A Linear Kernel for Independent Set Reconfiguration in Planar Graphs

Nicolas Bousquet

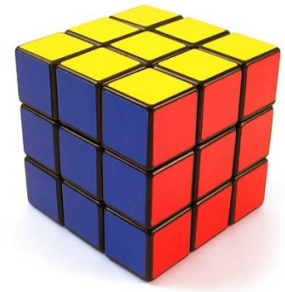
(Joint work with Daniel W. Cranston)

https://arxiv.org/abs/2506.03319

STACS'26

Reconfiguration

A one-player game is a puzzle : one player makes a series of moves, trying to accomplish some goal.



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Giving my current position, can I reach a fixed target position ?

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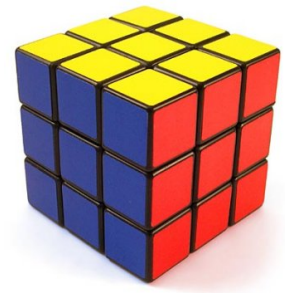

Question :
Giving my current position, can I reach a fixed target position ?

- Important problems in random sampling, bioinformatics, discrete geometry, games...etc... for decades.

Main questions

- **Reachability problem.** Given two configurations, is it possible to transform the one into the other ?
- **Connectivity problem.** Given any pair of configurations, is it possible to transform the one into the other ?
- **Minimization.** Given two configurations, what is the length of a shortest sequence ?

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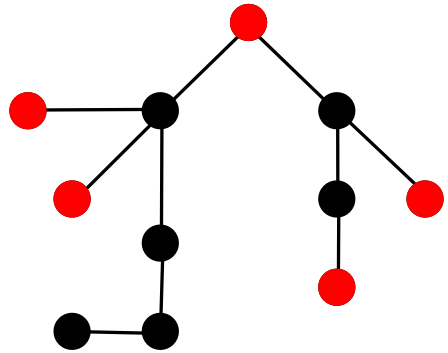
- **Algorithmics.** Can we efficiently solve these questions ? (In polynomial time, FPT-time...).

Reconfiguration of independent sets



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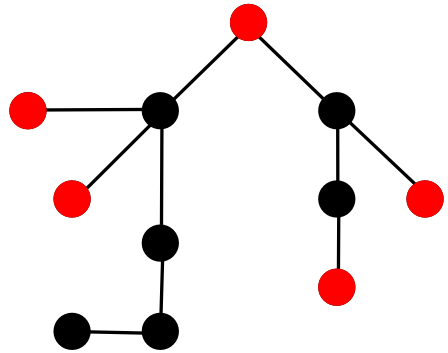

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(keeping an IS)

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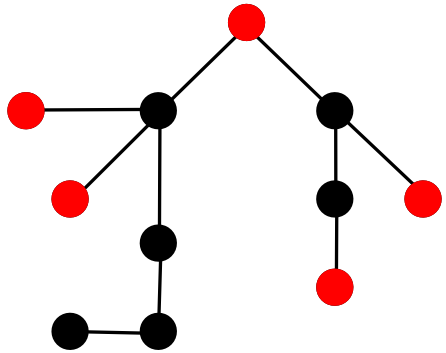

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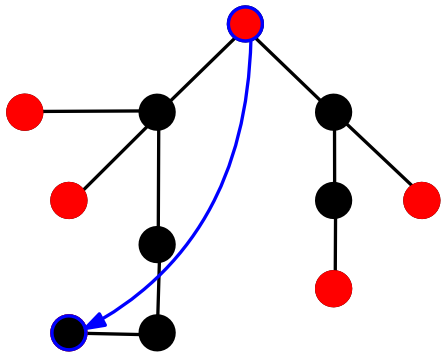

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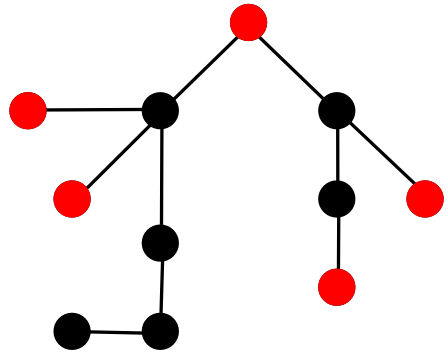

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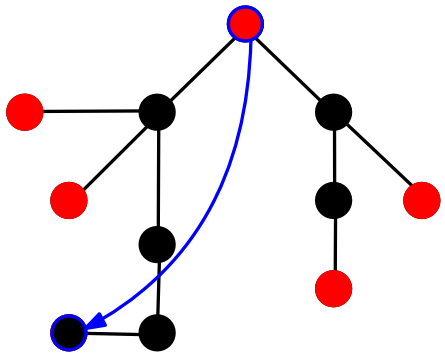

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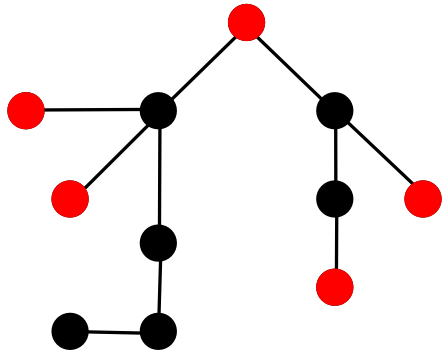

