Optimal Average Disk-Inspection via Fermat's Principle

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How the Problem Came to Be?

Bellman's 1955 Lost-in-a-Forest Shoreline Search – Known Distance

Specs

- Unit-speed hiker
- Forest shape is a **half-space**
- Hiker at **center** of unit Disk
- **Known** starting point/distance to boundary

Objective

- Minimize wst or avg escape time

The Average-Case Disk-Inspection Problem

Average Disk-Inspection Made Formal

Inspection curve $\boldsymbol{T}$ ($O \to A \to Z$ contains disk) at unit speed.

Pont X on disk chosen uniformly at random.

$I_X :=$ Time until until X is inspected.

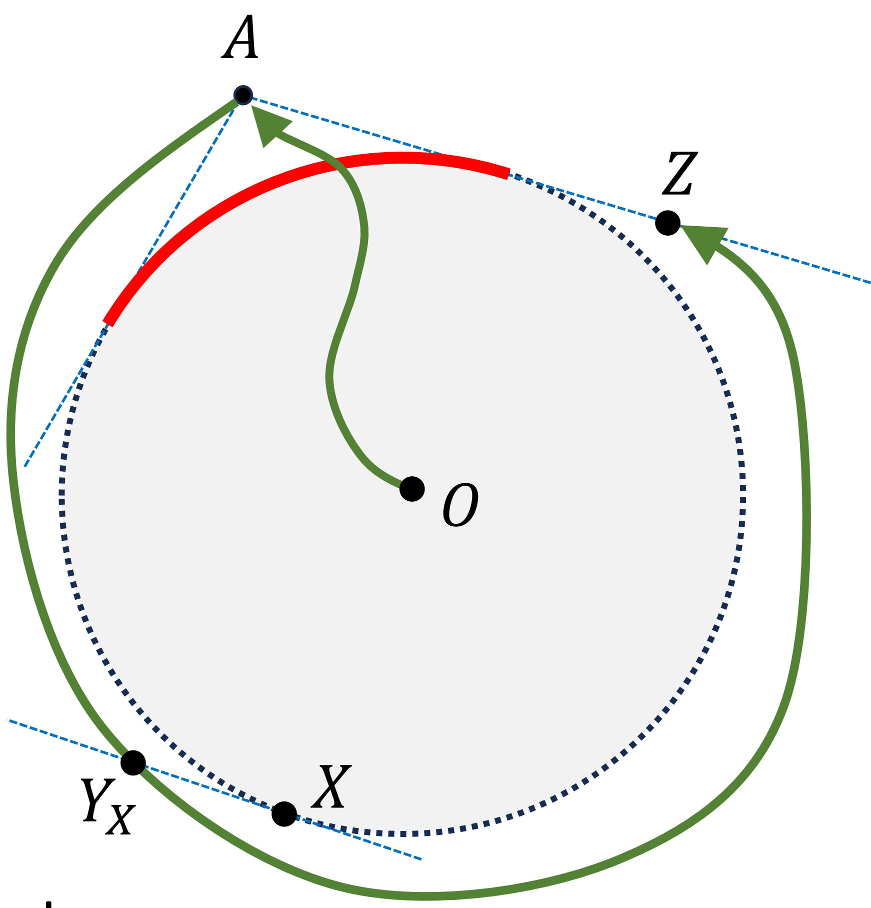
(or hit tangent shoreline).

$= |O \to A \to Y_X|$

OBJECTIVE: $\min_{\boldsymbol{T}} \boldsymbol{E}_X[\boldsymbol{I}_X]$

Equivalently:

Find cost-efficient randomized algorithm against worst-case adversary.



What is **NOT** New?

