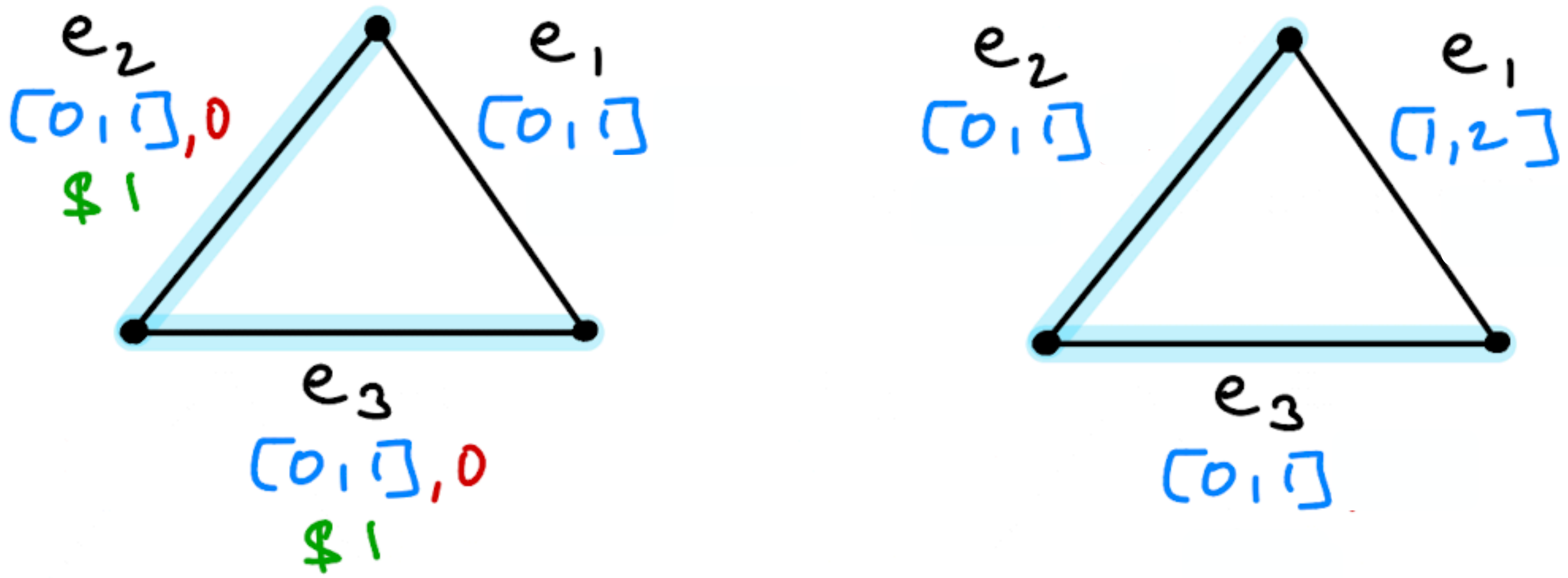

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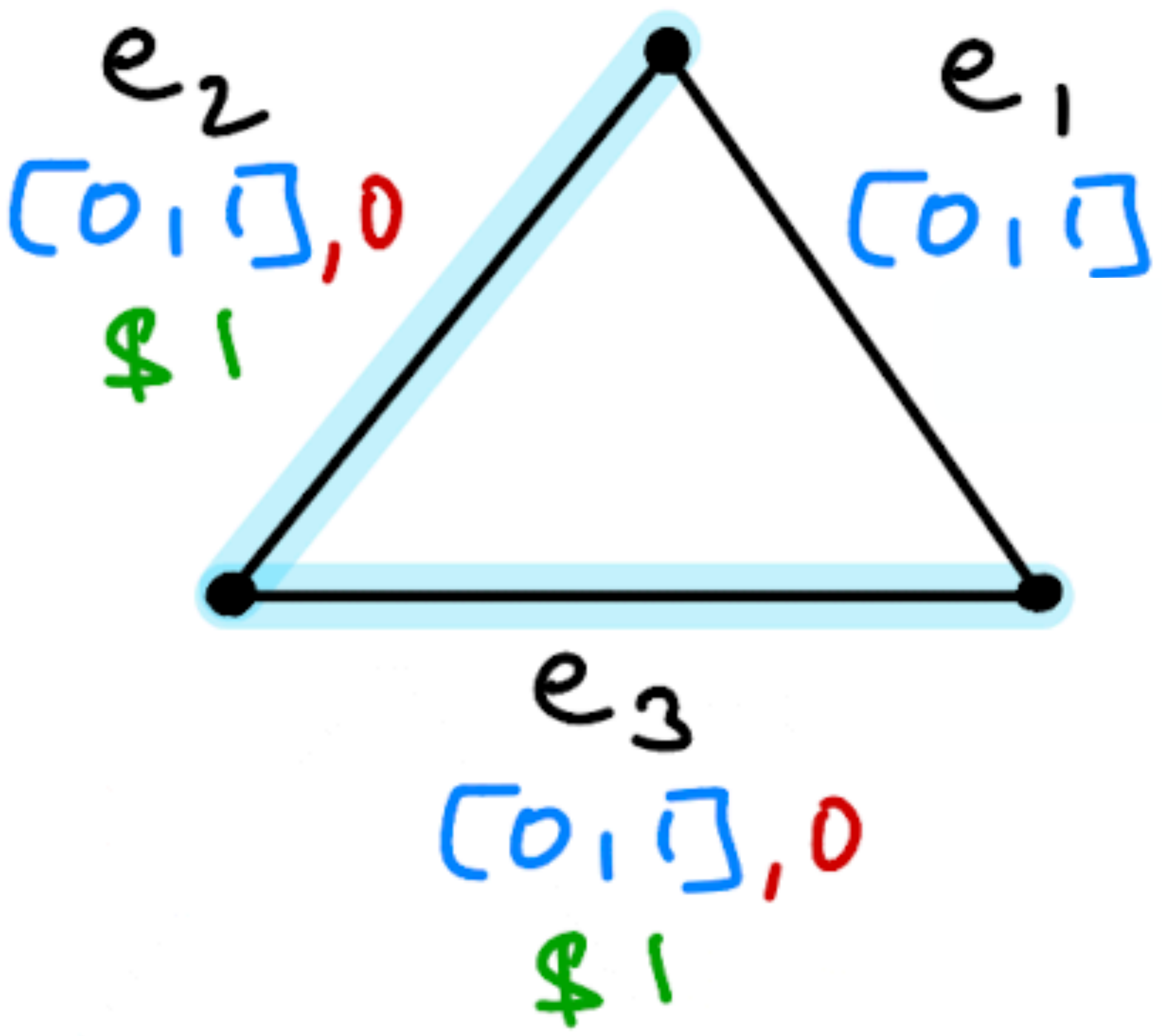