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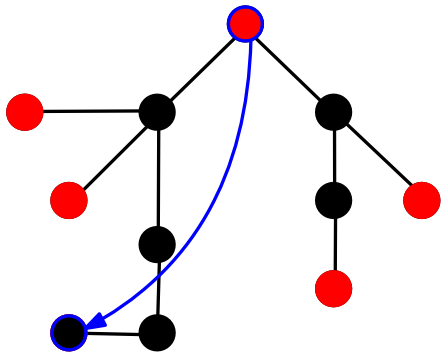

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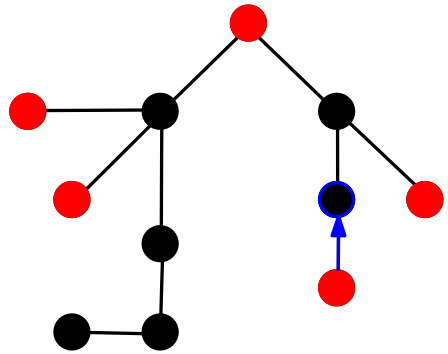

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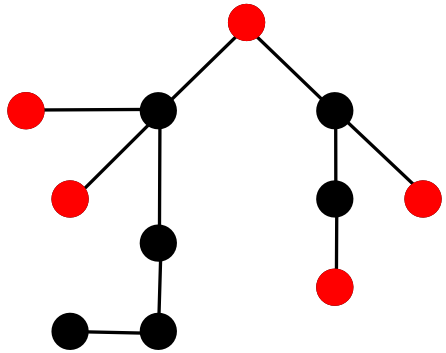

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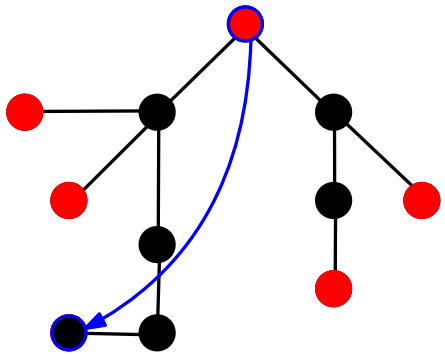

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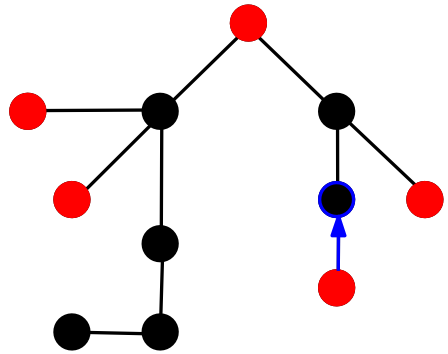

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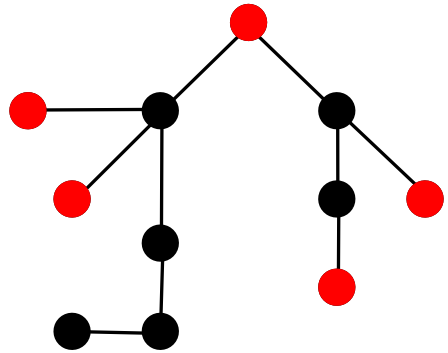

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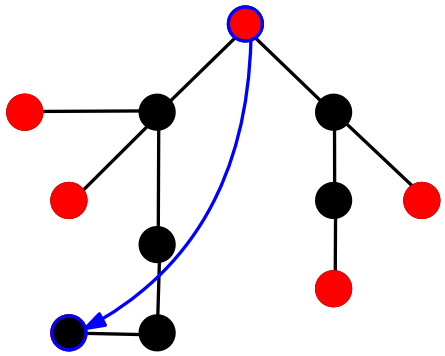

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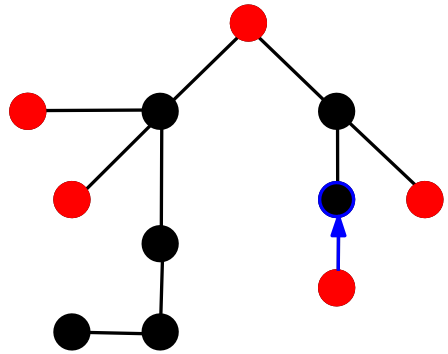

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Today : Focus on TJ.

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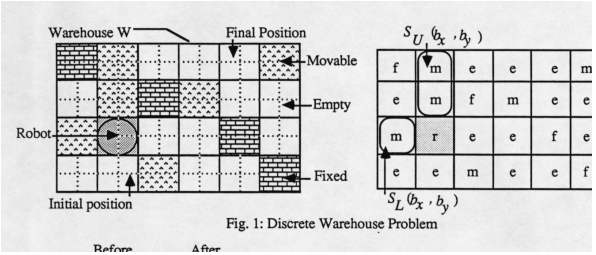