Literature on Search Games

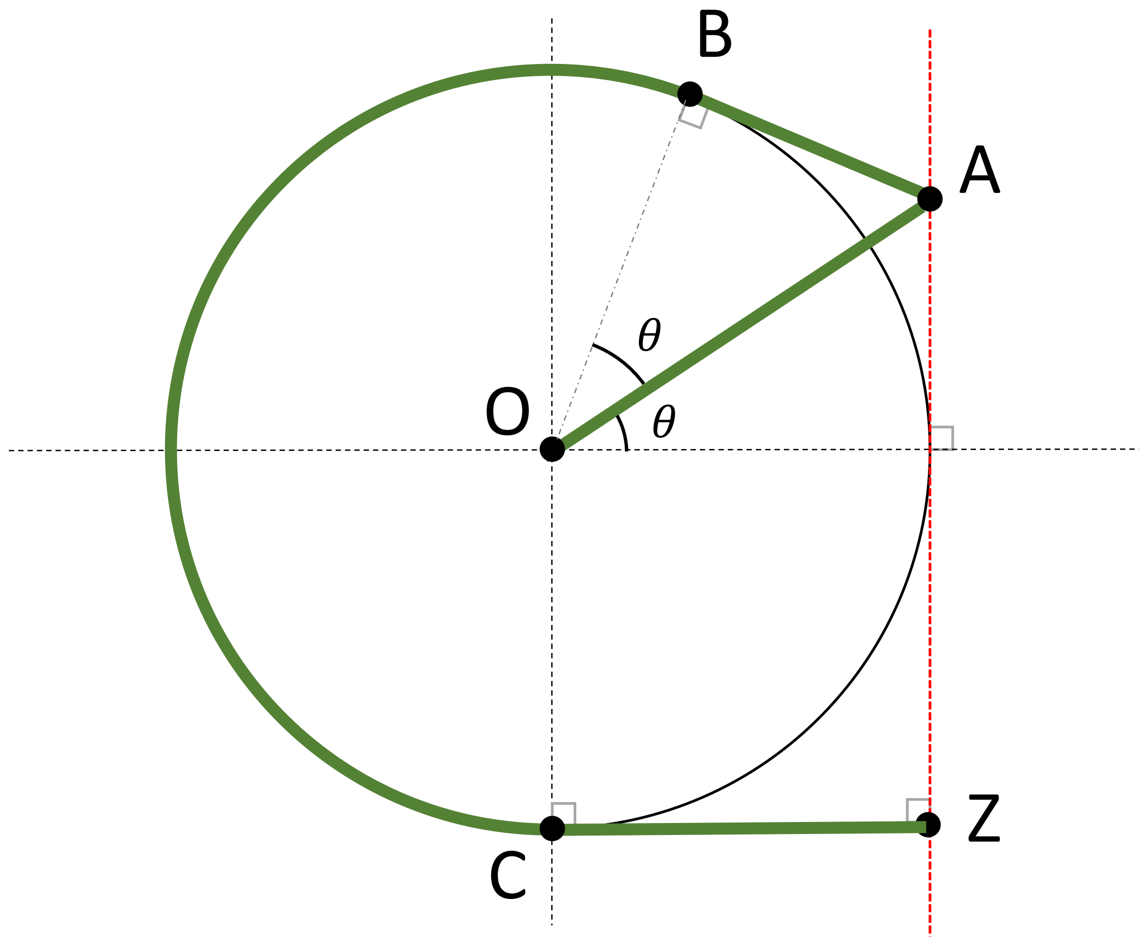
Optimal Worst-Case Disk-Inspection

Theorem (Isbell 1957)

$\theta = \frac{\pi}{3}$ gives optimal **worst-case** (shortest length) trajectory of cost

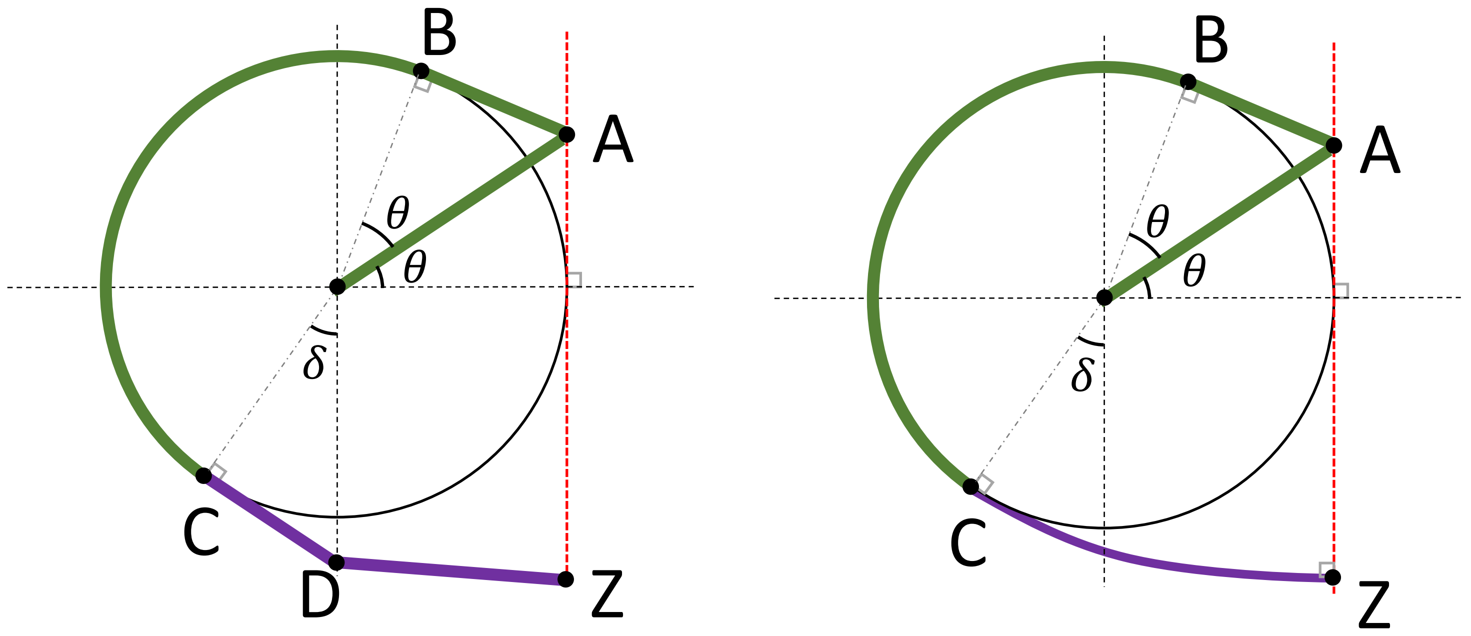
$$1 + \sqrt{3} + \frac{7\pi}{6} \approx 6.39724 \ldots$$

Opt Trajectory touches the disk!



New (optimal) cost
$3.549259\ldots$

Heuristics to Average Disk-Inspection



Theorem (Bluss 1961) $\theta \approx 33^o 56'$ gives average cost $3.00799 + \boldsymbol{d}/2$, where **$\boldsymbol{d}$ is the cost contribution of component $\boldsymbol{C} \to \boldsymbol{Z}$.**

Best heuristic proposed gives average cost **$3.63489\ldots$**

Conjecture: Best trajectory must touch the disk, as above.

