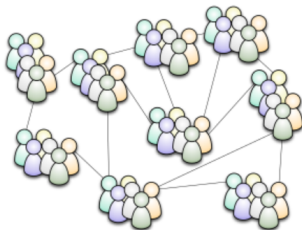


Modularity of preferential attachment graphs

Katarzyna Rybarczyk and Małgorzata Sulkowska*



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Wrocław University
of Science and Technology



ADAM MICKIEWICZ
UNIVERSITY
POZNAŃ

Outline of the talk

- 1 Definitions
 - ▶ Modularity
 - ▶ Preferential attachment graph
- 2 Conjectures and the result
- 3 Proof
 - ▶ Step 1
 - ▶ Step 2
 - ▶ Martingales
- 4 Future work

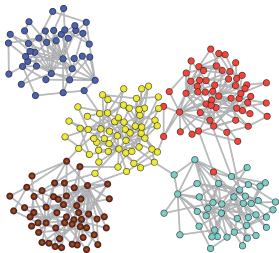
Modularity

Modularity

A quantitative parameter measuring the presence of **community structure** in a graph.

Application: algorithms for community detection in networks

- detecting users with similar interests
- localizing web pages on related topics
- network compression and visualization



high modularity



low modularity

Modularity score of a partition

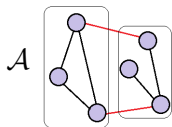
Modularity score of a partition \mathcal{A} of G (Newman, Girvan, 2004)

$G = (V, E)$ - graph with at least one edge

\mathcal{A} - partition of V (into communities)

$$\text{mod}_{\mathcal{A}}(G) = \sum_{S \in \mathcal{A}} \left(\frac{e(S)}{e(G)} - \left(\frac{\text{vol}(S)}{\text{vol}(G)} \right)^2 \right)$$

where $e(S) = |\{e \in E : e \subseteq S\}|$ $\text{vol}(S) = \sum_{v \in S} \text{deg}(v)$
($\text{deg}(v)$ - degree of v in G)



$$\text{mod}_{\mathcal{A}}(G) = \left(\frac{3}{7} - \left(\frac{8}{14} \right)^2 \right) + \left(\frac{2}{7} - \left(\frac{6}{14} \right)^2 \right) = \frac{10}{49}$$

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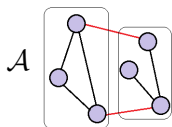
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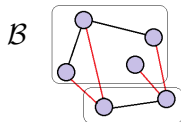
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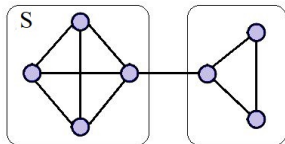
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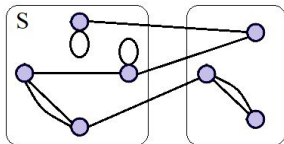
$$\text{mod}_{\mathcal{B}}(G) = \left(\frac{2}{7} - \left(\frac{8}{14} \right)^2 \right) + \left(\frac{1}{7} - \left(\frac{6}{14} \right)^2 \right) = -\frac{4}{49}$$

Modularity score of a partition

$$\text{mod}_{\mathcal{A}}(G) = \sum_{S \in \mathcal{A}} \left(\frac{e(S)}{e(G)} - \left(\frac{\text{vol}(S)}{\text{vol}(G)} \right)^2 \right)$$



$$\frac{e(S)}{e(G)}$$



$$\mathbb{E} \left[\frac{e(S)}{e(G^*)} \right] = \left(\frac{\text{vol}(S)}{\text{vol}(G)} \right)^2$$

Single summand:

- $\frac{e(S)}{e(G)}$ - fraction of edges within S
- $\left(\frac{\text{vol}(S)}{\text{vol}(G)} \right)^2$ - expected fraction of edges within S if we considered a random multigraph G^* on V with the expected degree sequence given by G

Modularity

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Modularity of G (Newman, Girvan, 2004)

$$\text{mod}(G) = \max_{\mathcal{A}} \text{mod}_{\mathcal{A}}(G)$$

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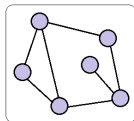
Modularity of G (Newman, Girvan, 2004)

$$\text{mod}(G) = \max_{\mathcal{A}} \text{mod}_{\mathcal{A}}(G)$$

Fact

$$\text{mod}(G) \in [0, 1).$$

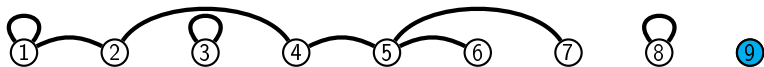
Consider
 $\mathcal{A} = \{V\}$.