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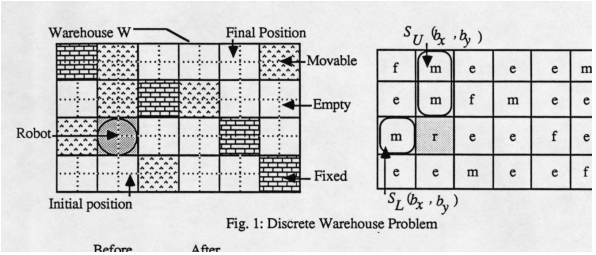

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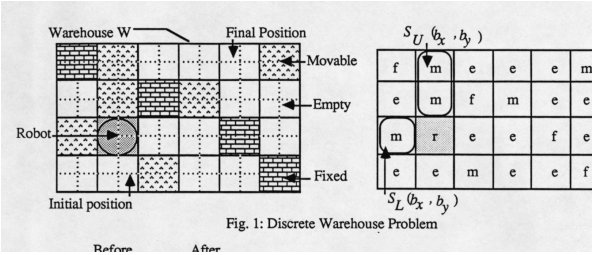

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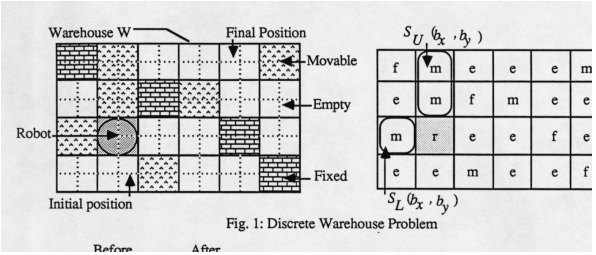

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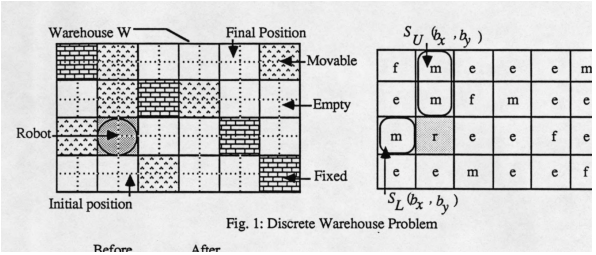

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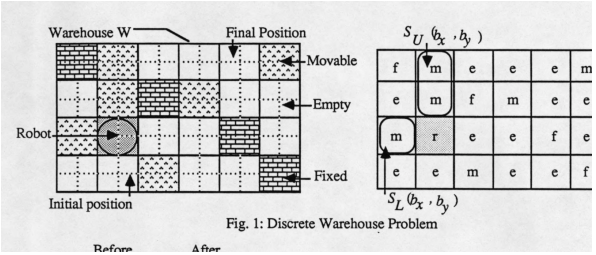

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[Wrochna '17] ... and on bounded bandwidth graphs !

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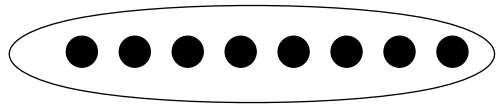
Question : Linear kernel ?

→ **This paper :** Positive answer !

Warm up : A quadratic kernel



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  $\Rightarrow$ Partition of $V \setminus (I_s \cup I_t)$ into $O(k)$ classes

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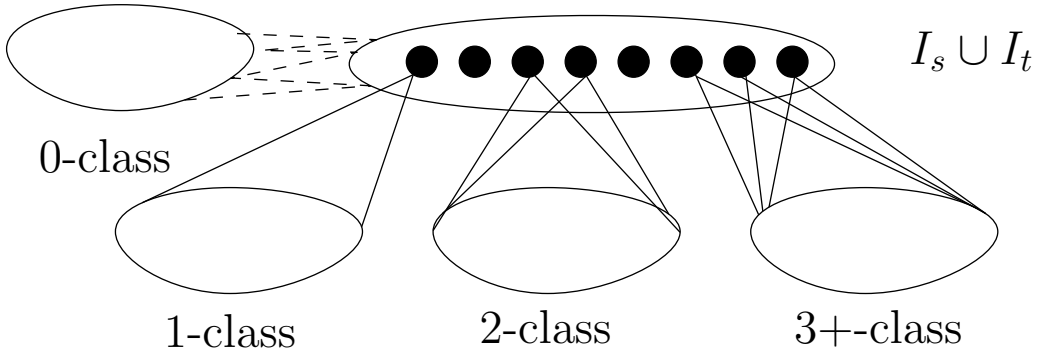

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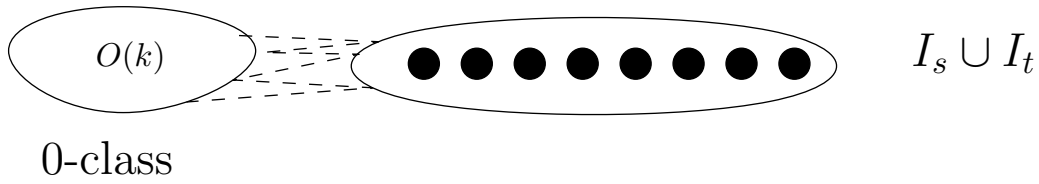