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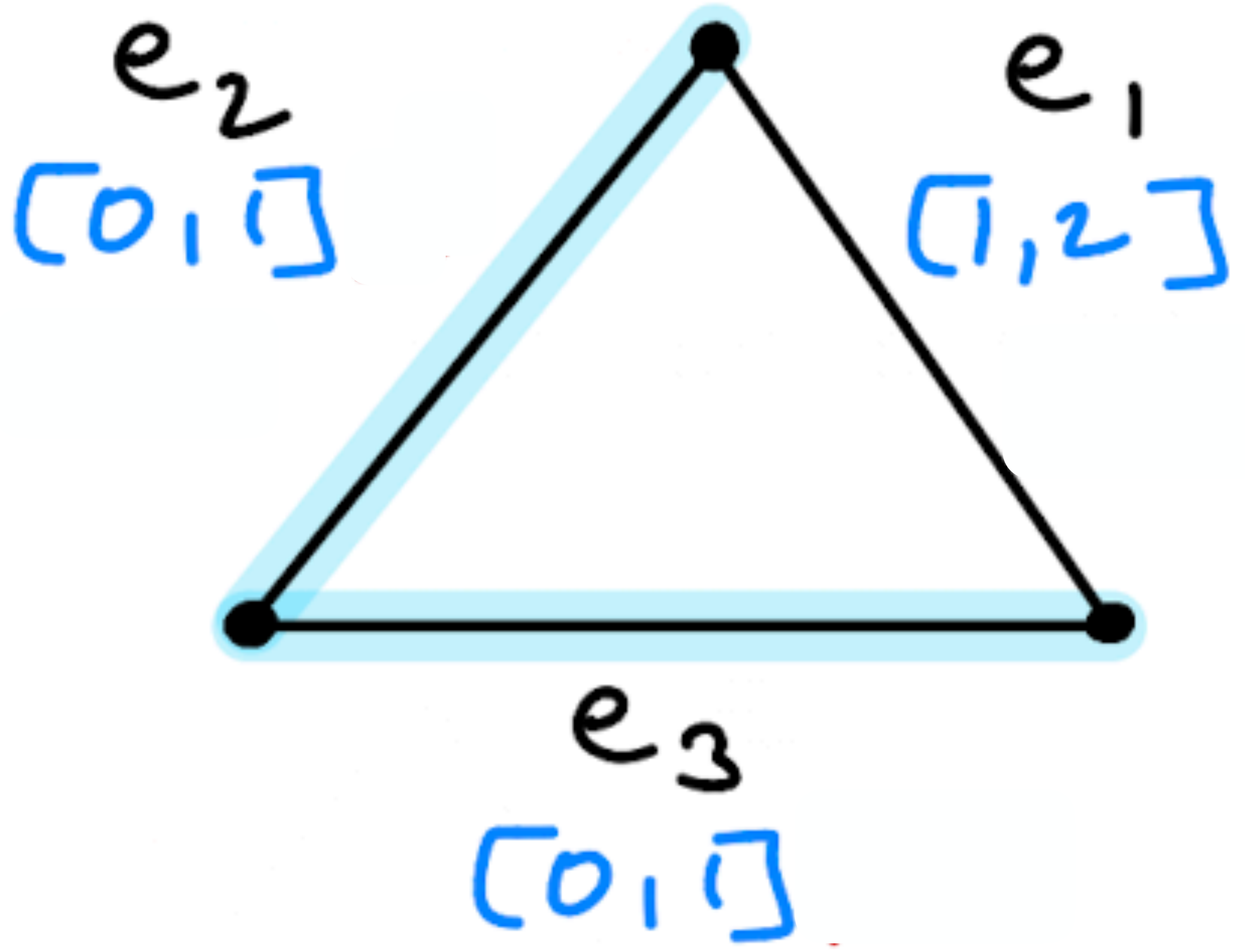