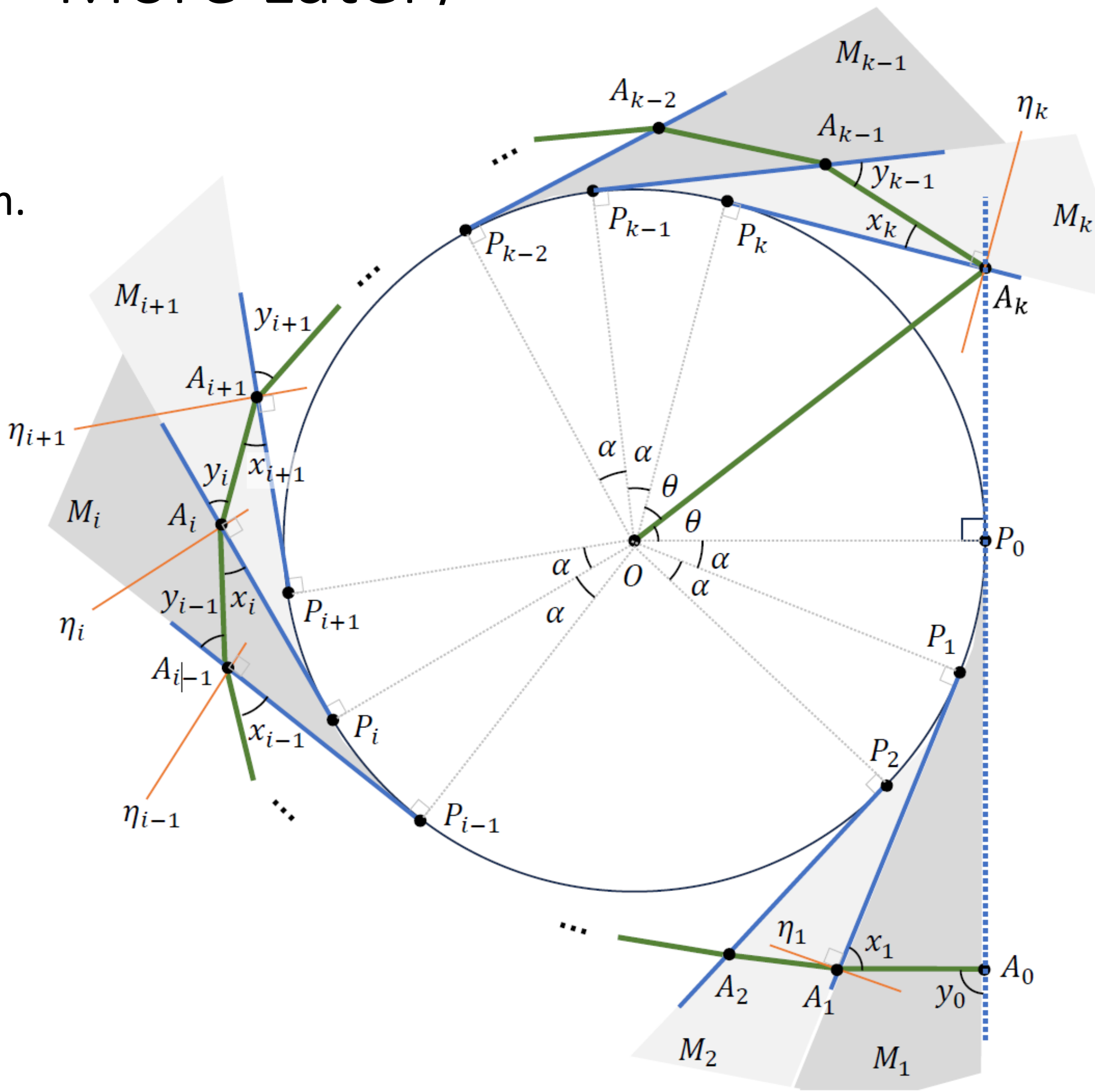
New (optimal) cost
$\mathbf{3.549259\ldots}$

More Systematic Heuristics
(Reduction to a Discretized Problem – More Later)

Theorem (Conley, G. 2024)
Non-Linear Program on $\Theta(k)$ variables solves discretized problem.
If discrete solution does not touch disk, then
Average Disk-Inspection Cost $\leq \left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{k}}\right)\cdot(\boldsymbol{Discrete\ Cost})$

Computational Results:
- $k = 2000$
- Trajectory of NLP solution does not touch disk.
- Induced cost $\mathbf{3.5509015\ldots}$

Drawbacks:
- Expensive to solve (non-convex) **NLP**
- Unable to certify global optimality.



What **is** New?

New Contribution

Theorem [G. 2026]

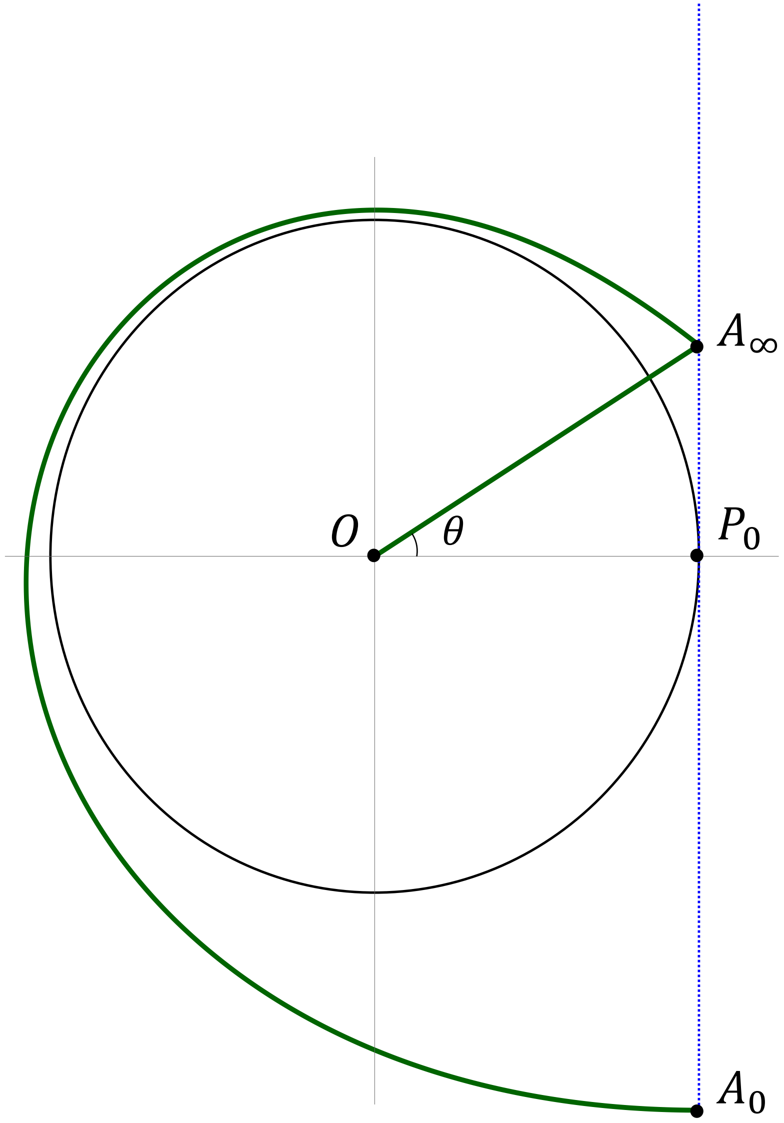
Optimal avg case cost to Disk-Inspection is

$$\mathbf{3.549259\ldots}$$

with ≥ 6 digits of accuracy.

