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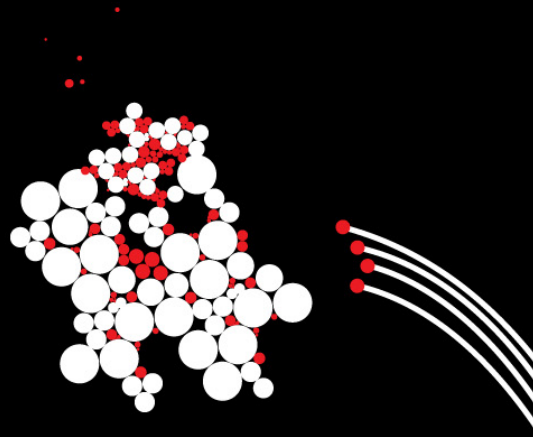
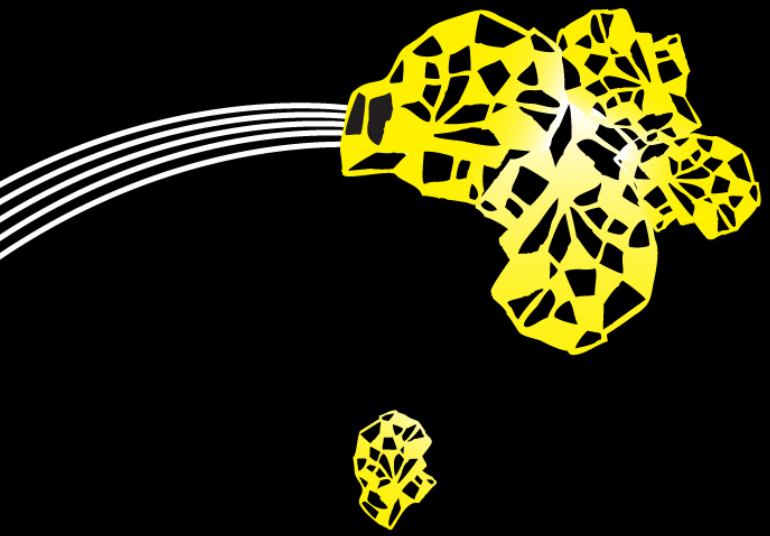


Lower bounds for ranking-based pivot rules

Yann Disser[1], Georg Loho[2,3], *Matthew Maat*[2] and Nils Mosis[1]

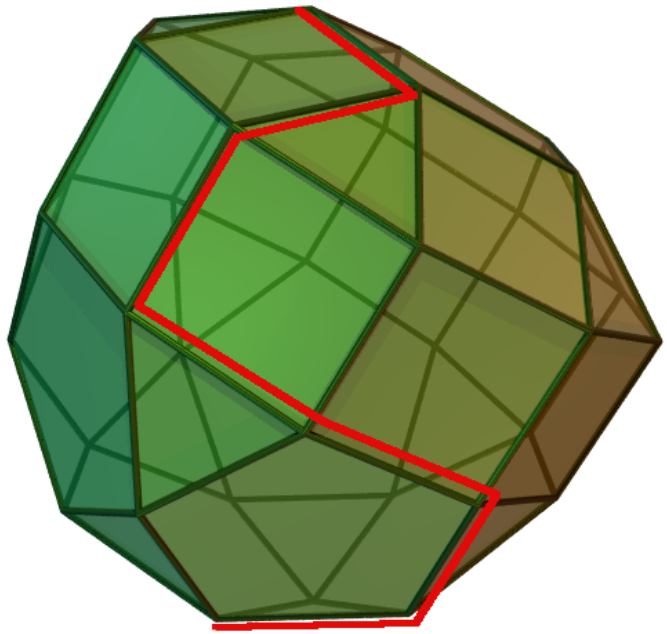
[1]TU Darmstadt
[2]University of Twente
[3]FU Berlin

March 2026, Grenoble



Linear programming and the simplex method

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$



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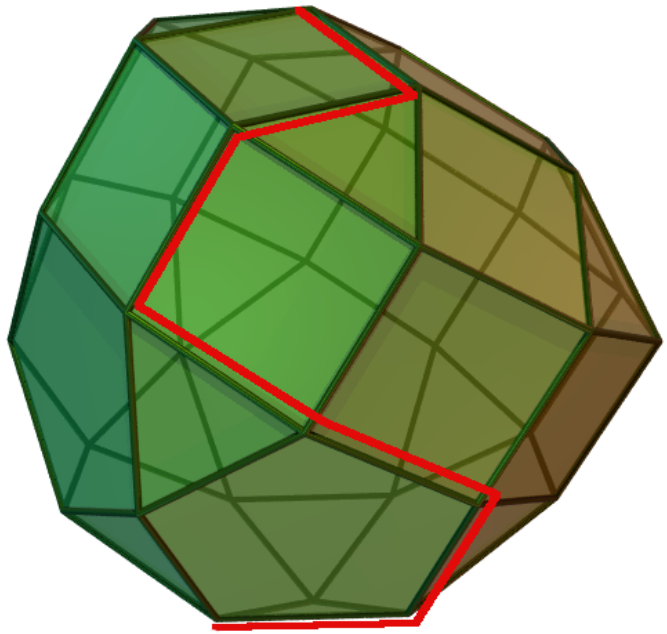
- Move along edge polyhedron



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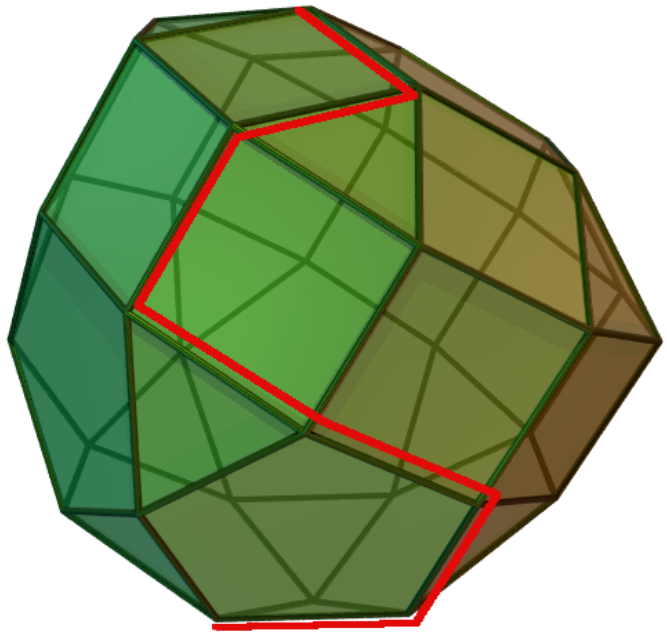
- Simplex: start with vertex of polyhedron
- Move along edge polyhedron
- Or: basis B (columns of A)
- Replace column of B/exchange one basic variable