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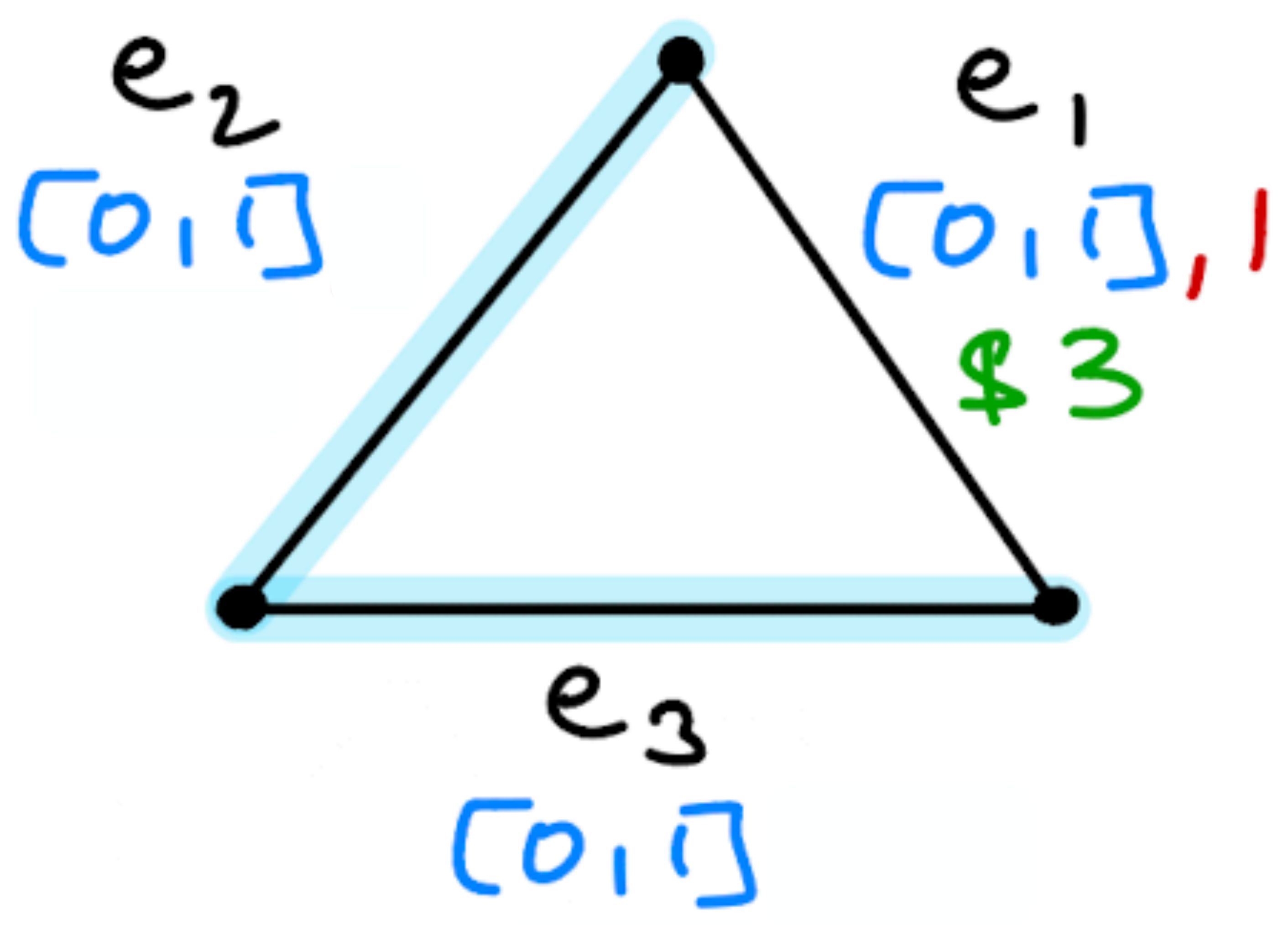
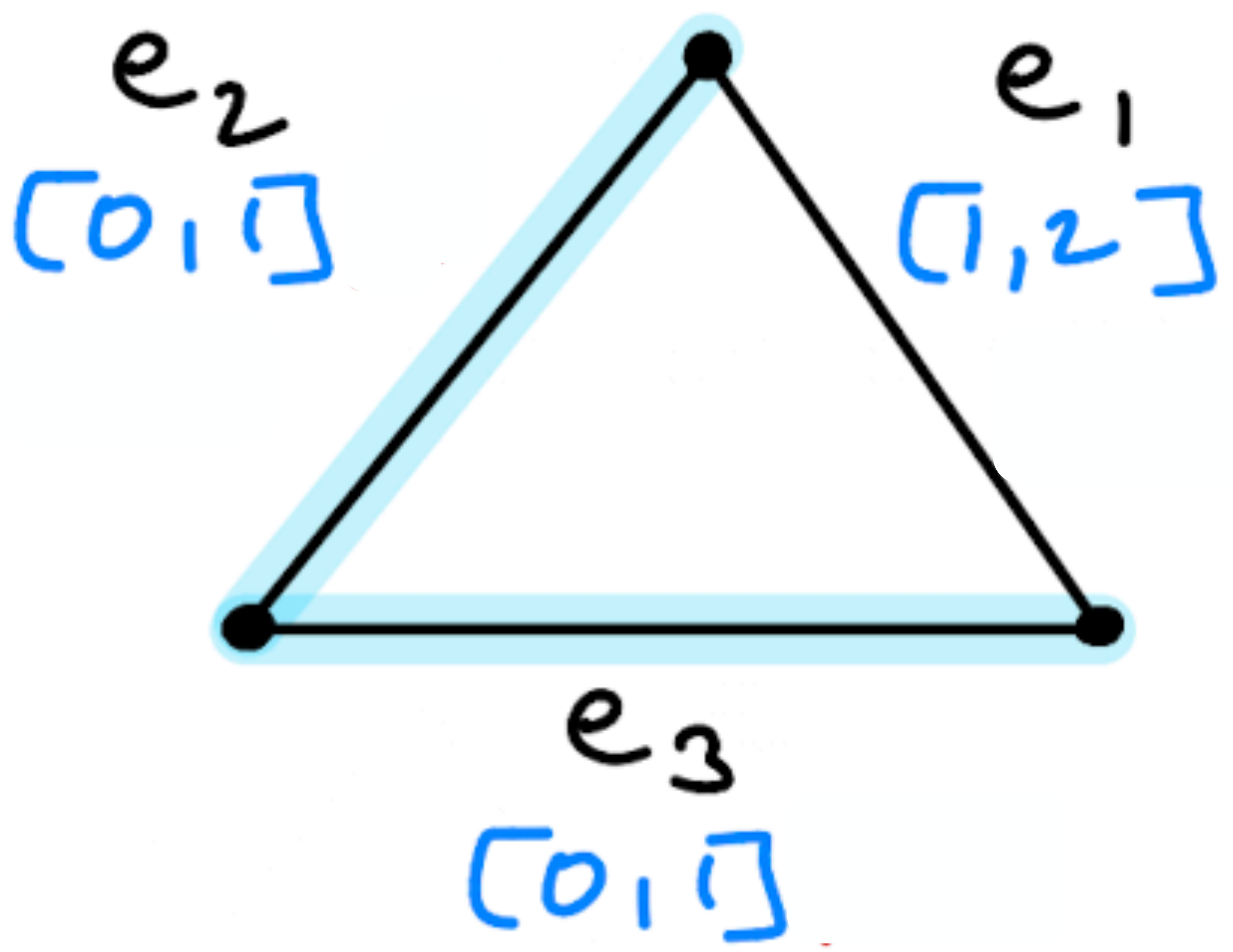
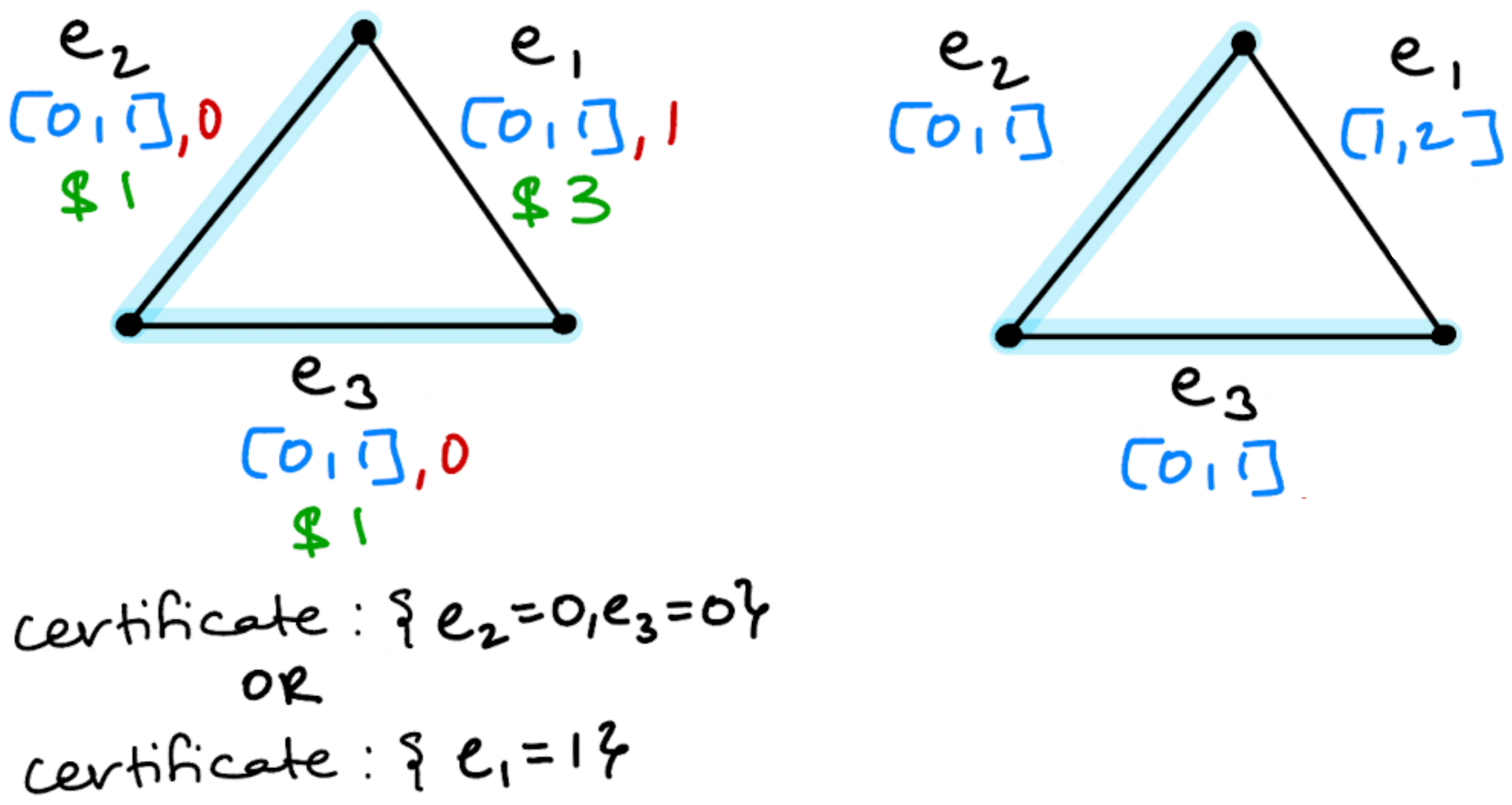