Timeline reminder

1961: 3.63489…	Upper bound only
2024: 3.550901 …	Upper bound only
2026: 3.549259 …	Upper/Lower bound



"More" Formally

(An Optimal Control Optimization Problem)

Theorem ODE system on $\psi, \tau\colon [0,1] \to \mathbb{R}$ with parameter $\boldsymbol{\tau_0} \geq 0$.

$$\psi'(x) = -2\pi + \frac{\cot \psi(x)}{x}$$

$$\frac{1}{2\pi}\boldsymbol{\tau}'(\boldsymbol{x}) = \boldsymbol{\tau}(\boldsymbol{x}) \cot \psi(x) - 1$$

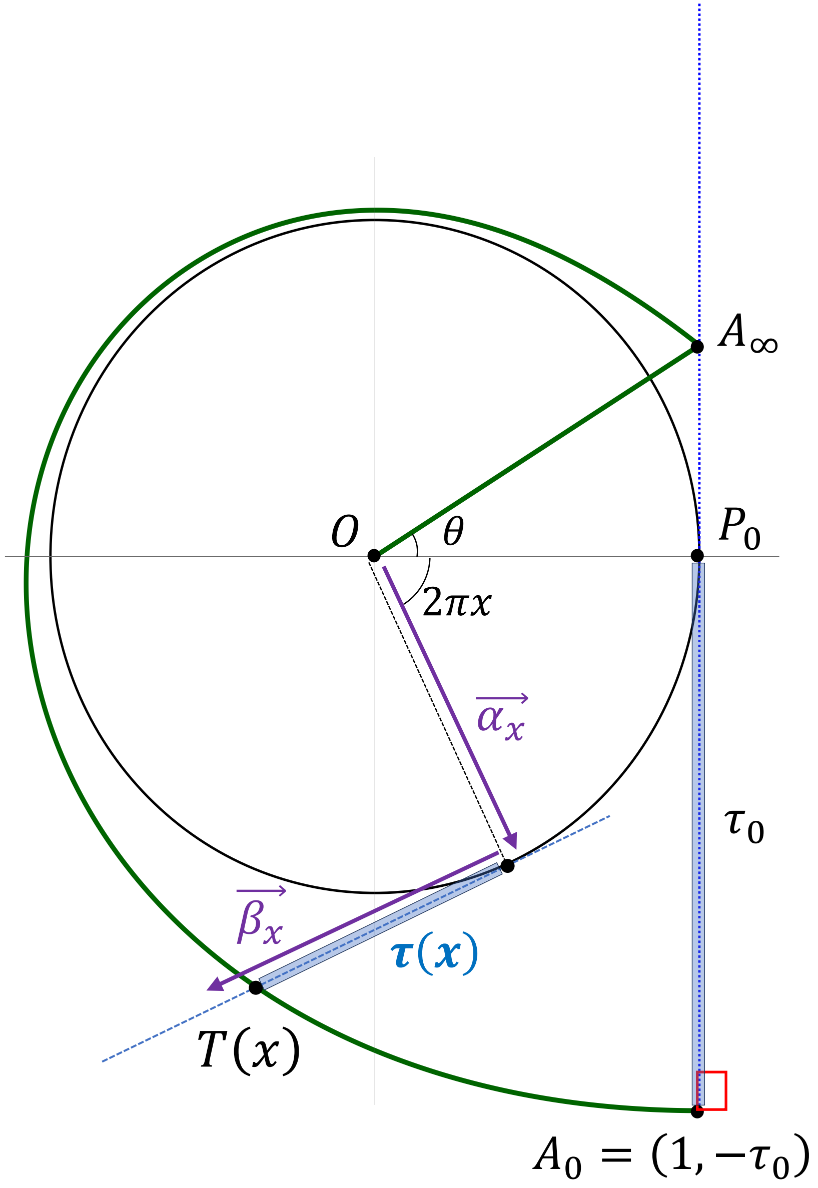
Initial conditions $\psi(0) = \pi/2$, $\boldsymbol{\tau(0) = \tau_0}$

Choose $\boldsymbol{\tau_0} \geq 0$, so that $\boldsymbol{T}\colon [\mathbf{0}, \mathbf{1}] \to \mathbb{R}^{\mathbf{2}}$ **does NOT touch the disk**