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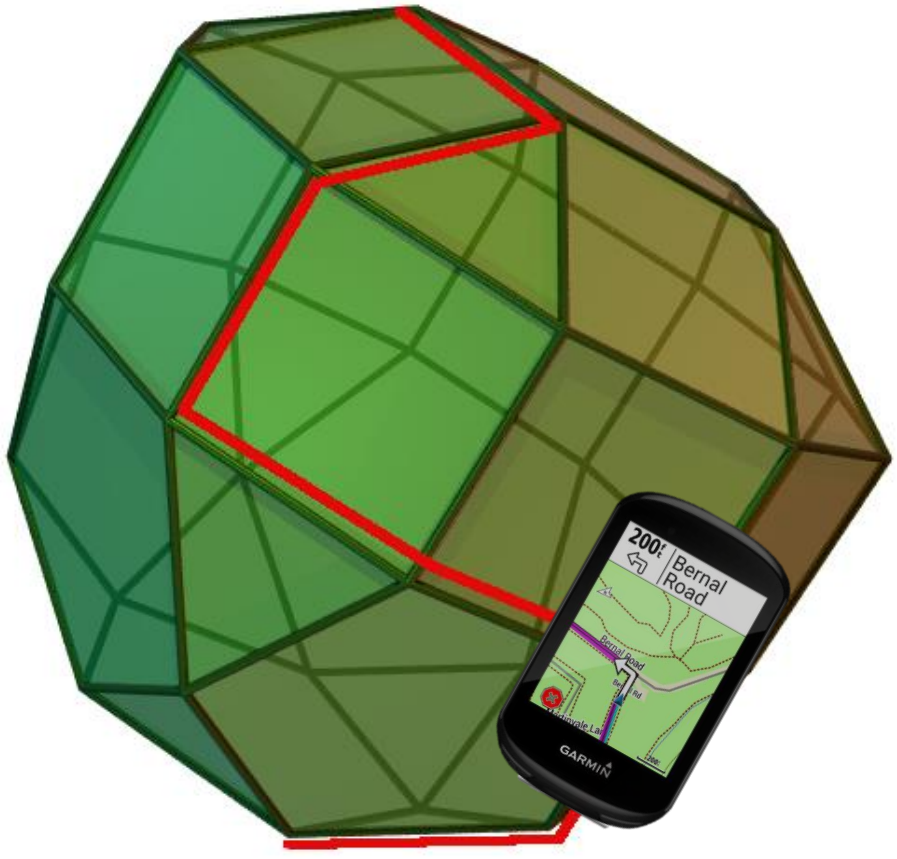
Pivot rules