certificate : { $e_1 = 1$ }

More Examples



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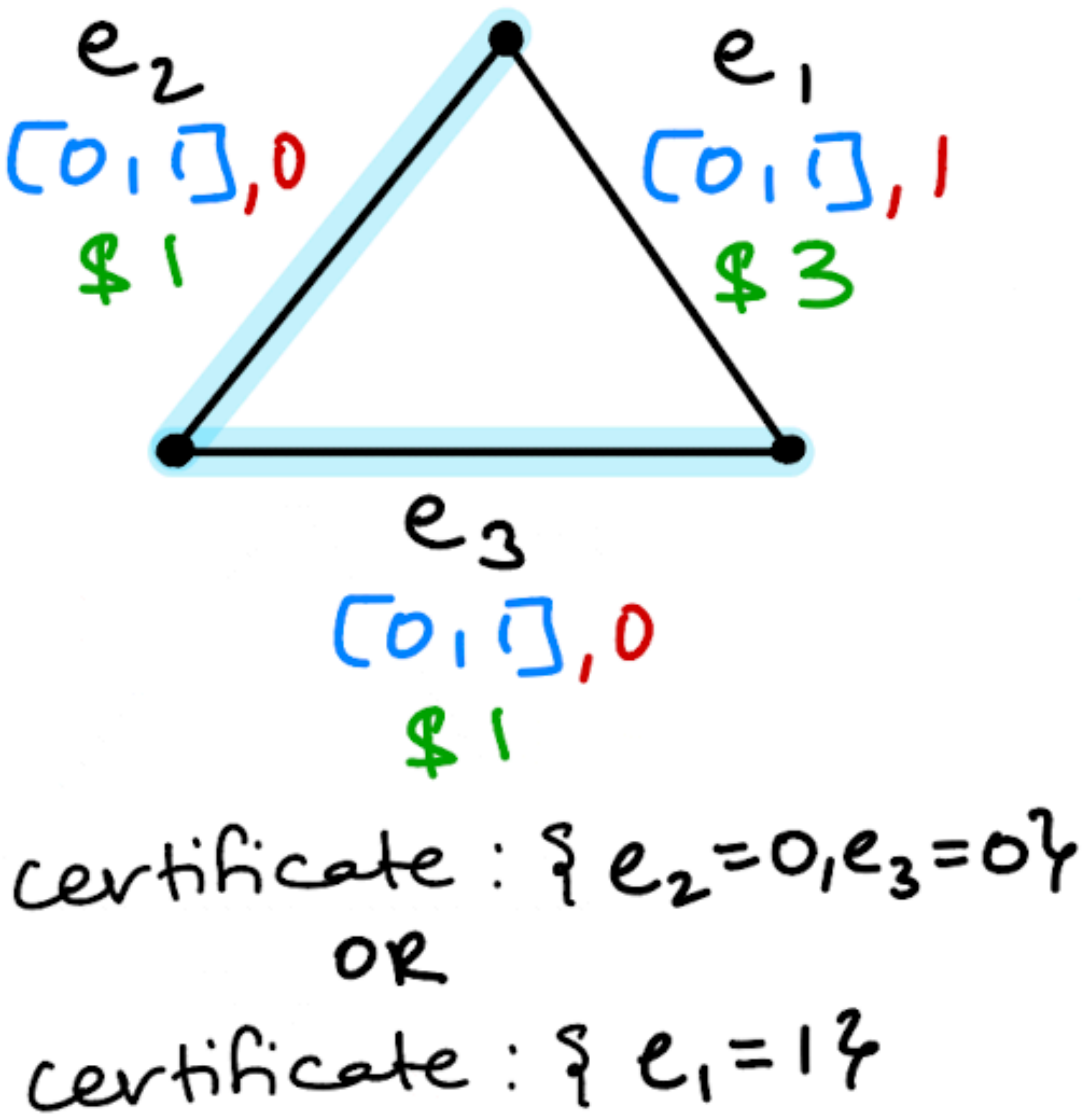
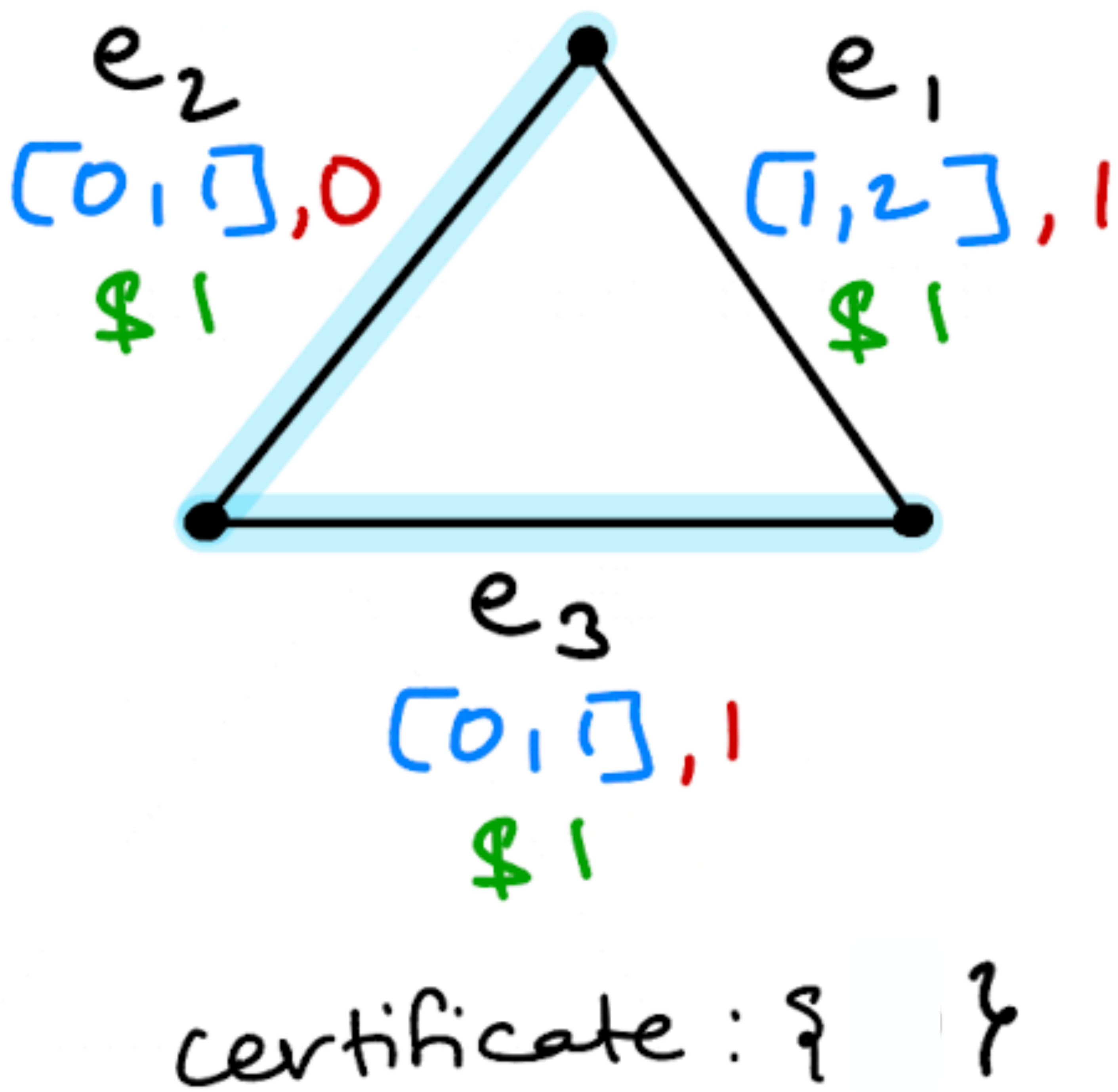