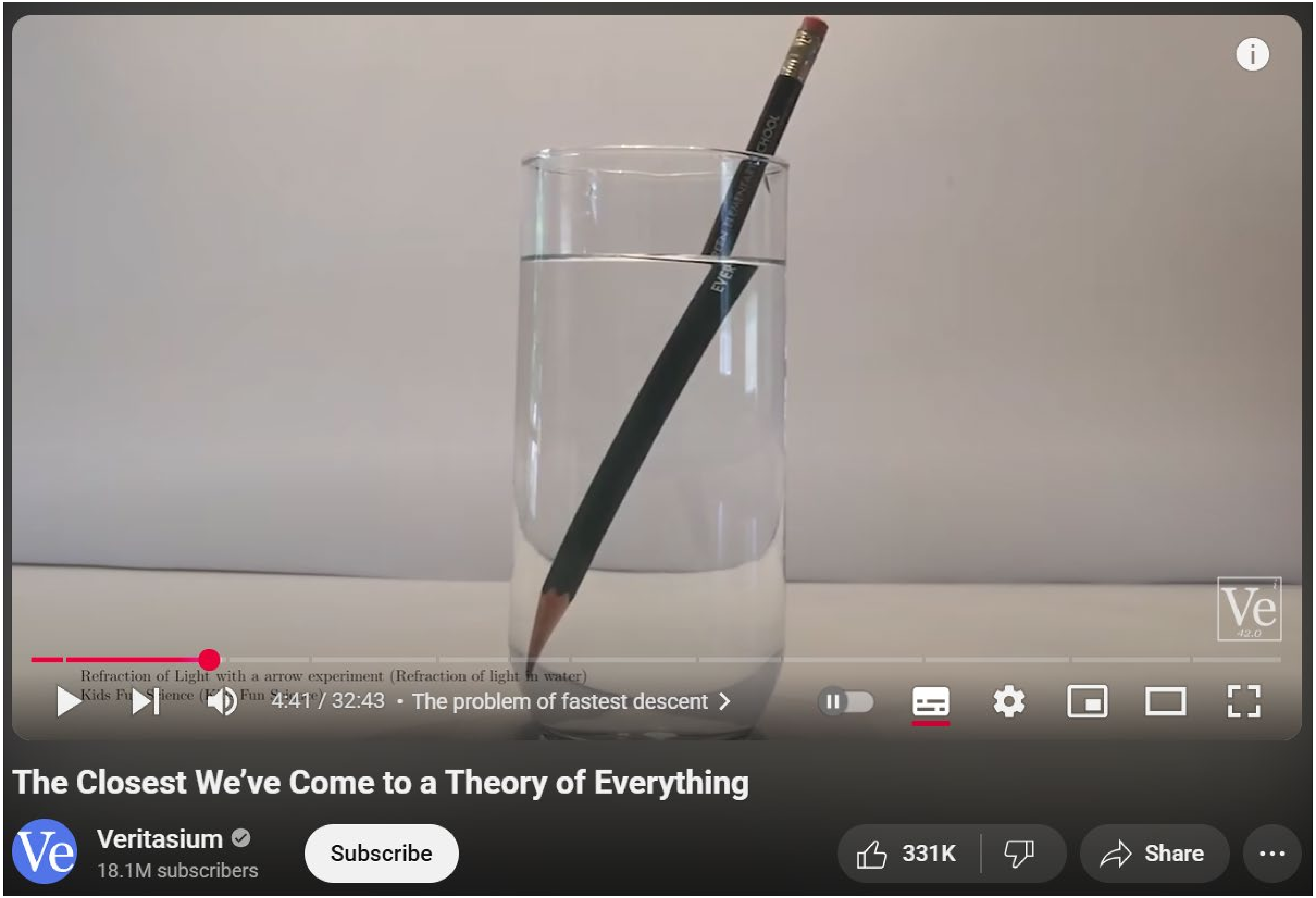
$$\boldsymbol{T}(\boldsymbol{x}) = \begin{pmatrix} \cos 2\pi x - \boldsymbol{\tau}(\boldsymbol{x}) \sin 2\pi x \\ -\sin 2\pi x - \boldsymbol{\tau}(\boldsymbol{x}) \cos 2\pi x \end{pmatrix}$$

and minimizing

$$\frac{1}{2\pi} \log \left(\frac{1 + \sin\theta}{1 - \sin\theta} \right) + \frac{1 - \theta/\pi}{\cos(\theta)} + 2\pi \int_0^{1-\frac{\theta}{\pi}} \frac{x\,\tau(x)}{\sin(\psi(x))} \mathrm{d}x$$



Proof of the Optimality Result


Refraction of Light with a arrow experiment (Refraction of light in water)
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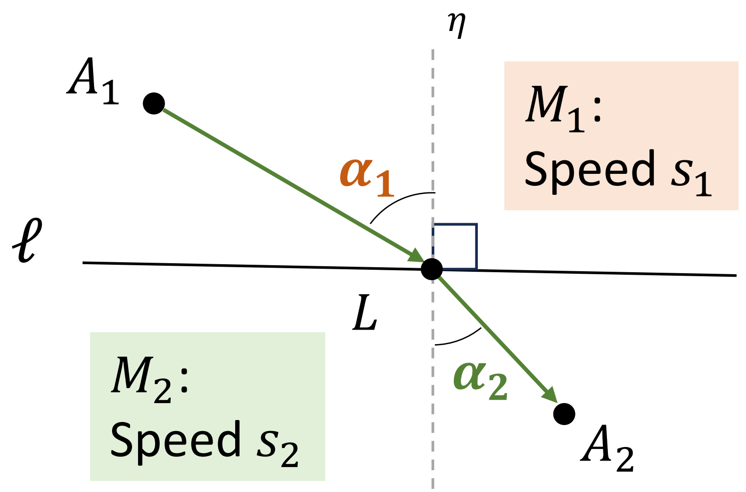
Fermat's Principle (of Least Time)

Snell's Law

A ray of light travelling through media M_1, M_2 with speeds s_1, s_2 satisfies

$$\frac{\sin \boldsymbol{\alpha_1}}{\sin \boldsymbol{\alpha_2}} = \frac{\boldsymbol{s_1}}{\boldsymbol{s_2}}$$



Fermat's Principle

Path taken by ray between two given points is the path that can be traveled in the least time.

Observation

Least-time path between two points passing through media M_1, M_2 with speeds s_1, s_2 is the one satisfying Snell's Law.