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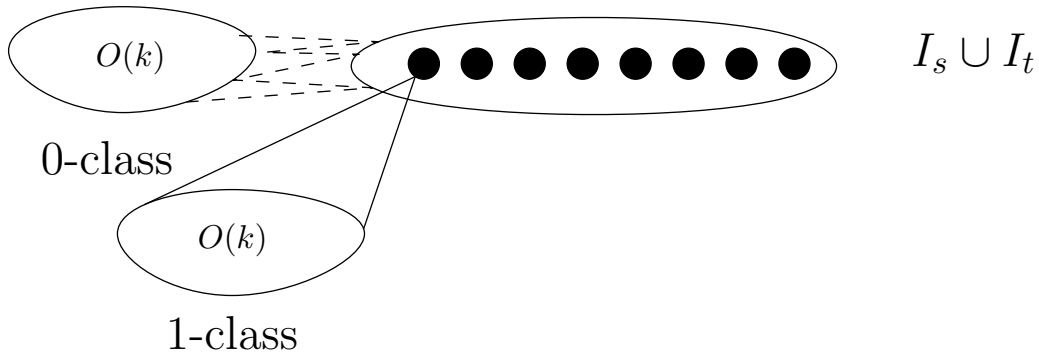

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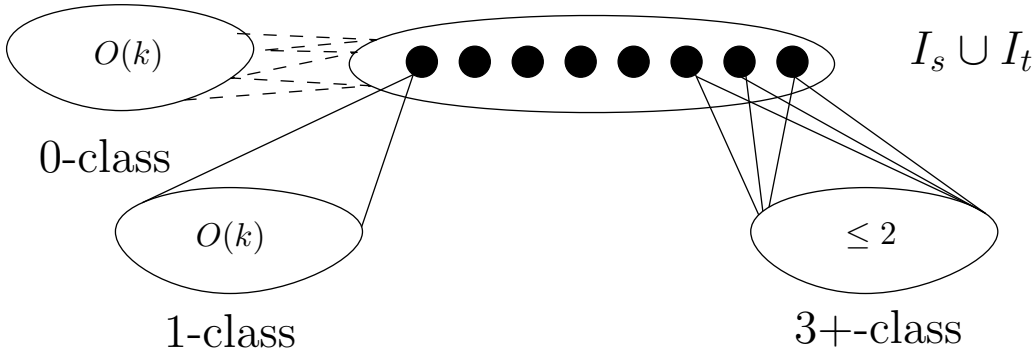

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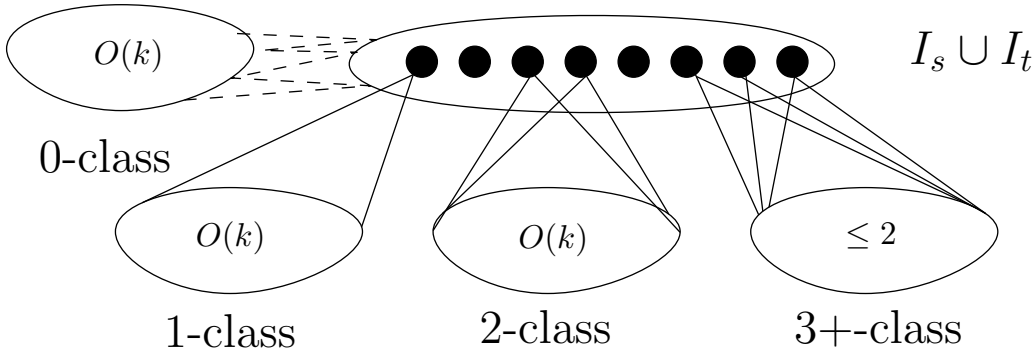

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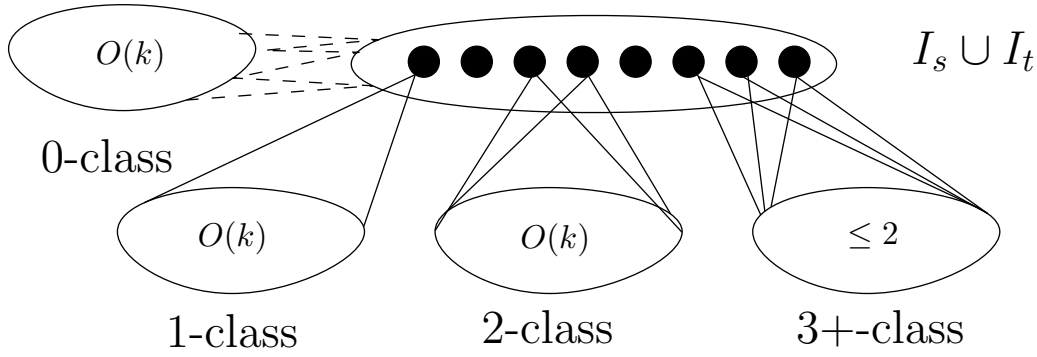

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Conclusion : $O(k)$ classes of size $O(k) \Rightarrow O(k^2)$ kernel.