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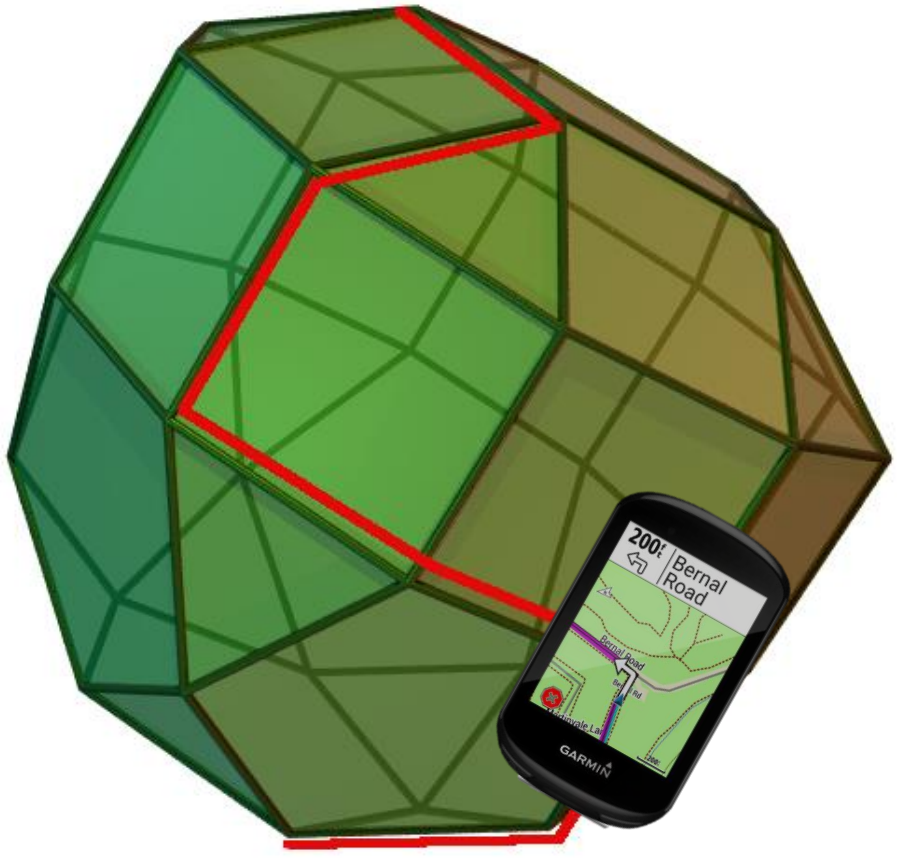
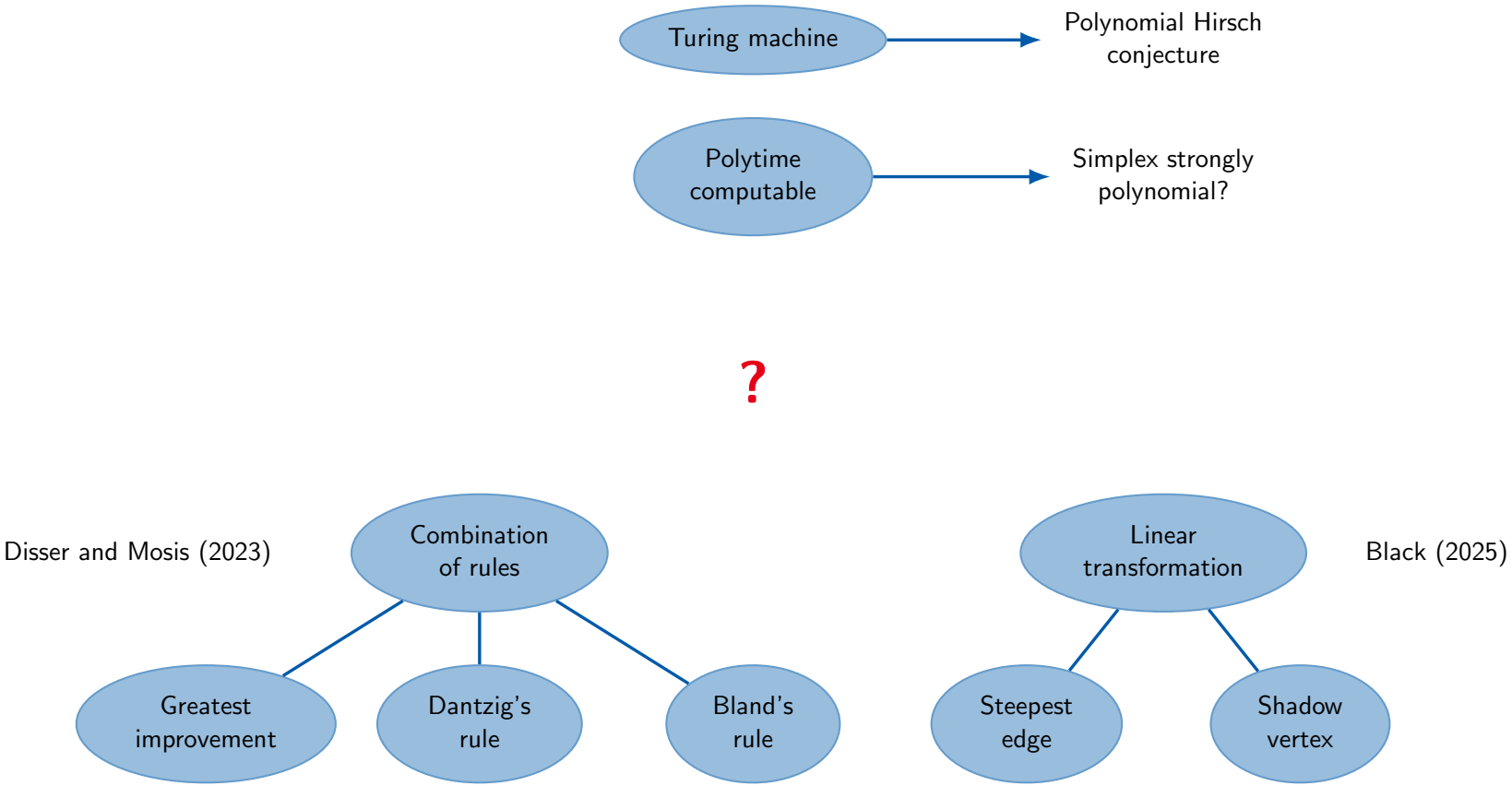


- Is there a pivot rule that guarantees reaching the optimum in (strongly) polynomial time?

Pivot rule models



Pivot rule models

- $\min \boldsymbol{c}^T \boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$ with $A \in \mathbb{R}^{m \times n}$.
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$$\Pi_{m,n} \colon \mathcal{L}_{m,n} \times \mathcal{P}([n]) \times [H_{m,n}] \to \mathcal{P}([n]) \times [H_{m,n}]$$
$$((A, \boldsymbol{b}, \boldsymbol{c}), B, h) \mapsto (S, h')$$

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- Split pivot rule up

$$\Pi_{m,n}((A, \boldsymbol{b}, \boldsymbol{c}), B, h) = D^{\Pi}_{m,n}(I^{\Pi}_{m,n}((A, \boldsymbol{b}, \boldsymbol{c}), B), h)$$

With $I^{\Pi}_{m,n}$ **information function** and $D^{\Pi}_{m,n}$ **decision function**.

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  With $I^{\Pi}_{m,n}$ **information function** and $D^{\Pi}_{m,n}$ **decision function**.
- Classify pivot rules based on a fixed information function $I^{\Pi}_{m,n}$.

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- Examples:
  - Steepness of edges
  - Indices of improving variables
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Information functions

- Examples:
  - Steepness of edges
  - Indices of improving variables
  - A ranking by some metric
- Ranking-based pivot rules: $I^{\Pi}_{m,n}$ encodes information from one or more rankings[2]
- We show lower bounds for:
  - $I^{\Pi}_{m,n}$ is ranking of indices (*index-based* rule)
  - Memoryless rules where $I^{\Pi}_{m,n}$ has rankings of indices, reduced cost, objective improvement **at the same time**.