The Discrete Problem

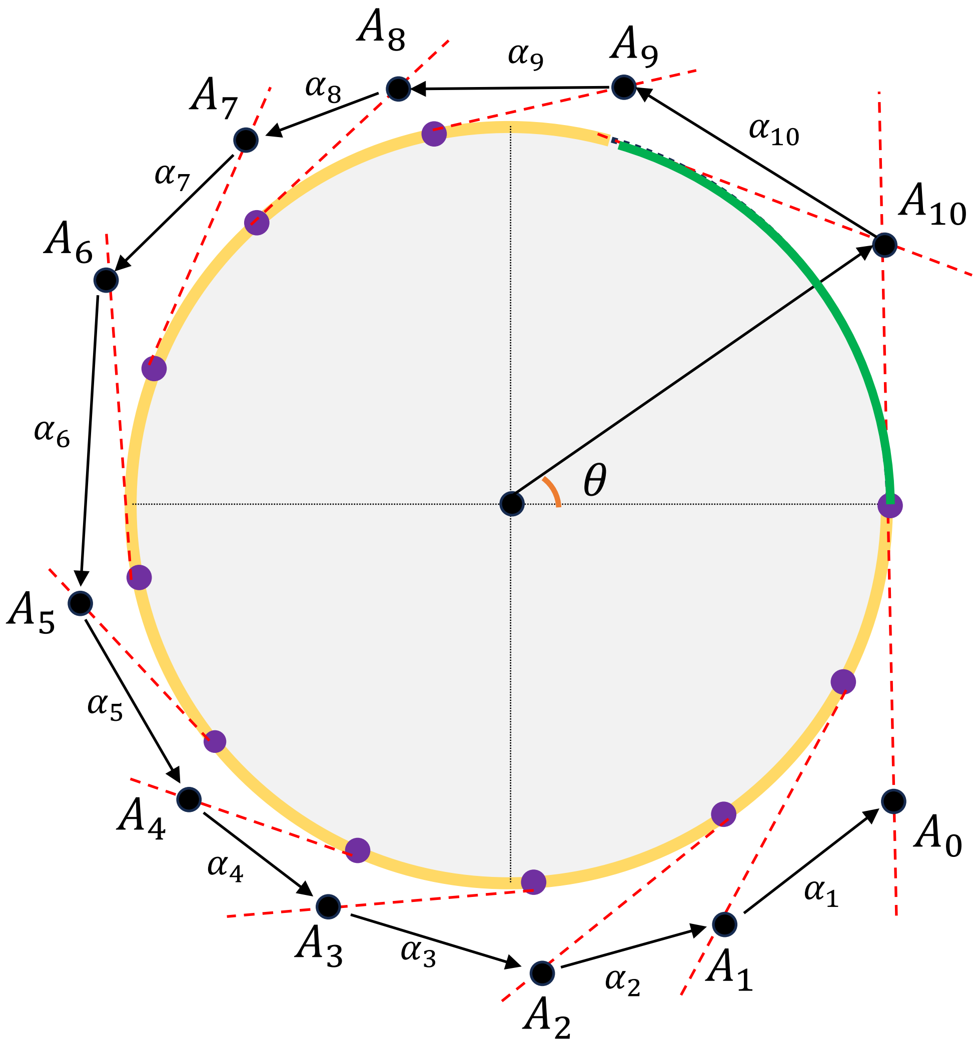
- Initial deployment angle θ.
- Remaining arc to inspect $2\pi - 2\theta$.
- Inspect $k = 10$ equi-spaced points.
- Choose k points on tangent lines.

Average Cost

$$\frac{1}{10}\left(\begin{array}{c}\alpha_{10} + (\alpha_{10} + \alpha_9) + (\alpha_{10} + \alpha_9 + \alpha_8) + \\ \ldots + (\alpha_{10} + \cdots + \alpha_1)\end{array}\right)$$

$$= \frac{1}{10}(1\alpha_1 + 2\alpha_2 + \cdots + 9\alpha_9 + 10\alpha_{10})$$

Solve NLP to minimize cost.



Disk-Inspection As An Optics Problem

NLP

$$\min\ 1\alpha_1 + 2\alpha_2 + \cdots + 9\alpha_9 + 10\alpha_{10}$$
$$s.t.\ A_i \text{ “sufficiently away from disk”}$$

Weighted average coefficients

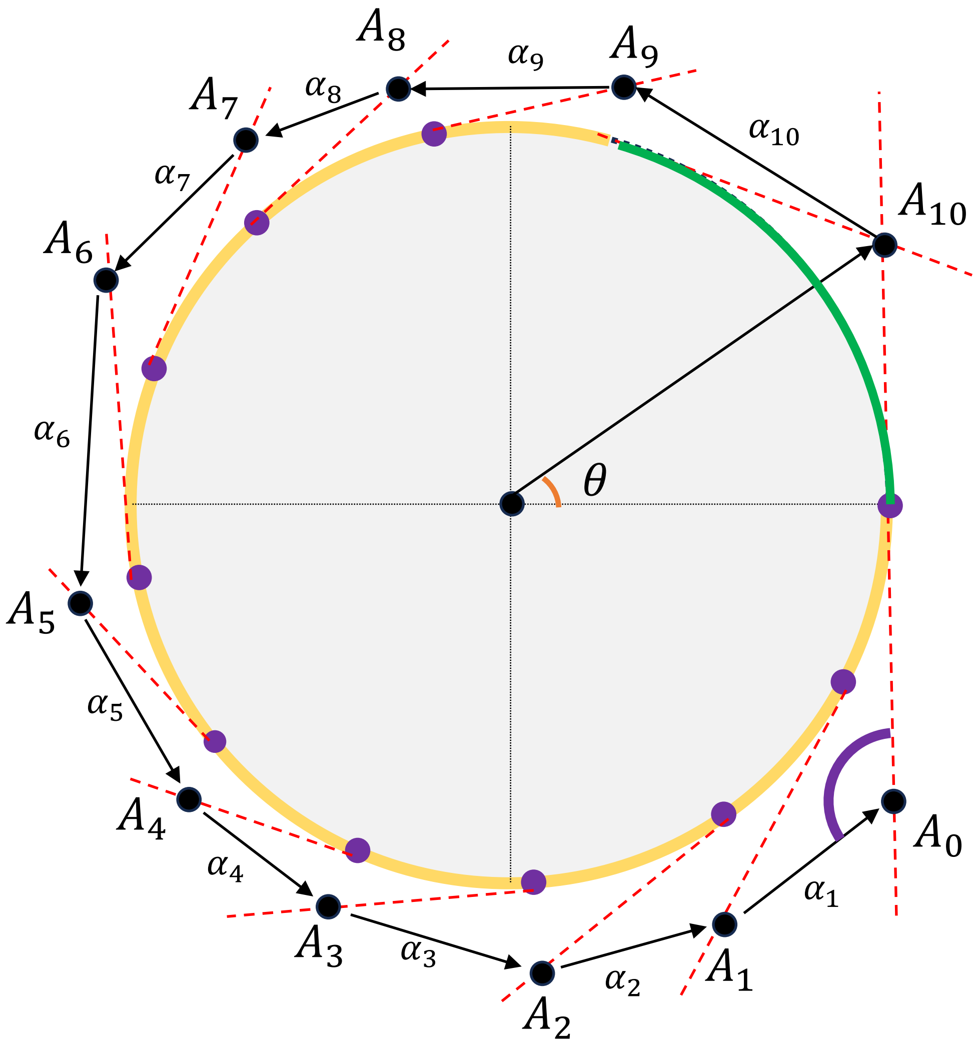
$\approx$ speed factors (speed $s_i = 1/i$)