Outline of the talk

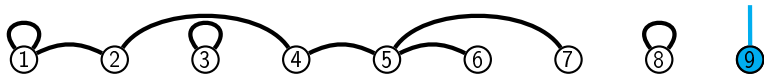
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Preferential attachment graph G_n^h



- 1 Construct a "tree" T_{hn}
 - ▶ Mini-vertices come one by one: 1, 2, 3, ...

Preferential attachment graph G_n^h



- 1 Construct a "tree" T_{hn}
 - ▶ Mini-vertices come one by one: $1, 2, 3, \dots$
 - ▶ The mini-vertex $t + 1$ chooses the neighbor $N(t)$ from $1, \dots, t$ proportionally to its degree in T_t :

$$\mathbb{P}(N(t) = s) = \begin{cases} \frac{\deg_{T_t}(s)}{2t+1} & \text{for } 1 \leq s \leq t \\ \frac{1}{2t+1} & \text{for } s = t + 1. \end{cases}$$

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- 2 Construct G_n^h by merging mini-vertices $\{h(i-1) + 1, \dots, h(i-1) + h\}$ into a vertex i for $i \in \{1, 2, \dots, n\}$.

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$$T_{hn} = (\tilde{V}, \tilde{E}), \quad |\tilde{V}| = hn, \quad |\tilde{E}| = hn$$

$$G_n^h = (V, E), \quad |V| = n, \quad |E| = hn$$

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Modularity of random graphs - previous work

Graph G	$\text{mod}(G)$ [whp*]
(1) Binomial $G(n, p)$, $p \in [1/n, 0.99]$	$\Theta(1/\sqrt{np})$
(2) Random r -regular, $r \geq r_0$	
(3) Random cubic	
(4) Random planar	
(5) Preferential G_n^h	
(6) Preferential G_n^h	

(1) McDiarmid, Skerman, 2020; Rybarczyk, Sulowska, 2025+

*whp - with probability tending to 1 as n (the order of the graph) tends to infinity

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$$\left(\frac{1}{\sqrt{\text{avg deg}(G)}} \right)$$

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Conjectures and the result

Conjecture 1 (Prokhorenkova, Prałat, Raigorodskii, 2017)

Let G_n^h be a preferential attachment graph. Then whp

$$\text{mod}(G_n^h) = \Theta(1/\sqrt{h}).$$

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Conjecture 2 (Prokhorenkova, Prałat, Raigorodskii, 2017)

Let G_n^h be a preferential attachment graph. Then whp

$$\text{mod}(G_n^h) \xrightarrow{h \rightarrow \infty} 0.$$

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Theorem (Rybarczyk, S., 2026)

Let G_n^h be a preferential attachment graph. Then whp

$$\text{mod}(G_n^h) = O\left(\sqrt{\ln h}/\sqrt{h}\right).$$

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Theorem (Rybarczyk, S., 2026)

Let G_n^h be a preferential attachment graph. Then for every $\varepsilon > 0$ whp

$$\text{mod}(G_n^h) \leq \frac{(1 + \varepsilon)f(h)}{\sqrt{h}},$$

where

$$f(h) = 6g(h) + 4\sqrt{2\ln 2} - g(h)^2/\sqrt{h}$$

with

$$g(h) = \frac{1}{6}\sqrt{2\ln 2(9\ln h + 8\ln 2)} + (2/3)\ln 2 + 2.$$

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The proof

- Step 1: no need to consider all the partitions!

Lemma (Dinh, Thai, 2011)

G – graph with at least one edge, $k \in \{2, 3, \dots\}$

$$\text{mod}(G) \leq \frac{k}{k-1} \max_{\mathcal{A}: |\mathcal{A}| \leq k} \text{mod}_{\mathcal{A}}(G)$$

The proof

- Step 1: no need to consider all the partitions!

