Maximum Reachability Orientation of Mixed Graphs

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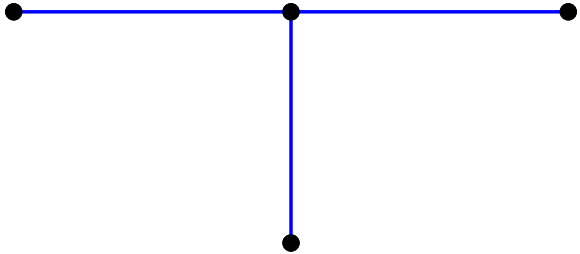
Basic Definitions

An orientation of a graph is a digraph obtained by assigning an orientation to every edge.



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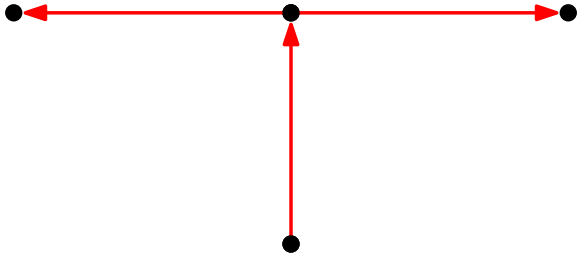

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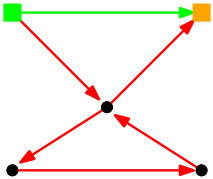

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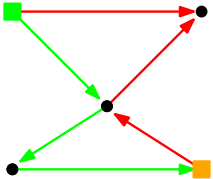

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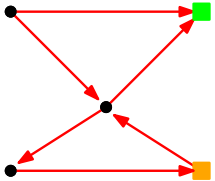

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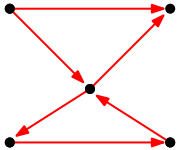
- we use $R(D)$ for $\sum_{(u,v)\in P(D)} \kappa_D(u, v)$.



$R(D) = 13$

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Maximum Reachability Orientation (MRO)

- Input: A graph G,
- Objective: Find an orientation $\vec{G}$ of G that maximizes $R(\vec{G})$.

Previous work

Theorem (Hakimi, Schmeichel, and Young; 1997)

MRO can be solved in polynomial time.

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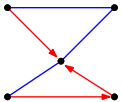
Definitions

- a mixed graph G consists of $V(G)$, an edge set $E(G)$ and an arc set $A(G)$,



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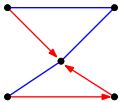

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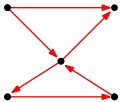

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MMRO is NP-complete.

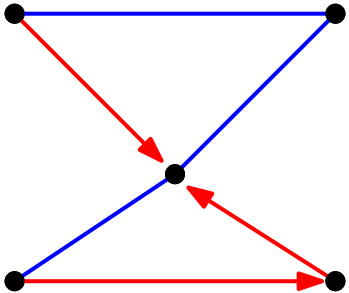
Theorem

Unless $P = NP$, there is no polynomial-time $\frac{581773}{581774}$-approximation algorithm for MMRO.

Small number of arcs

Definition

Given an instance G of MMRO, we use k for $|A(G)|$.



$k = 3$

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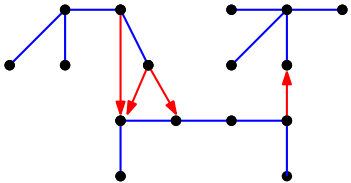
Theorem

For every $\epsilon > 0$, a $(1-\epsilon)$-approximation for MMRO can be computed in $f(k,\epsilon)n^{O(1)}$.

Preprocessing

Definition

An undirected component of G is a component of $G \setminus A(G)$.



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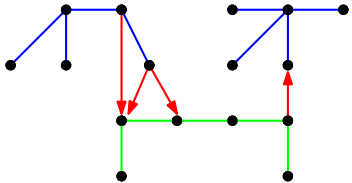

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We simplify so that undirected components

- are trees (contract cycles),
- contain at most two vertices incident to arcs (guess certain edges).

Algorithm for one Undirected Component

Objective

Want a small number of orientations one of which is always good.