[2] And some symmetry requirements

Theorems

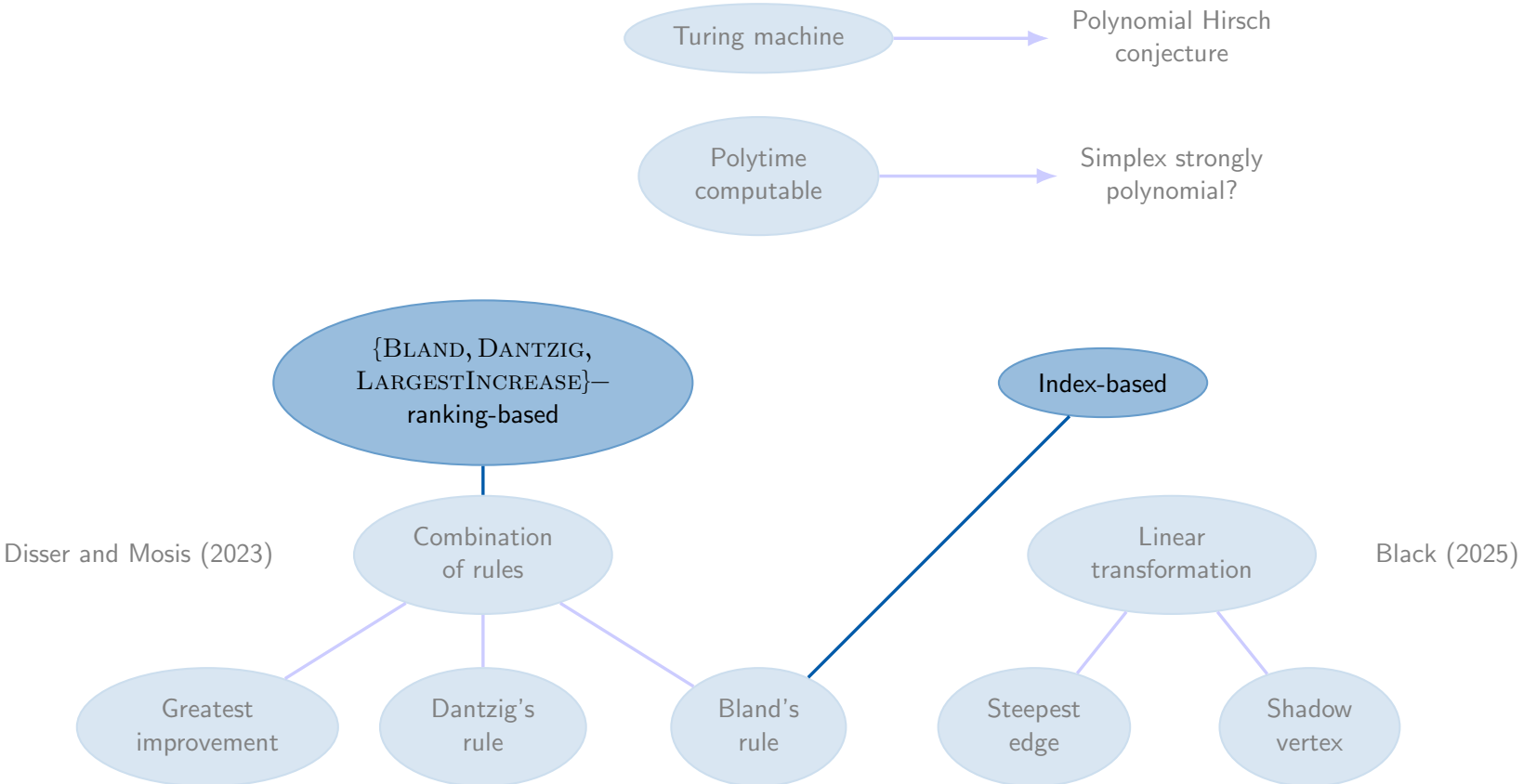
Theorem (Disser, Loho, M., Mosis)

Using an index-based pivot rule that uses $o(N/\log(N))$ memory states, the simplex algorithm needs a number of steps that is superpolynomial in N in the worst case, with N the input size.

Theorem (Disser, Loho, M., Mosis)

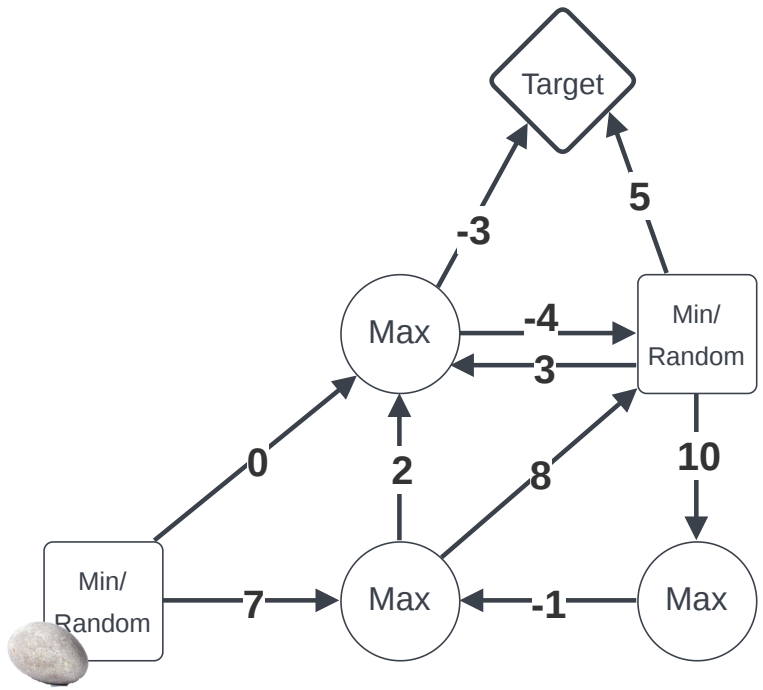
Using a memoryless $\{\textsc{Bland}, \textsc{Dantzig}, \textsc{LargestIncrease}\}$-ranking-based pivot rule, the simplex algorithm takes $\Omega(2^{\sqrt{N}})$ iterations in the worst case, with N the input size.

Pivot rule models



LSP and weak unichain MDP

- Directed graph with edge costs, Max/Min or Max/Rand nodes
- Move pebble to target vertex



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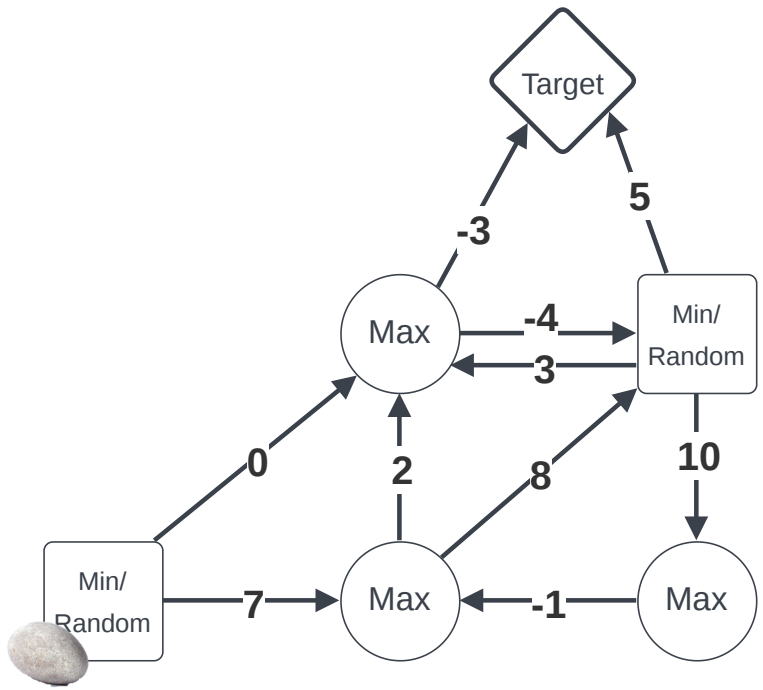