The Verification Objective

- Find an MST T and an associated min-cost certificate Q
  - Q must verify that T is an MST assuming that for edges not in Q, only their uncertainty areas are known (not their actual weights)
  - T is a cheapest-to-verify MST among all MSTs of G
    - no other MST of G has a cheaper certificate

MST Verification Problem

- **Input:** Connected graph $G = (V, E)$, with the following values for each edge $e \in E$
  - uncertainty area $A_e = [L_e, U_e]$ (a closed real interval)
  - weight w_e such that $w_e \in A_e$
  - cost c_e such that $c_e \geq 0$
- **Output:** T, Q where
  - T is a cheapest-to-verify MST of G
  - Q is a min-cost certificate for T

Theorem: There is a polynomial-time algorithm that solves the MST verification problem

MWB Verification Problem

- **Input:** Matroid $M = (E, \mathcal{I})$, with the following values for each element $e \in E$
  - uncertainty area $A_e = [L_e, U_e]$ (a closed real interval)
  - weight w_e such that $w_e \in A_e$
  - cost c_e such that $c_e \geq 0$
- **Output:** B, Q where
  - B is a cheapest-to-verify MWB of M
  - Q is a min-cost certificate for B

Theorem: There is a polynomial-time algorithm that solves the MWB verification problem