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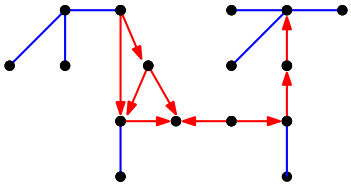
Algorithm Sketch

- take a simplified instance and one undirected component,



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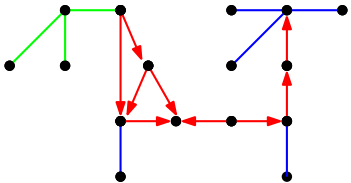

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- take a simplified instance and one undirected component,
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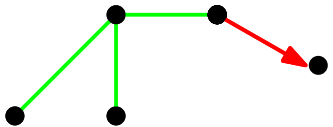

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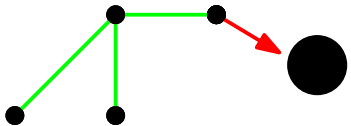

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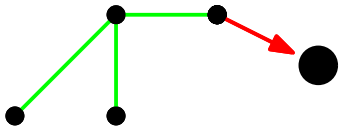

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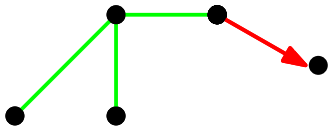

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- take a simplified instance and one undirected component,
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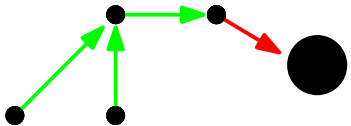
- create small number of simple instances,
- solve these instances and store orientations.



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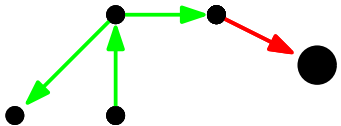

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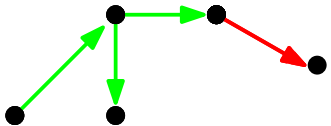

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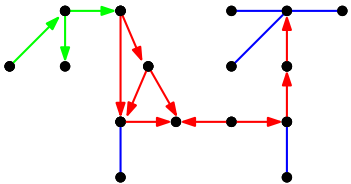

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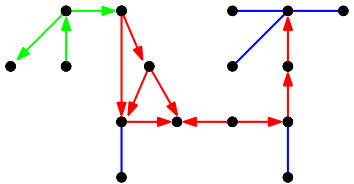

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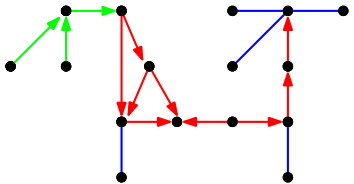

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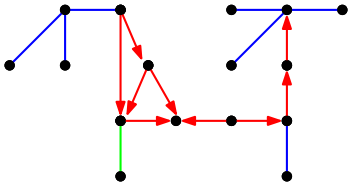

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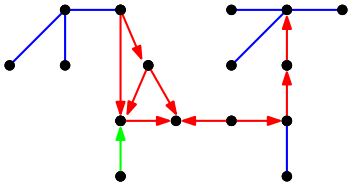

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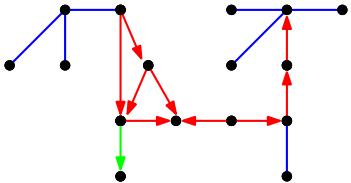

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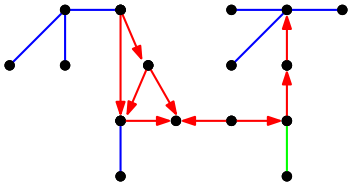

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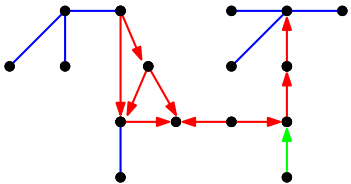

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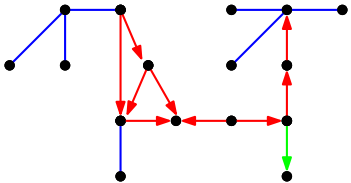

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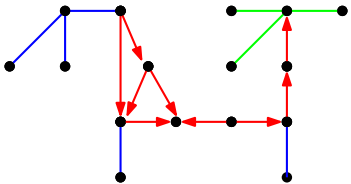

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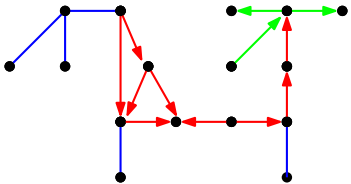

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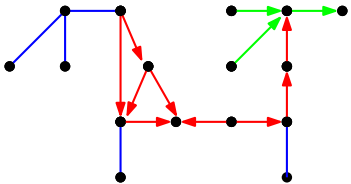

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- try all possible combinations,
- keep the best one.

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Open problems

- constant factor approximation in polynomial time,
- FPT algorithm (i.e. $f(k)n^{O(1)}$),
- similar parameterization for other problems.