LSP and weak unichain MDP

- Directed graph with edge costs, Max/Min or Max/Rand nodes
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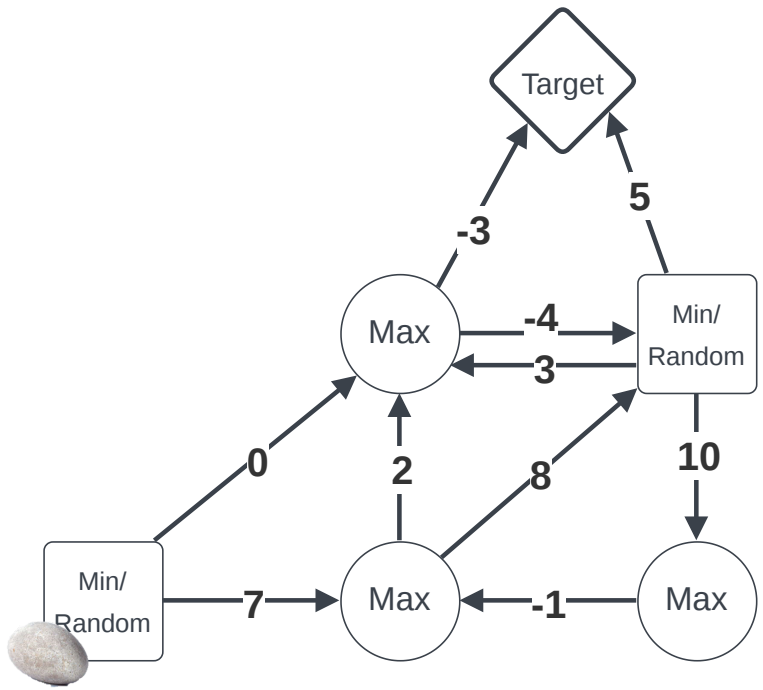
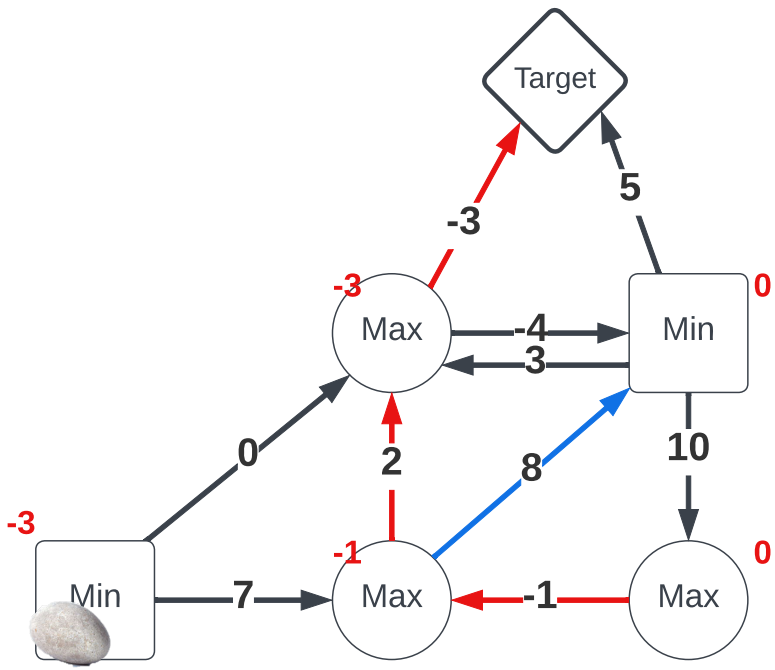
- Player goals: maximize/minimize weight of path to target
- Find optimal strategies and value



Strategy improvement

- Fix an admissible strategy for Maximizer



Strategy improvement

- Fix an admissible strategy for Maximizer
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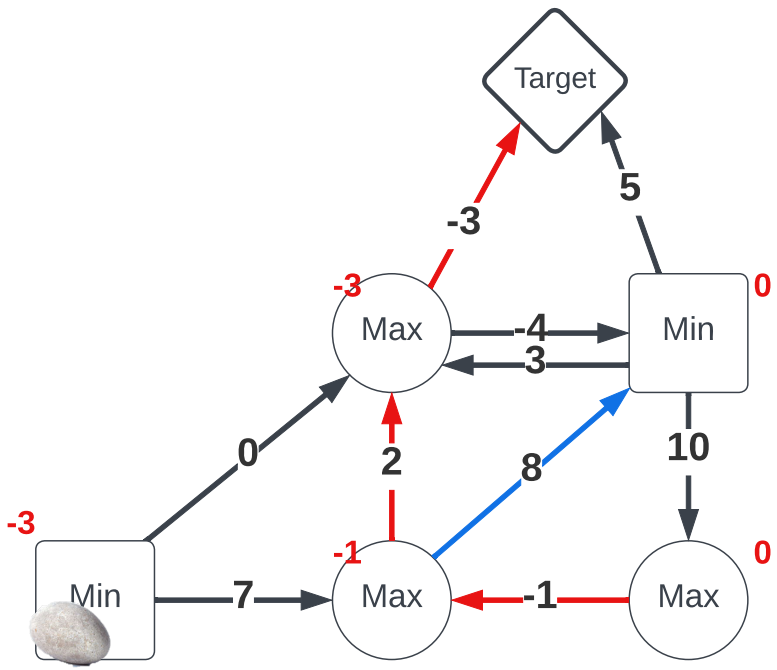