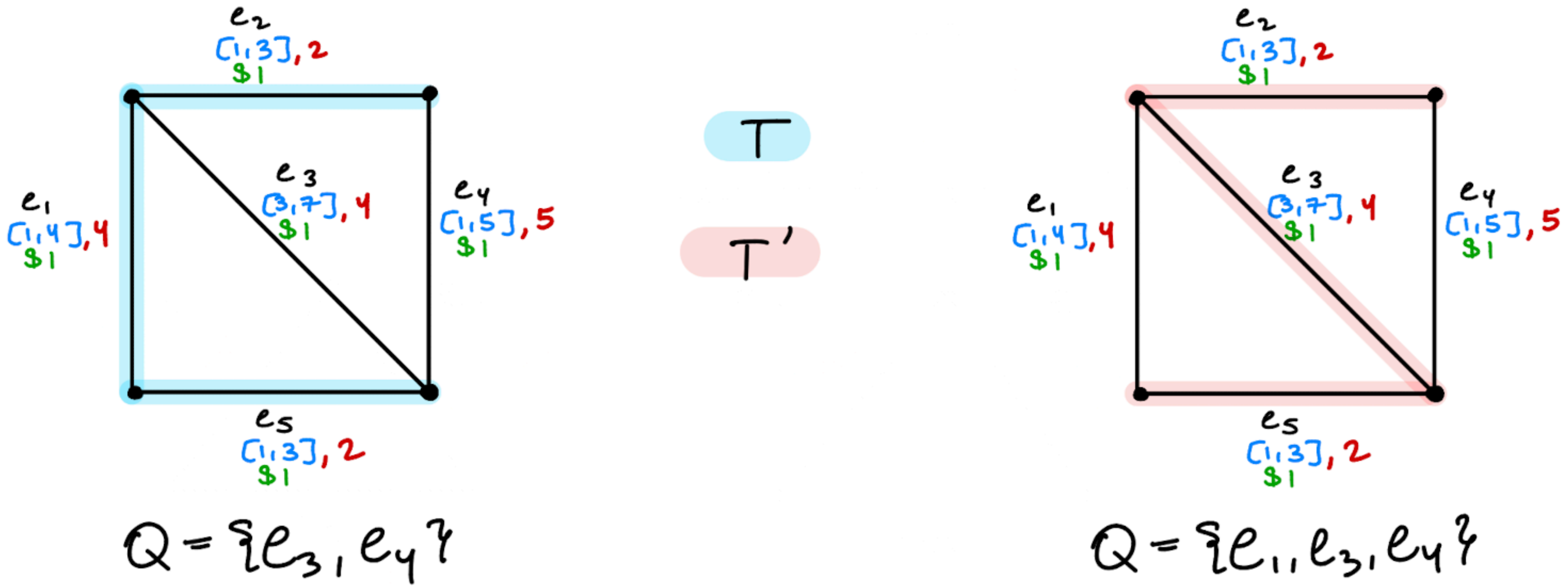
Related Work

- [Erlebach and Hoffman, 2014] gave a polynomial time algorithm for a related MST verification problem:
  - Open (L_e, U_e) uncertainty intervals
- The MST problem for closed uncertainty intervals (our problem) is significantly more challenging
  - Any algorithm that works for closed intervals also works for open intervals
  - With open intervals [Erlebach and Hoffman, 2014] all MSTs have the same verification cost
    - Can output arbitrary MST T, doesn't matter which one
  - With closed intervals different MSTs can have different verification costs!
    - Need to find a cheapest-to-verify MST

MST Verification in Online Settings

- **Worst case online setting:** Competitive ratio compares online query costs to a minimum-cost offline certificate
    - **Open Intervals:** 2-competitive ratio [Erlebach et. al, 2016]
    - **Closed Intervals:** Ratio degrades to n [Gupta et. al, 2016]
- In worst case online setting, we achieved 2-competitive ratio for closed intervals by providing the algorithm with a cheapest-to-verify MWB
    - Complexity in closed interval case stems from the search for a cheapest-to-verify MWB, not the search for its min-cost certificate
- Leveraged these structural insights to develop an algorithm that utilizes untrusted MWB predictions to beat worst-case bounds

Impact of Closed Intervals: Different Verification Costs



MST Verification Algorithm Overview

- **Phase 1: Compute a cheapest-to-verify MST T of G**
  - Contract or delete extreme case edges ($w_e = L_e$ or $w_e = U_e$) according to certain rules
    - Each rule application transforms the current graph into a smaller resulting graph: $G \to G' \to G'' \to \ldots \to G^*$
    - Let K be the set of contracted edges

    # Remaining edges in G^* satisfy $L_e < w_e < U_e$

    # Equivalent to just having open intervals as in [Erlebach and Hoffman, 2014]
  - Compute any MST T^* of G^*
  - $T := T^* \cup K$ is a cheapest-to-verify MST of G
- **Phase 2: Compute a min-cost certificate for T**
  - Reduce to a weighted bipartite vertex cover problem [Erlebach and Hoffman, 2014]
    - minimum weight vertex cover gives a min-cost certificate that verifies T

Phase 1

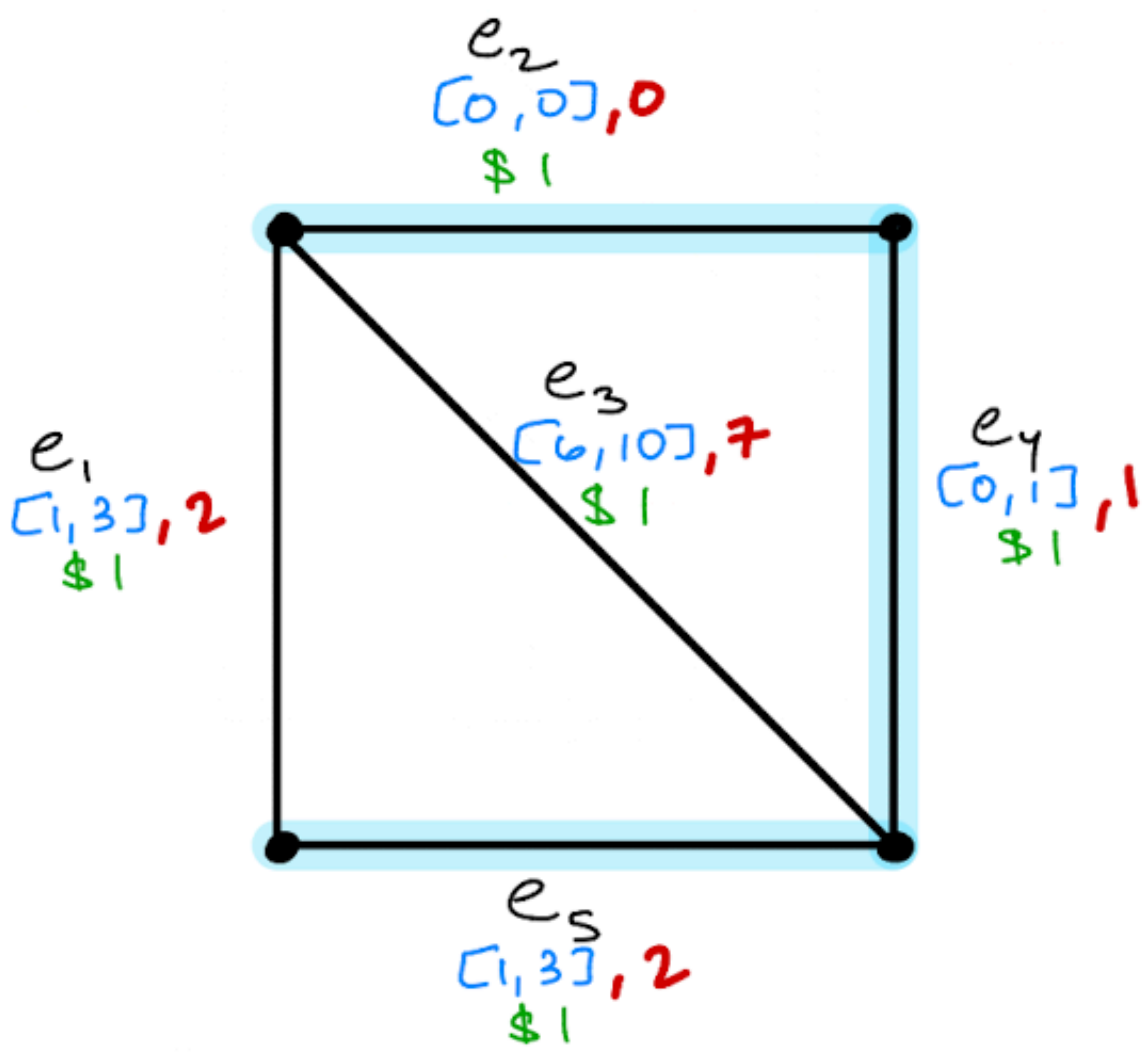
Phase 1: Contractions and Deletions

- We use properties of MSTs to derive contraction (K) and deletion (D) rules
- Phase 1 applies K-rules and D-rules to eliminate all extreme edges
  - Edges in K will be included in output MST
  - Edges in D will be excluded
  - Tie-break using query costs
- Rules ensure contractions and deletions are consistent with a cheapest-to-verify MST