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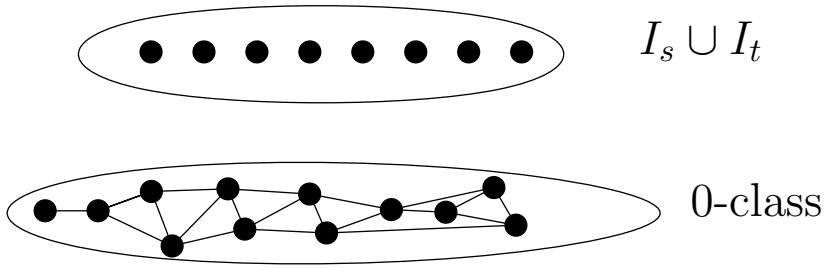

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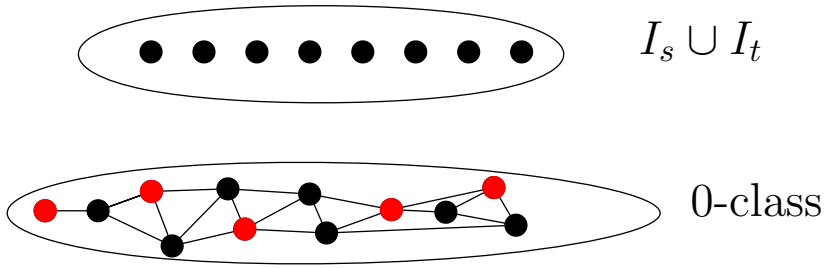

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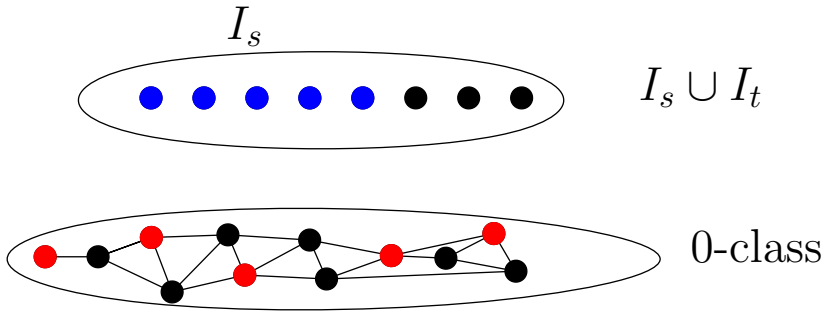

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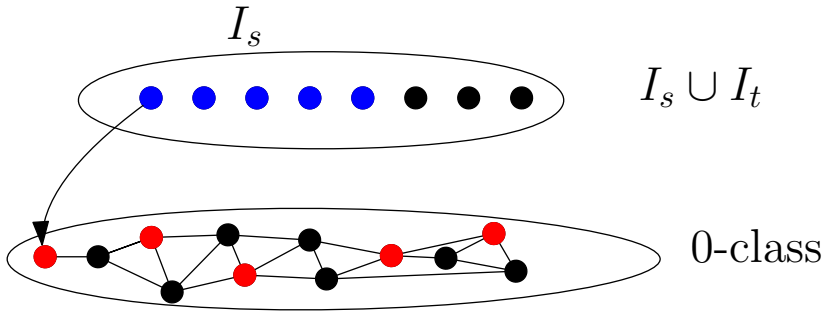

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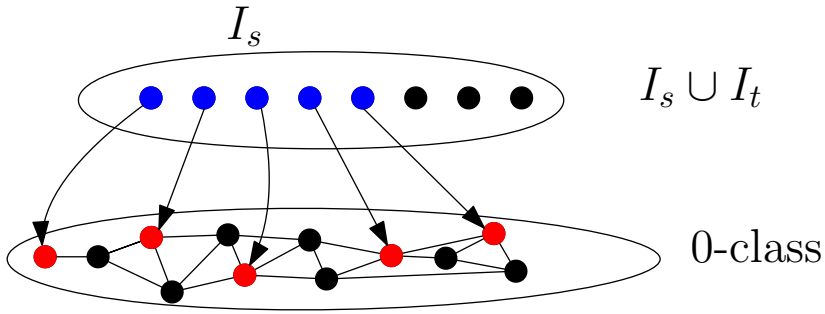

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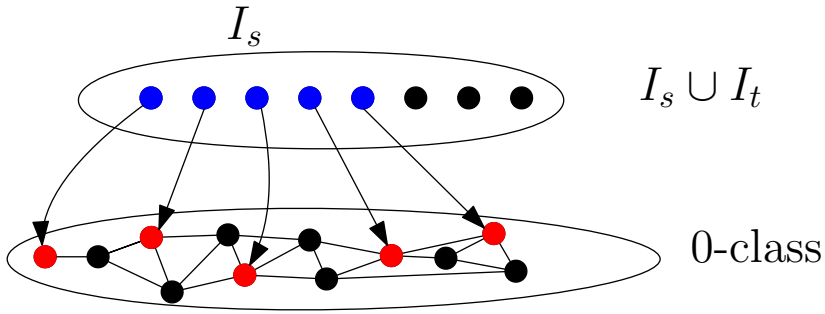

Claim :

Size of the 0-class $\geq 4k \Rightarrow$ Project I_s and I_t on the 0-class
$\Rightarrow$ yes-instance

Sketch of the proof

- Select an IS of size k arbitrarily in the 0-class
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- $I_s \rightsquigarrow \bullet \rightsquigarrow I_t$