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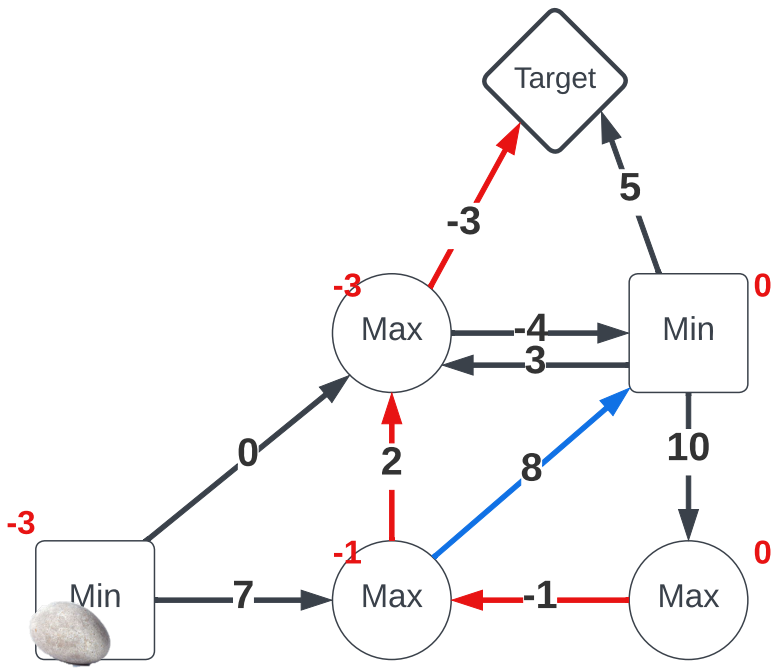

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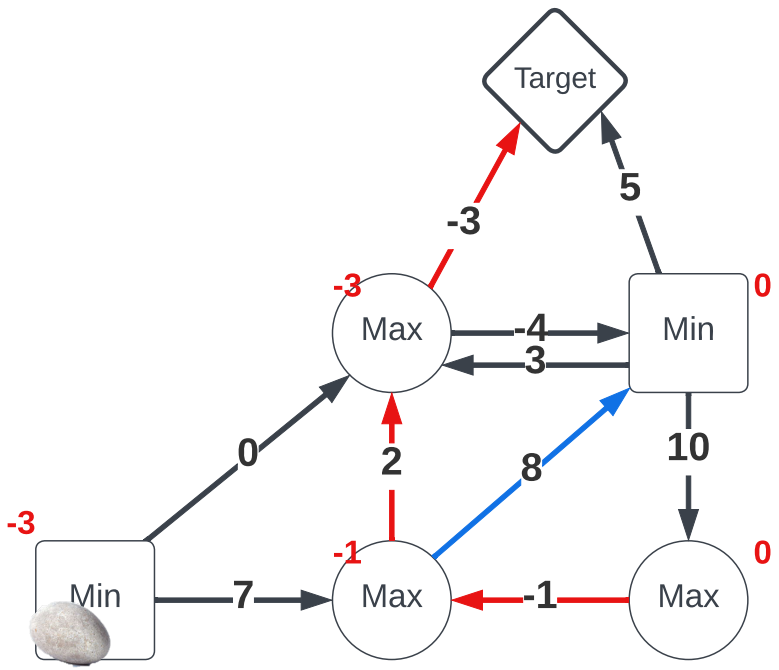
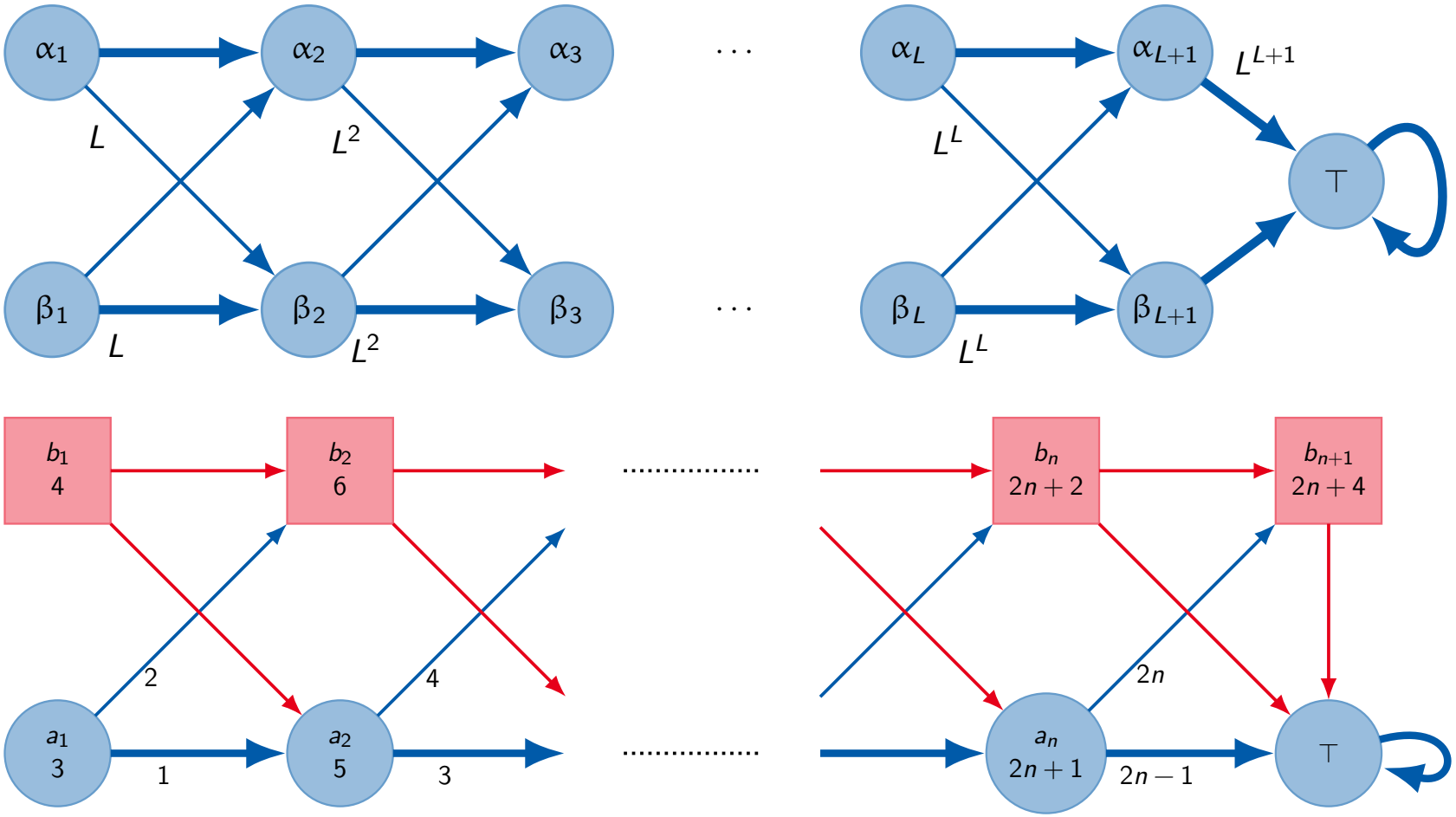
- Equivalent to simplex method in related LP
  (M. (2026+), Schewe (2009), Friedmann (2011))