Example Rules

- **Example K-rule (1):** Contract edge e (add to set K) if
  - It belongs to some MST T of current induced multigraph G'
  - $w_e = U_e$ and $U_e \neq L_e$
  - e is a min-weight edge in the set of edges that can replace e to form a new Spanning Tree
  - e has highest query cost among similar extreme edges that can replace e to form a new Spanning Tree
- **Example K-rule (2):** Contract an edge e (add to set K) if
  - It belongs to some MST T of current induced multigraph G'
  - e is the unique minimum weight edge connecting the two components formed by $T \backslash \{e\}$

Contraction and Deletion Example



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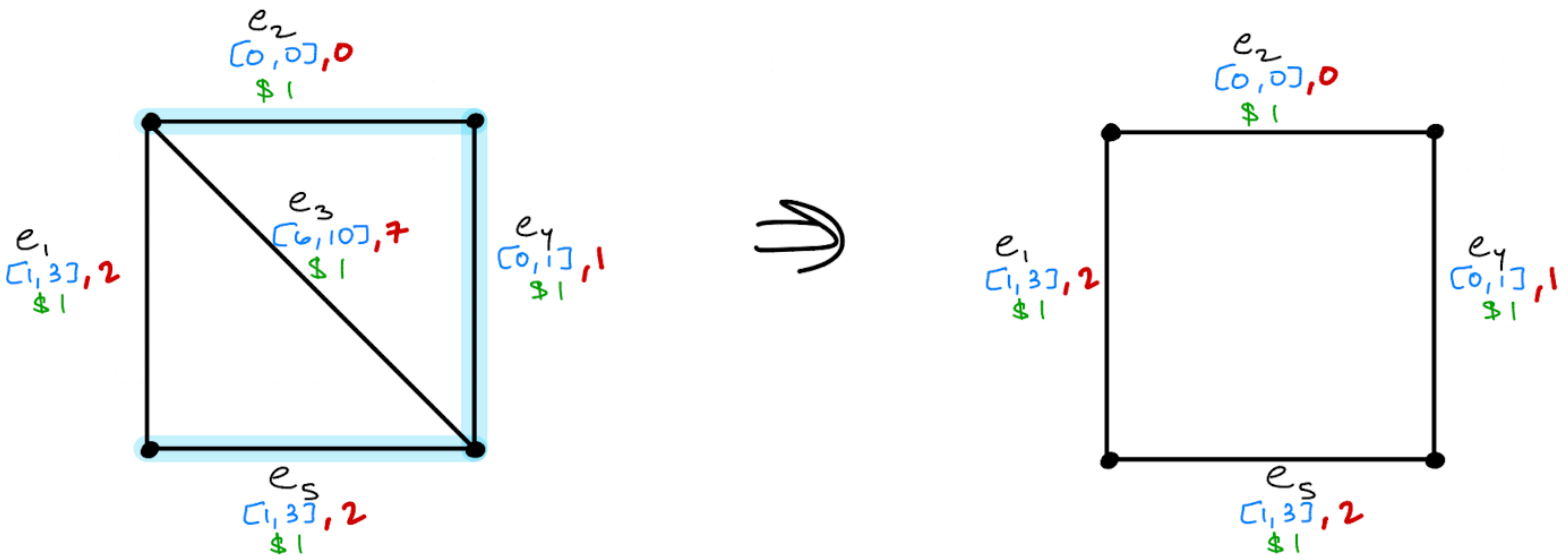

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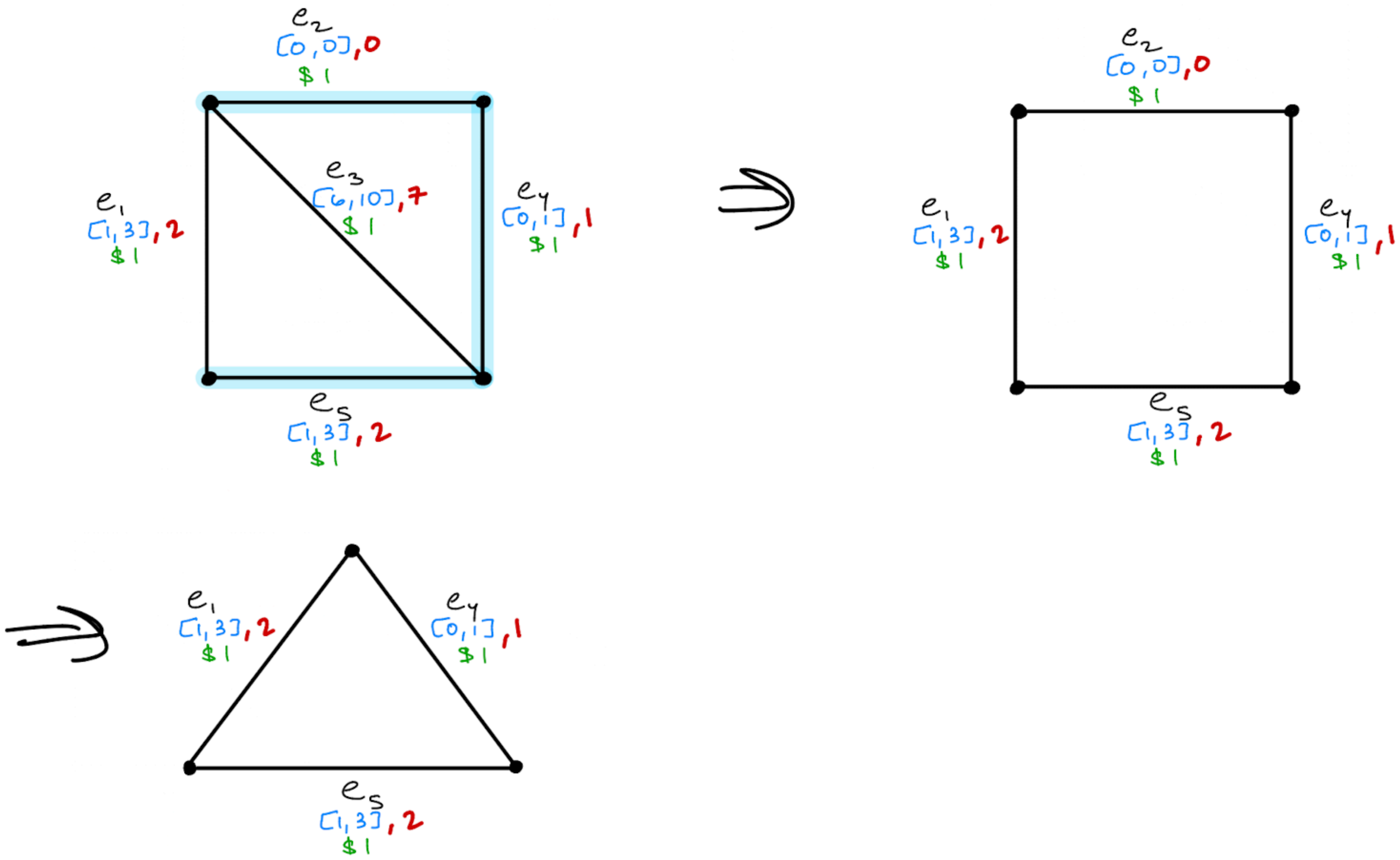