Objective: $1\alpha_1 + 2\alpha_2 + \cdots + 3\alpha_9 + 10\alpha_{10}$

$$= \frac{\alpha_1}{1/1} + \frac{\alpha_2}{1/2} + \cdots + \frac{\alpha_9}{1/9} + \frac{\alpha_{10}}{1/10}$$

Corollary Avg-Optimal trajectory (if not touching disk) is the optics path

- satisfying Snell-Descartes Law,
- connects deployment point A_{10} to **last tangent line**, at **angle $\pi/2$**.



A Recurrence System for the Discrete Problem

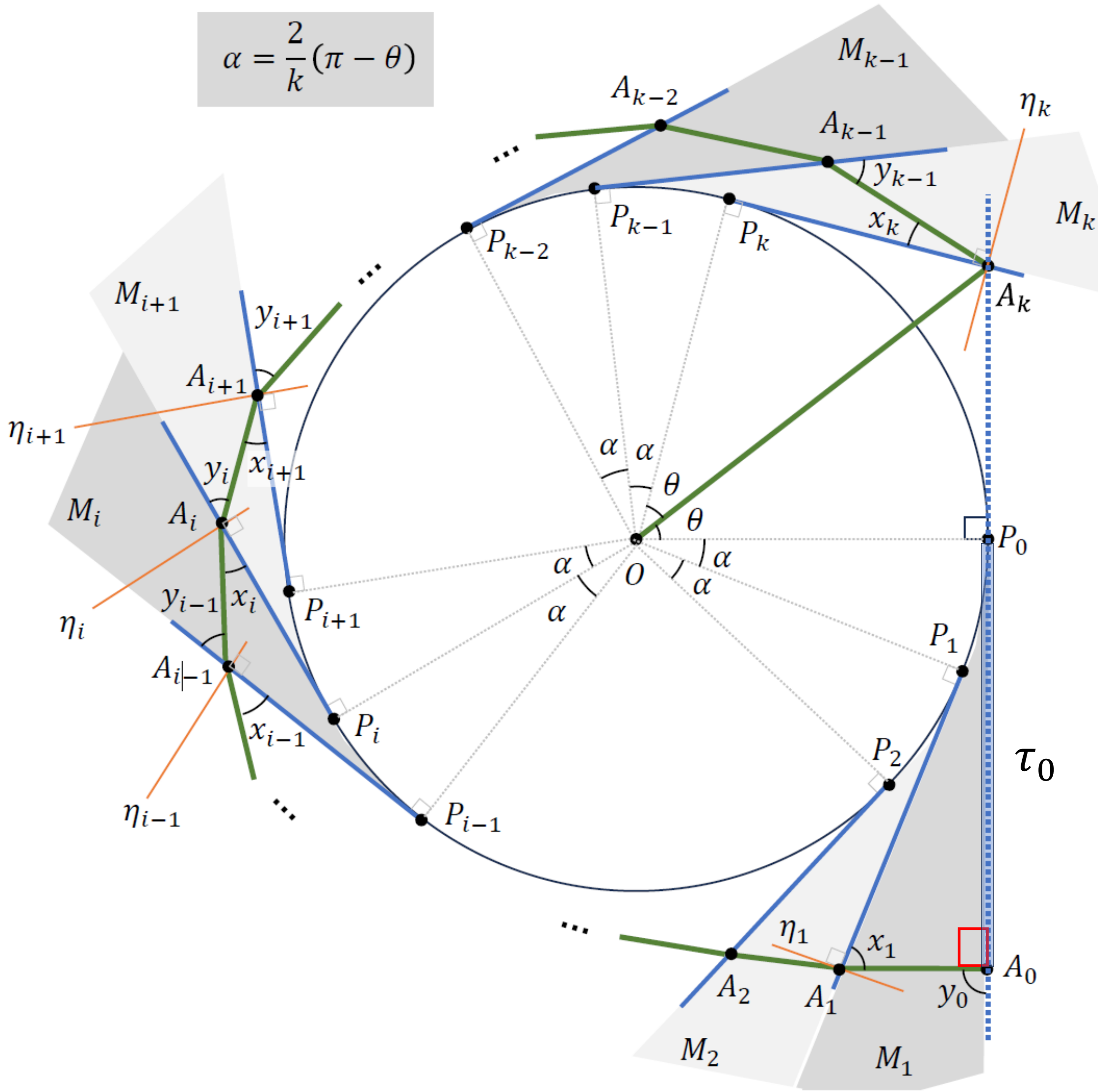
$t_i := |P_i A_i|,\ d_i := |A_{i-1} A_i|$

- $x_i = y_{i-1} - \alpha$
- $y_i = \arccos\left(\frac{i}{i+1}\cos(y_{i-1} - \alpha)\right)$ **(Snell's Law)**
- $t_i = \left(t_{i-1} - \tan\frac{\alpha}{2}\right)\frac{\sin y_{i-1}}{\sin x_i} - \tan\frac{\alpha}{2}$
- $d_i = \left(t_{i-1} - \tan\frac{\alpha}{2}\right)\frac{\sin \alpha}{\sin x_i}$