Proof technique: binary counter games



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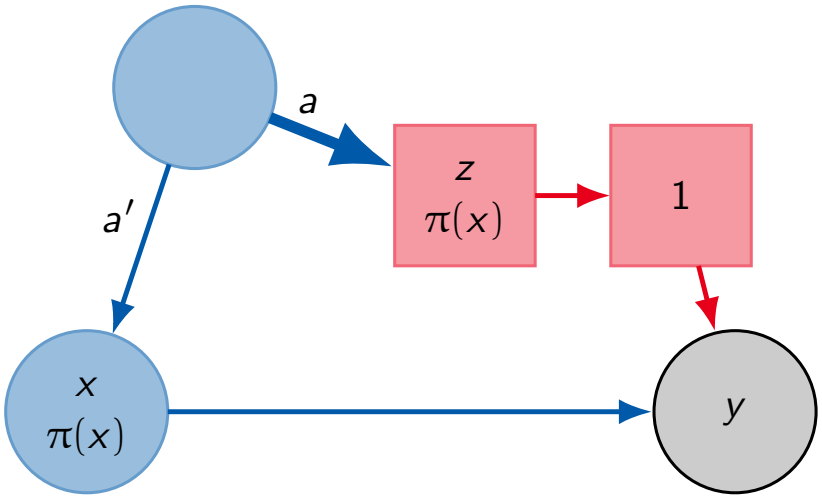
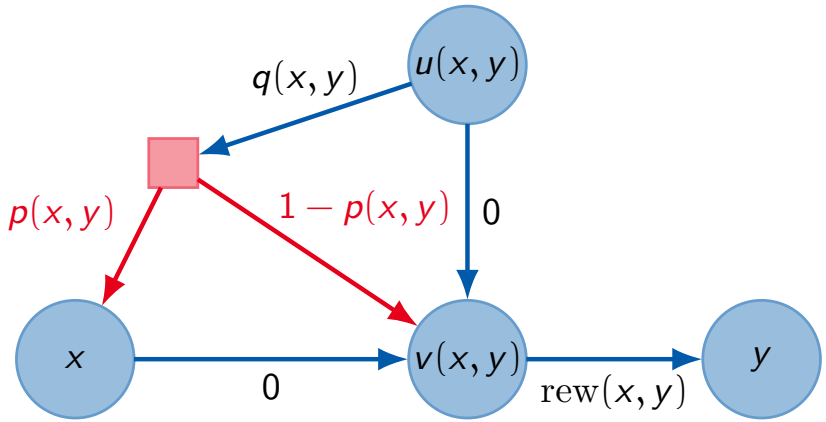
- Turn LSP/MDP into equivalent LP
- Constant number of improving moves





Proof technique: index-based rules

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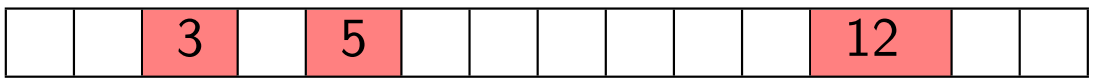
- Rule uses order of indices and memory
- Keep number of improving moves constant
  - Choice of j-th variable will repeat after at most $o(\frac{n}{\log(n)})$ steps.

h	1	2	3	4
j	3	5	12	3
h'	2	3	4	1



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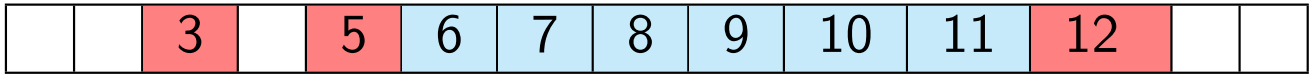

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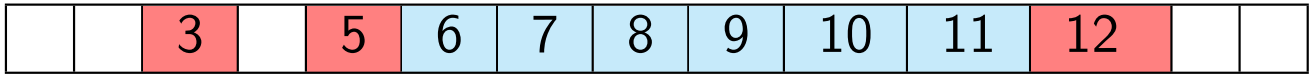

  - Make binary counter with its edges at $5, 6, \ldots, 11$ in Bland order: $\omega(\log(n))$ bits