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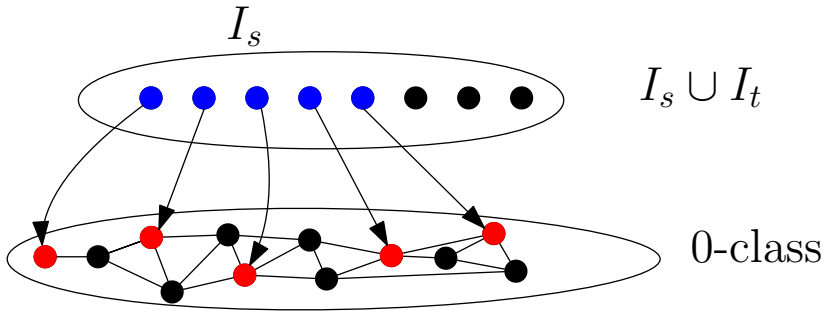

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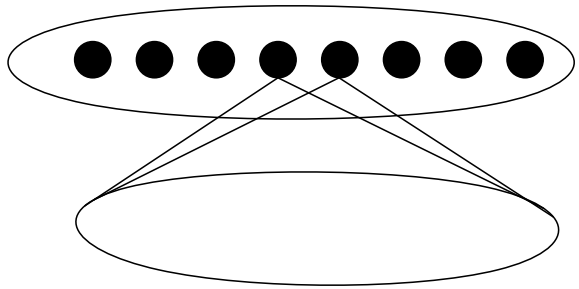
With a similar argument : $|\cup$ 1-classes$| = O(k)$

Reducing 2-classes (I)



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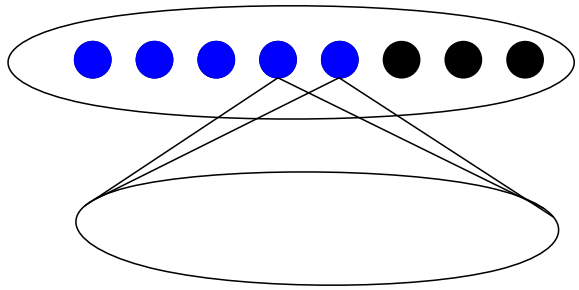

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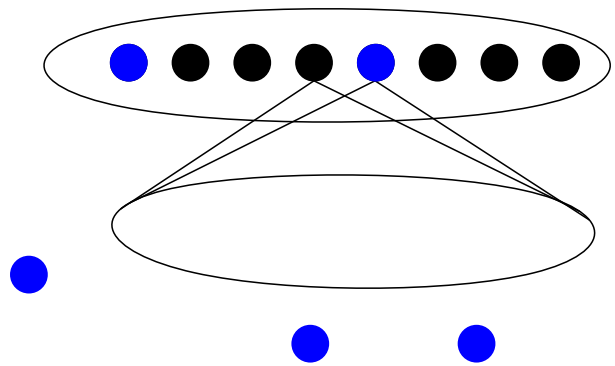

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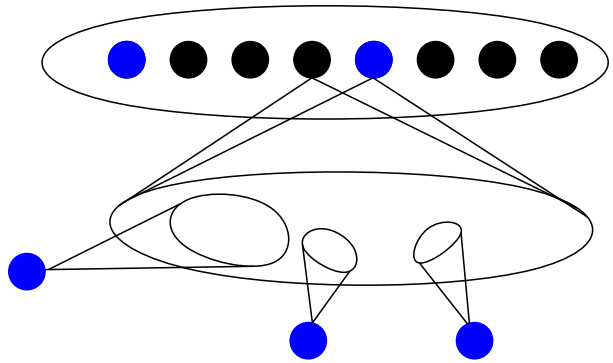

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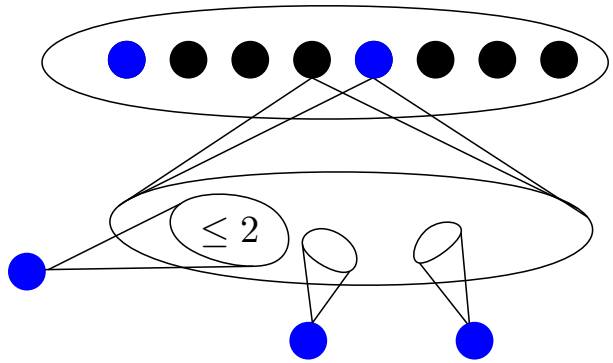

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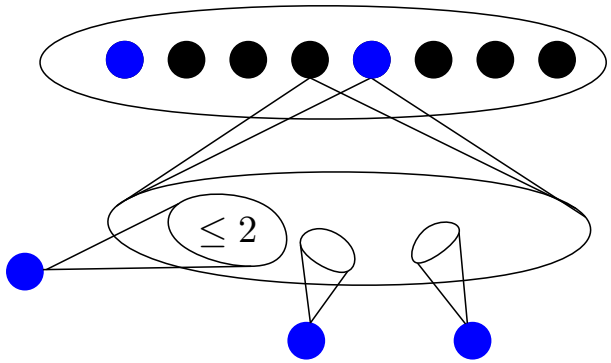