Initial Conditions:

- $y_0 = \pi/2$
- $t_0 = \tau_0$ **(algorithmic choice determining θ)**

(feasible as long as trajectory does not touch disk)



The Continuum Limit

Discrete Recurrence ($i = 1 \dots, k$)

- $x_i = y_{i-1} - \alpha$
- $y_i = arccos\left(\frac{i}{i+1} cos(y_{i-1} - \alpha)\right)$
- $t_i = \left(t_{i-1} - tan\frac{\alpha}{2}\right)\frac{sin\, y_{i-1}}{sin\, x_i} - tan\frac{\alpha}{2}$
- $d_i = \left(t_{i-1} - \tan\frac{\alpha}{2}\right)\frac{\sin\alpha}{\sin x_i}$

Initial Conditions:

$y_0 = \pi/2,\ t_0 = \tau_0$

Objective: $\sum_i i\, |A_{i-1}A_i| = \sum_i i\, d_i$

ODE system ($x \in [0, 1], \frac{i}{k} \to x$)

$$\psi'(x) = -2\pi + \frac{cot\, \psi(x)}{x}$$

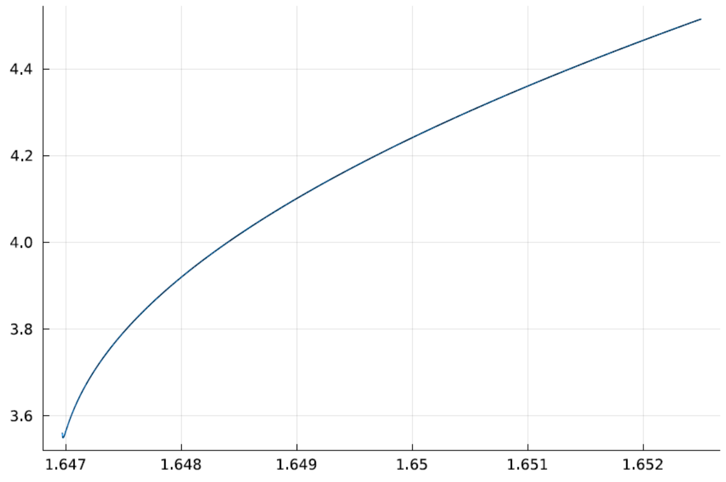
$$\frac{1}{2\pi}\tau'(x) = \tau(x)\, cot\, \psi(x) - 1$$

Initial Conditions:

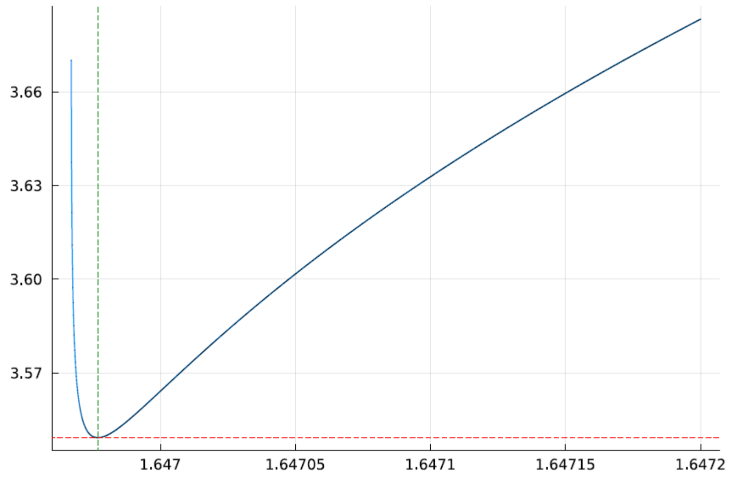
$\psi(0) = \pi/2, \tau(0) = \tau_0$

Objective: $\int_0^{1-\frac{\theta}{\pi}} \frac{x\, \tau(x)}{sin(\psi(x))}\, dx$

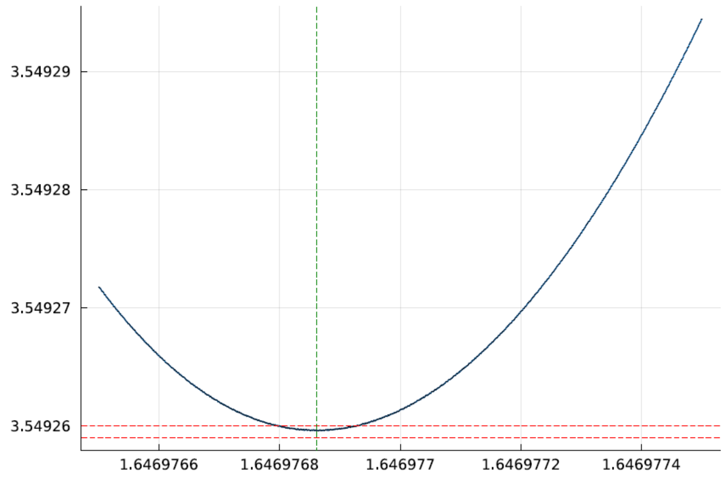
Trajectories Performance in Critical Regime



$\tau_0 \in [1.64697, 1.6525]$



$\tau_0 \in [1.64697, 1.6472]$



$\tau_0 \in [1.6469764, 1.6469774]$

Theorem Optimal avg Disk-Inspection cost is **3.549259 …** with $\geq$ 6 digits of confidence.

Corollary: Bluss' 1961 conjecture that best trajectory must touch the disk, is wrong.

New Directions

Some Open Problems

- 2D Object-Search Variations
  - Shoreline search (unknown distance)
  - Shoreline search (known distance)
  - Point search
- Objective/Completion variations
  - Cost of first finder (search)
  - Cost of last finder (evacuation)
  - Other variations
- Analysis Variations
  - Worst Case
  - Average Case
  - Trade-Offs
- Agents’ Specs Variations
  - Communication models
  - Speeds

Merci, des questions ?