Lemma (Dinh, Thai, 2011)

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Corollary

G – graph with at least one edge

$$\text{mod}(G) \leq 4 \max_{S \subseteq V} \left(\frac{e(S)}{e(G)} - \frac{\text{vol}(S)^2}{\text{vol}(G)^2} \right)$$

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- Step 2: concentration results for $e(S)$ and $\text{vol}(S)$

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$G_n^h = (V, E)$, $T_{hn} = (\tilde{V}, \tilde{E})$, $S \subseteq V$ and $\tilde{S} \subseteq \tilde{V}$ – counterparts

Observation: $e(S) = e(\tilde{S})$ and $vol(S) = vol(\tilde{S})$
depend on the arrival time of vertices; but how?

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Hint: relation between T_{hn} and a random graph $G^* = (V^*, E^*)$:

- $V^* = [hn]$
- edge $\{i, j\}$ present with probability $1/(2\sqrt{ij})$, independently of the other edges

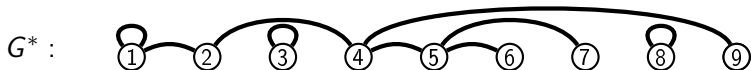
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(Bollobás and Riordan, 2004)

Concentration results for $e(S)$ and $\text{vol}(S)$

In $G^* = (V^*, E^*)$ for $\tilde{S} \subseteq V^*$

$$\mathbb{E}[e_{G^*}(\tilde{S})] = \left(\sum_{j \in \tilde{S}} \frac{1}{2\sqrt{j}} \right)^2 + O(\ln(hn))$$

$$\mathbb{E}[\text{vol}_{G^*}(\tilde{S})] = 2 \left(\sum_{j \in [hn]} \frac{1}{2\sqrt{j}} \right) \left(\sum_{j \in \tilde{S}} \frac{1}{2\sqrt{j}} \right) + O(\ln(hn))$$

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In G_n^h whp for all $S \subseteq V$

$$e(S) = \mu^2(\tilde{S}) + O\left(hn \sqrt{\ln h} / \sqrt{h}\right)$$

$$\text{vol}(S) = 2\sqrt{hn} \mu(\tilde{S}) + O\left(hn \sqrt{\ln h} / \sqrt{h}\right)$$

where $\mu(\tilde{S}) = \frac{1}{2}\sqrt{\pi} \sum_{j \in \tilde{S}} \frac{(2j-3)!!}{(2j-2)!!} \sim \sum_{j \in \tilde{S}} \frac{1}{2\sqrt{j}}$

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$$\mathbb{E}[e_{G^*}(\tilde{S})] = \left(\sum_{j \in \tilde{S}} \frac{1}{2\sqrt{j}} \right)^2 + O(\ln(hn))$$

$$\mathbb{E}[\text{vol}_{G^*}(\tilde{S})] = 2 \left(\sum_{j \in [hn]} \frac{1}{2\sqrt{j}} \right) \left(\sum_{j \in \tilde{S}} \frac{1}{2\sqrt{j}} \right) + O(\ln(hn))$$

In G_n^h whp for all $S \subseteq V$

$$e(S) = \mu^2(\tilde{S}) + O\left(hn \sqrt{\ln h} / \sqrt{h}\right)$$

$$\text{vol}(S) = 2\sqrt{hn} \mu(\tilde{S}) + O\left(hn \sqrt{\ln h} / \sqrt{h}\right)$$

where $\mu(\tilde{S}) = \frac{1}{2}\sqrt{\pi} \sum_{j \in \tilde{S}} \frac{(2j-3)!!}{(2j-2)!!} \sim \sum_{j \in \tilde{S}} \frac{1}{2\sqrt{j}}$

The exact formula for $\mu(\tilde{S})$ follows from martingale techniques.

The proof

Corollary

G – graph with at least one edge

$$\text{mod}(G) \leq 4 \max_{S \subseteq V} \left(\frac{e(S)}{e(G)} - \frac{\text{vol}(S)^2}{\text{vol}(G)^2} \right)$$

The proof

Corollary

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Theorem

$G_n^h = (V, E)$ – preferential attachment graph; then whp for all $S \subseteq V$

$$e(S) = \mu(\tilde{S})^2 + O(hn \sqrt{\ln h} / \sqrt{h})$$

$$\text{vol}(S) = 2\sqrt{hn}\mu(\tilde{S}) + O(hn \sqrt{\ln h} / \sqrt{h})$$

where \tilde{S} – counterpart of S in T_{hn} ,

$$\mu(\tilde{S}) = \frac{1}{2} \sqrt{\pi} \sum_{j \in \tilde{S}} \frac{(2j-3)!!}{(2j-2)!!} \sim \sum_{j \in \tilde{S}} \frac{1}{2\sqrt{j}}.$$

The desired result follows: whp $\text{mod}(G_n^h) = O(\sqrt{\ln h} / \sqrt{h})$.