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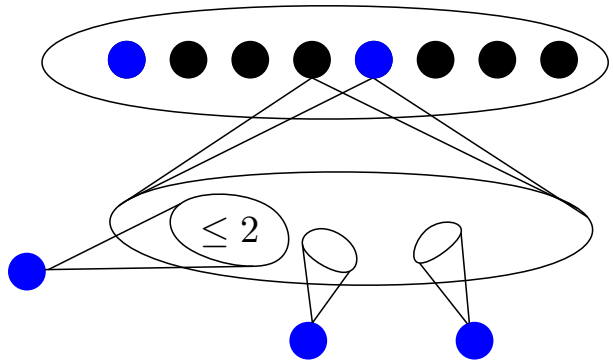

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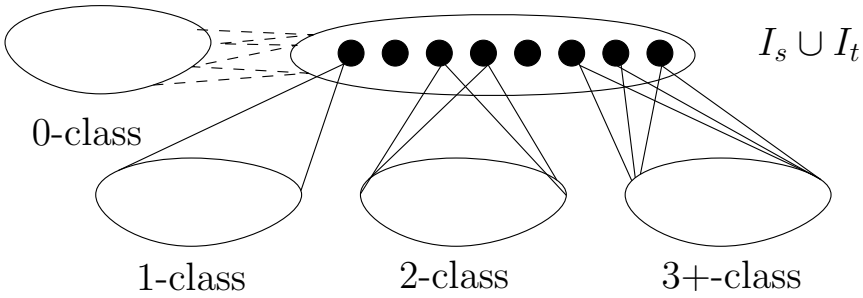
- Apply the same argument on I_t.

Roadmap towards a linear kernel



- The 0-class has size $O(k)$ as well as the union of the 1-classes
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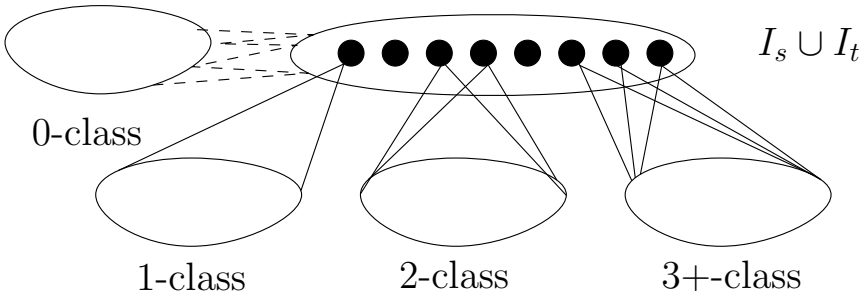

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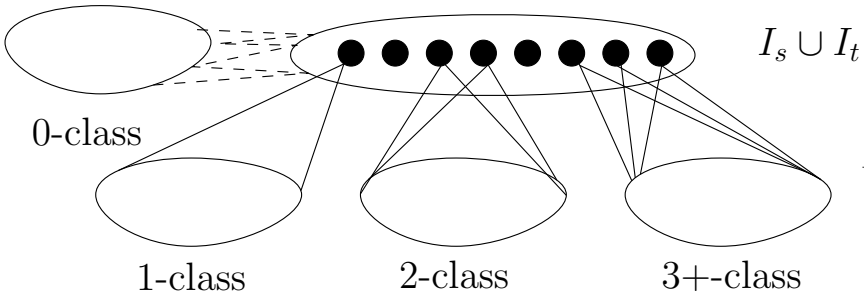

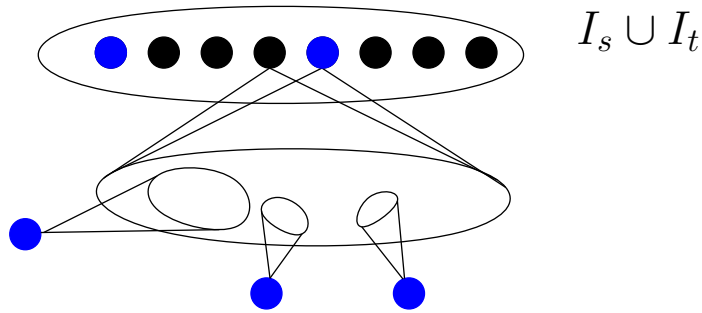

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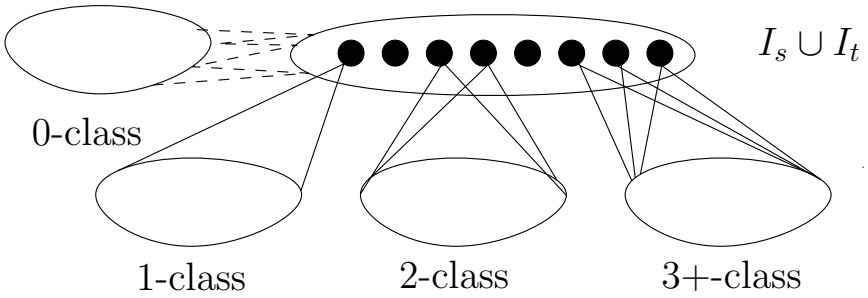