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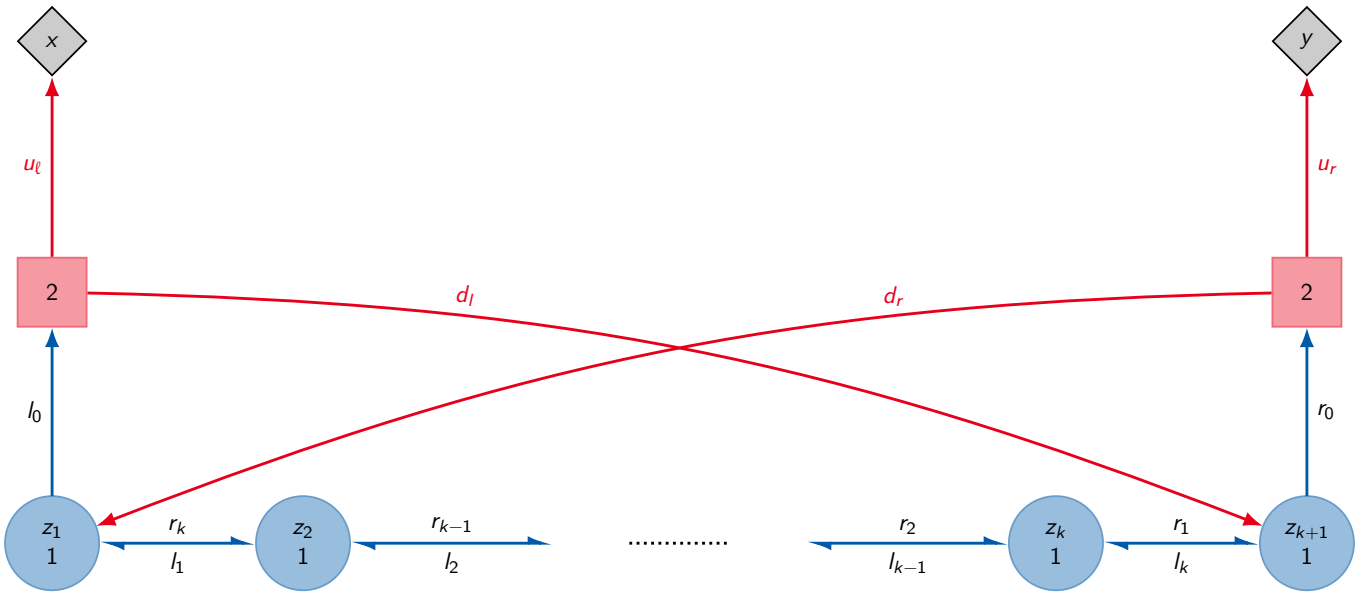
  - Decoy switches that get reactivated every cycle



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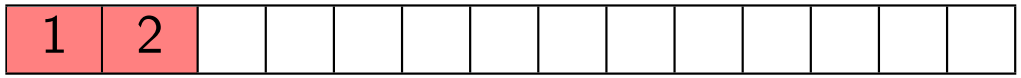

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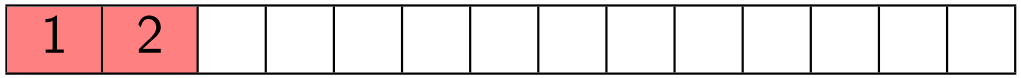

  - Make one big binary counter with $\omega(\log(n))$ bits, copy every edge $o(\frac{n}{\log(n)})$ times
- Binary counter takes $2^{\omega(\log(n))} = n^{\omega(1)}$ iterations

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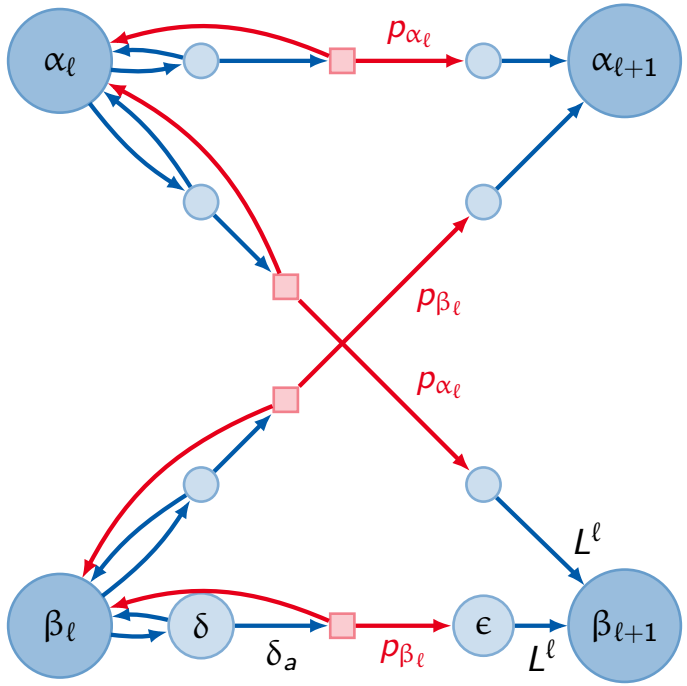
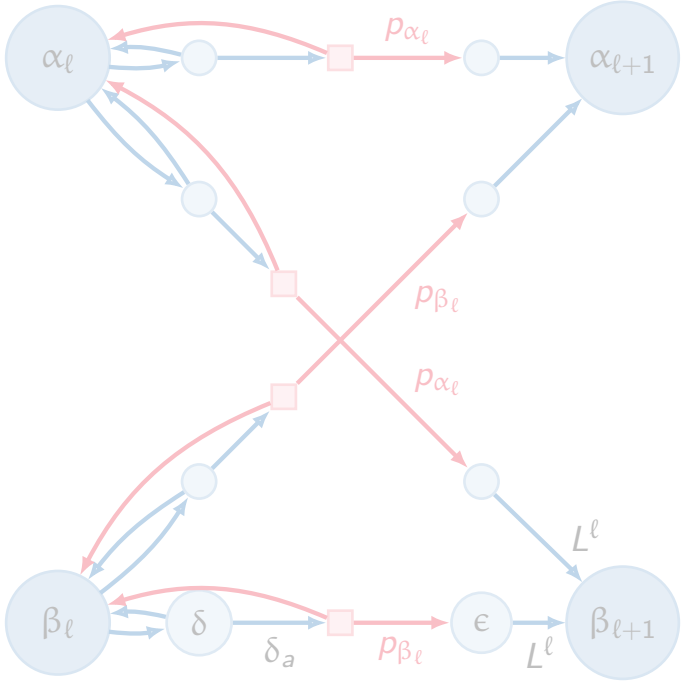


Proof technique: memoryless ranking-based rules

- Ranking by index, reduced cost and objective improvement
- Bland's rule, constant number of improving moves
- Make sure all rankings coincide:
- Pivot rule effectively only sees one ranking



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  - This behaves like Bland's rule
- In all cases, takes $\Omega(2^{\sqrt{N}})$ iterations

Contact/references I

Contact: m.t.maat@utwente.nl

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