Outline of the talk

- 1 Definitions
 - ▶ Modularity
 - ▶ Preferential attachment graph
- 2 Conjectures and the result
- 3 Proof
 - ▶ Step 1
 - ▶ Step 2
 - ▶ [Martingales](#)
- 4 Future work

Martingale techniques for $e(\tilde{S})$ and $vol(\tilde{S})$

① Start with $vol(\tilde{S})$.

- ▶ Set $Z_t = vol_{T_t}(\tilde{S}_t)$.
- ▶ Define a martingale*

$$\hat{Z}_t = c_t Z_t - \sum_{j \in \tilde{S}_t} c_{j-1}, \quad c_j = \frac{(2j-1)!!}{(2j)!!}.$$

- ▶ Apply Freedman's inequality to $\hat{Z}_{t/h}, \dots, \hat{Z}_t$.

* Inspired by Frieze, Prałat, Pérez-Giménez, Reiniger, 2019

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2 Continue with $e(\tilde{S})$.

- ▶ Set $X_t = e_{T_t}(\tilde{S}_t)$.
- ▶ Define a martingale

$$\hat{X}_t = X_t - \sum_{j=2}^t \mathbb{E}[X_j - X_{j-1} | \mathcal{F}_{j-1}],$$

\mathcal{F}_j - σ -algebra associated with all the events till time j .

- ▶ Apply Azuma-Hoeffding inequality to \hat{X}_t , use the concentration result for $vol(\tilde{S}_t)$.

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Future and related work

Theorem (Rybarczyk, S., 2026)

Let G_n^h be a preferential attachment graph. Then whp

$$\text{mod}(G_n^h) = O\left(\sqrt{\ln h}/\sqrt{h}\right).$$

Conjecture 1 (Prokhorenkova, Prałat, Raigorodskii, 2017)

Let G_n^h be a preferential attachment graph. Then whp

$$\text{mod}(G_n^h) = \Theta(1/\sqrt{h}).$$

Future and related work

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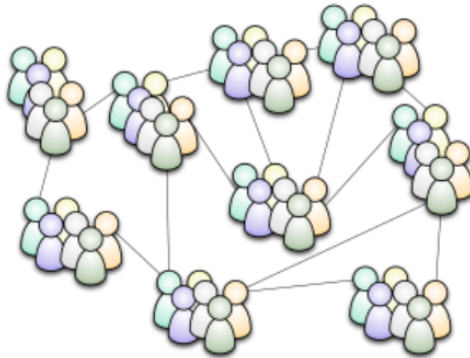
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Our theorem gives non-trivial bound for $h \geq 810$.

Bounds on $\text{mod}(G_n^h)$ for small h : independent work with McDiarmid, Rybarczyk, Skerman.



Thank you!

Martingale inequalities

Azuma-Hoeffding inequality

If X_0, X_1, \dots, X_n is a martingale and there exist b_1, \dots, b_n such that $|X_j - X_{j-1}| \leq b_j$ for each $j \in [n]$, then, for every $x > 0$,

$$\mathbb{P}[X_n \geq X_0 + x] \leq \exp \left\{ -\frac{x^2}{2 \sum_{j=1}^n b_j^2} \right\}.$$

Freedman's inequality

Let X_0, X_1, \dots, X_n be a martingale with respect to a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_n$. Set $A_k = \max_{i \in [k]} (X_i - X_{i-1})$ and $W_k = \sum_{i=1}^k \text{Var}[X_i - X_{i-1} | \mathcal{F}_{i-1}]$. Then for every $\lambda > 0$ and $W, A > 0$ we have

$$\mathbb{P}[\exists k \in [n] \quad X_k \geq X_0 + \lambda, W_k \leq W, A_k \leq A] \leq \exp \left\{ -\frac{\lambda^2}{2W + 2A\lambda/3} \right\}.$$