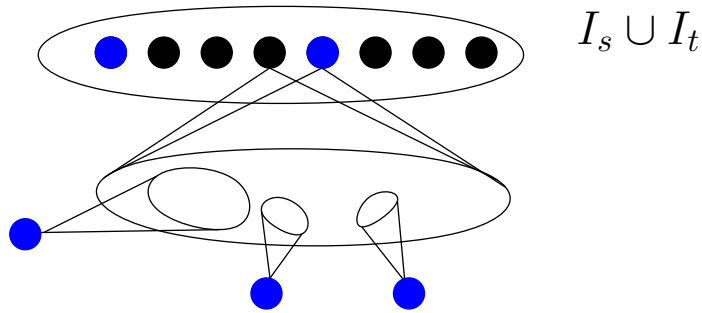

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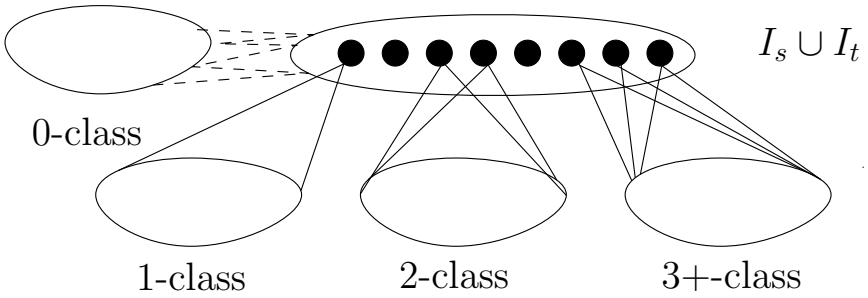

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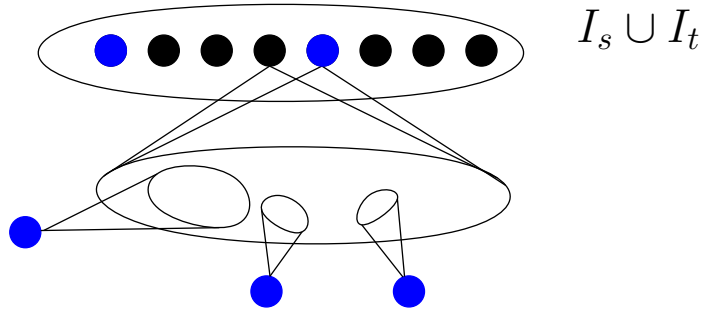

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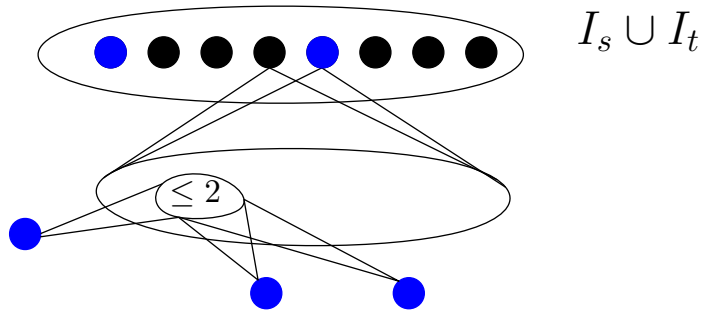
Informal claim : The size of $N(\bullet)$ in C is ≤ 2.



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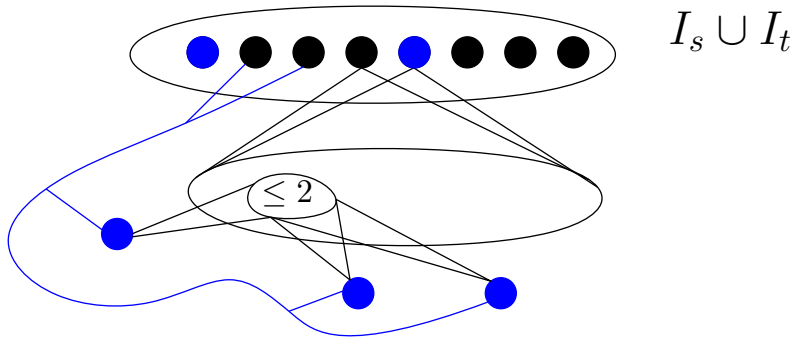

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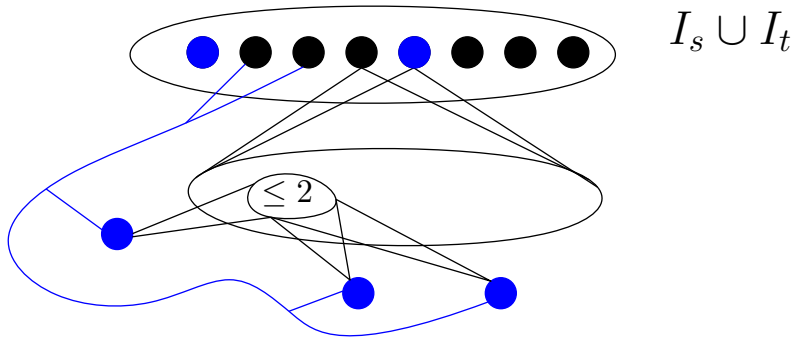

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Thanks for your attention !