List Coloring Ordered Graphs with Forbidden Induced Subgraphs

Marta Piecyk[1] & Paweł Rzążewski[2]

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Grenoble, STACS 2026

List-k-Coloring

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Remark 1: An instance of List-k-Coloring can be seen as an instance of List-$(k+1)$-Coloring.

Remark 2: If all lists have size at most 2, then the problem can be solved in polynomial time. (Edwards '86)

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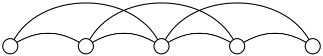

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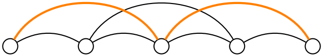

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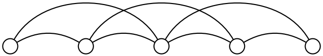
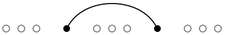


Contains induced  Does not contain induced 

List-k-Coloring H-free ordered graphs

Forbidden H	$k = 3$	$k = 4$	$k \geqslant 5$
	[1]		
			
			
			
			
			
			
other			

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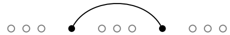	[1]		
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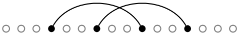			
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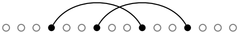			
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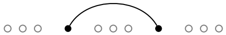	[1]		
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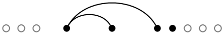
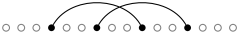
	[1]		
			
			
			
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★ Our results

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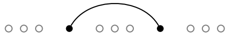	[1]		★
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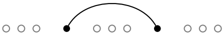	[1]		★
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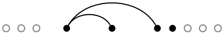
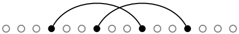
	[1]	★	
			
			
			
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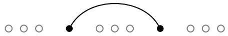	[1]		★
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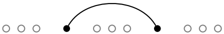	[1]		★
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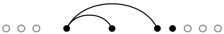	[1]	★	
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Theorem. LIST-4-COLORING is polynomial-time solvable on  -free ordered graphs.

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(R3) If there is $v \in V(G)$ with $L(v) = \{a\}$ for some $a \in [4]$, then remove a from list of every neighbor of v, and remove v from G.

List-4-Coloring  -free cnd.

Guess first vertex x_i in each color $i \in [4]$.

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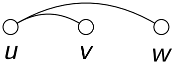

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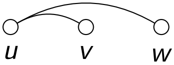

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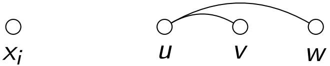

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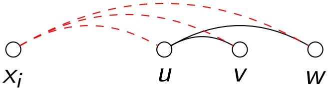

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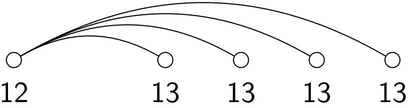

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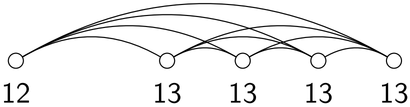

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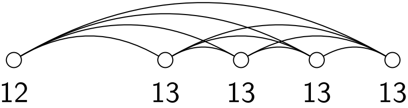

K_5!

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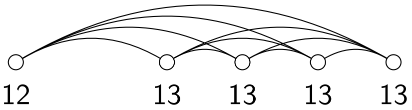

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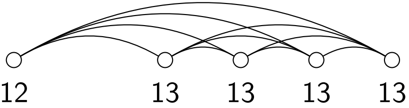

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Remark: All bad vertices have lists of size 2.

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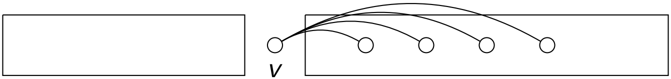
Let v be the first good vertex. We guess coloring on v and its forward neighbors.



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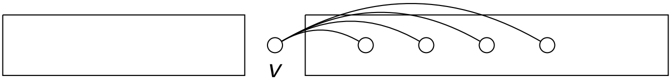

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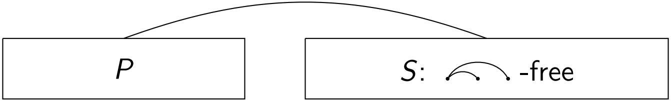

- Suffix S is a chordal graph.

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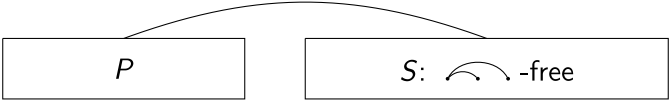

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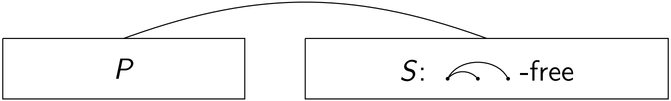

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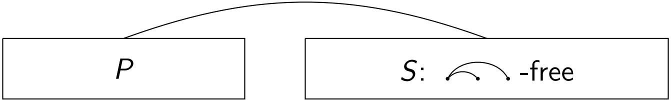

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- By additional guesses, we can assume that there are at most two distinct lists on P (and disjoint), say $\{1,2\}$ and $\{3,4\}$.
- It turns out that we can remove some edges with lists $\{1,2\}$-$\{3,4\}$, so that the remaining graph is chordal.

Open questions

- Determine the complexity of List-k-Coloring in the remaining cases.

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- In particular: List-3-Coloring on  -free graphs.

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- Subexponential-time algorithms?
- What if we consider non-list coloring?
- Other problems on ordered graphs with forbidden induced subgraphs.