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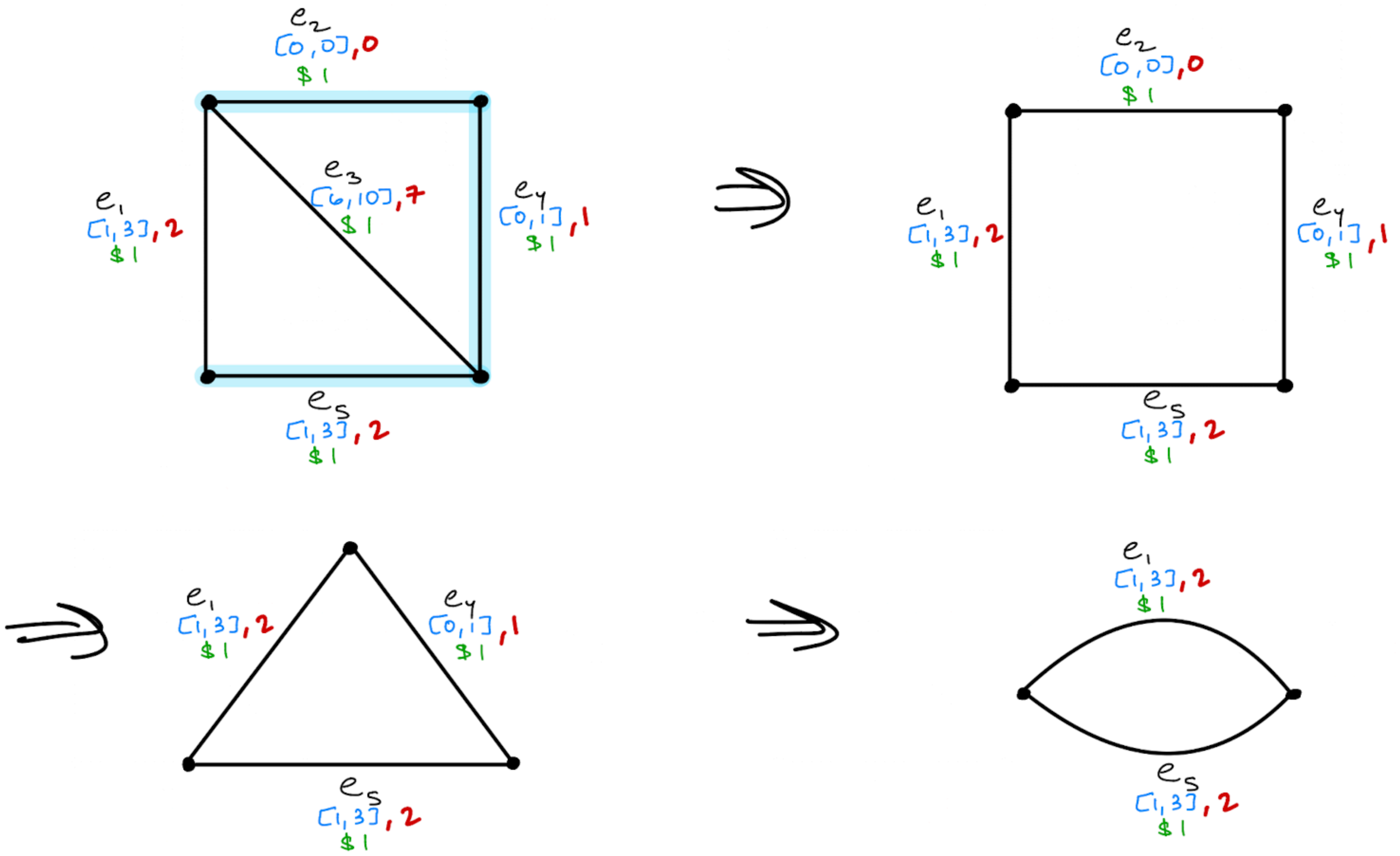

Algorithm Runs in Polynomial Time

- **Phase 1:**
  - Checking the conditions for applying K and D rules can be done in polynomial time
  - Output is a cheapest-to-verify MST T
- **Phase 2:**
  - Constructs an instance of Bipartite Weighted Vertex Cover from G and T
  - Solves it using known poly-time algorithm
  - Solution gives min-cost certificate for T

From Offline to Online

- Solving the (offline) verification problem is often a prerequisite to solving online versions of the problem
- **Open problem:** develop a polynomial time algorithm solving the stochastic version of the MST problem
  - Weight of an edge comes from a known distribution
    - Edge weights are independent
  - Need to query an edge to reveal its weight
  - **Goal:** minimize expected query cost

Questions?

References

- [1] Thomas Erlebach and Michael Hoffmann. Minimum spanning tree verification under uncertainty. In Dieter Kratsch and Ioan Todinca, editors, Graph-Theoretic Concepts in Computer Science, volume 8747 of Lecture Notes in Computer Science, pages 164–175. Springer International Publishing, Cham, 2014. doi:10.1007/978-3-319-12340-0_14.

- [2] T. Erlebach, M. Hoffmann, D. Krizanc, M. Mihalák, R. Raman, Computing minimum spanning trees with uncertainty, in: S. Albers, P. Weil (Eds.), 25th International Symposium on Theoretical Aspects of Computer Science, STACS'08, in: LIPIcs, vol. 1, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Germany, 2008, pp. 277–288.

- [3] Thomas Erlebach, Michael Hoffmann, and Frank Kammer. Query-competitive algorithms for cheapest set problems under uncertainty. Theor. Comput. Sci., 613:51–64, 2016. doi:10.1016/J.TCS.2015.11.025.

- [4] Manoj Gupta, Yogish Sabharwal, and Sandeep Sen. The update complexity of selection and related problems. Theory Comput. Syst., 59(1):112–132, 2016.