

An Improved Version of Hmelevskii's Theorem on Three-Variable Word Equations

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Introduction

We are interested in word equations, and the structure of their solution sets (definitions on the following slides).

Some old results on word equations:

- Constant-free, one or two variables: only periodic solutions.
- Constant-free, three variables: a parametric solution.
- Constant-free, four variables: not always a parametric solution.
- General case: EDTOL description.
- Various other results as well.

Our goal is to improve the constant-free three-variable case.

Outline

- 1 Definitions
- 2 Old and new results
- 3 Some ideas used in the proof
- 4 Conclusion

Words and morphisms

- The set of all words over an alphabet Σ is denoted by Σ^* .
- Σ^* is a free monoid and can be extended to a free group $(\Sigma^{\pm 1})^*$ by adding inverses.
- G-morphism: a mapping $h : (\Sigma_1^{\pm 1})^* \rightarrow (\Sigma_2^{\pm 1})^*$ such that $h(uv) = h(u)h(v)$ for all $u, v \in (\Sigma_1^{\pm 1})^*$.
- M-morphism: a morphism h that maps words to words, that is, $h(\Sigma_1) \subseteq \Sigma_2^*$.
- The set of M-morphisms is denoted by $\text{MM}(\Sigma_1, \Sigma_2)$.

Word equations

Let Ξ be an alphabet of variables and Σ an alphabet of constants.

- A *constant-free word equation* is a pair $E = (L, R) \in \Xi^* \times \Xi^*$.
- A *solution* of E is an M-morphism $h \in \text{MM}(\Xi, \Sigma)$ such that $h(L) = h(R)$.
- The solution set of E is denoted by $\text{Sol}(E)$.
- The equation E is *nontrivial* if $L \neq R$.
- A solution h is *periodic* if the images of all variables are powers of a common word.

Three-variable equations

- We are particularly interested in the three-variable case $\Xi = \{x, y, z\}$.
- We use the shorthand notation $[u, v, w]$ for the morphism $h : \{x, y, z\}^* \rightarrow \Sigma^*$ defined by $h(x) = u$, $h(y) = v$, $h(z) = w$.

Example

Consider the equation $E = (xy, yz)$. The morphism $h = [uv, (uv)^j u, vu]$, where $u, v \in \Sigma^*$ and $j \in \mathbb{N}$, is a solution of E :

$$h(xy) = uv(uv)^j u = (uv)^j uvu = h(yz).$$

All solutions of E are of this form or of the form $[\varepsilon, u, \varepsilon]$, where ε is the empty word.

Parametric solutions: first example

We introduce parametric solutions through two examples (formal definition in the article).

Example

The solution set of the equation (xy, yz) is

$$\begin{aligned} & \{[uv, (uv)^j u, vu] \mid u, v \in \Sigma^*, j \in \mathbb{N}\} \\ & \cup \{[\varepsilon, u, \varepsilon] \mid u \in \Sigma^*, j \in \mathbb{N}\}. \end{aligned}$$

The equation (xy, yz) has a parametric solution with

- two word parameters (u and v),
- one numerical parameter (j),
- nesting level one.

Parametric solutions: second example

Example

The solution set of the equation $(xyyz, zyyx)$ is

$$\begin{aligned} & \{[(uvv)^i u, v, (uvv)^j u] \mid u, v \in \Sigma^*, i, j \in \mathbb{N}\} \\ & \cup \{[vu((uv)^{k+2}u)^i uv, (uv)^{k+1}u, vu((uv)^{k+2}u)^j uv] \mid u, v \in \Sigma^*, i, j, k \in \mathbb{N}\} \\ & \cup \{[u^i, u^j, u^k] \mid u, v \in \Sigma^*, i, j, k \in \mathbb{N}\}. \end{aligned}$$

The equation $(xyyz, zyyx)$ has a parametric solution with

- two word parameters (u and v),
- three numerical parameters (i , j and k),
- nesting level two.

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Hmelevskii's theorem

Theorem (Hmelevskii 1971)

Every constant-free three-variable word equation E has a parametric solution.

The number of word parameters is two, unless the equation is trivial (this is related to the well-known defect theorem).

An analysis of the proof reveals that

- the number of numerical parameters is at most logarithmic w.r.t. the length of E ,
- the nesting level is at most logarithmic w.r.t. the length of E .

Entire systems of equations

For a morphism $h \in \text{MM}(\{x, y, z\}, \Sigma)$, let $\text{Eq}(h)$ be the set of equations it is a solution of. $\text{Eq}(h)$ is called an *entire system*.

Theorem (Saarela 2024)

Every nontrivial entire system of three-variable equations has a parametric solution with

- *two numerical parameters,*
 - *nesting level one.*
- The parametric solutions can be given explicitly, but there are several different cases.
 - The proof relies on some old results, like the classification of three-generator subsemigroups of a free semigroup by Budkina and Markov.

Our new main result

Theorem

Every constant-free three-variable word equation has a parametric solution with

- *three numerical parameters,*
- *nesting level two.*

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Idea from an earlier article

The following is essentially a reformulation of a result in [Saarela 2024].

Lemma

Let E be a nontrivial constant-free equation on $\{x, y, z\}$. Let P be the set of periodic solutions of E . Then

$$\text{Sol}(E) = \bigcup_{k=1}^9 \bigcup_{h \in H_k} \text{Sol}(\text{Eq}(h)) \cup P,$$

where every H_k is either a finite set or, up to a permutation of the variables, of one of the following forms:

$$\begin{aligned} \{[a, a^p b a^q, (uv)^i u] \mid i \in \mathbb{N}\} & \quad \text{for some } p, q \in \mathbb{N}, u, v \in \Sigma^*, \\ \{[a, a^p b (a^m b)^i a^q, a^r b (a^m b)^j a^s] \mid i, j \in \mathbb{N}\} & \quad \text{for some } m, p, q, r, s \in \mathbb{N}. \end{aligned}$$

Morphism ϕ

Let ϕ be the G-morphism defined by

$$\phi(a) = aba, \quad \phi(b) = a^{-1}.$$

With the help of ϕ , we can combine several results from [Saarela 2024] into one.

Theorem

For every entire system S , there is h such that $S = \text{Eq}(h)$ and

$$\text{Sol}(\text{Eq}(h)) = \text{MM}(\{a, b\}, \Sigma) \circ \{\phi^j \circ h \mid 0 \leq j < \text{deg}_b(h)\} \cup P,$$

where deg_b has a relatively simple definition (see article) and P is the set of periodic solutions of $\text{Eq}(h)$.

Final steps

The proof of the main theorem can be completed by analyzing the morphisms

- $\phi^j \circ [a, a^p b a^q, (uv)^i u],$
- $\phi^k \circ [a, a^p b (a^m b)^i a^q, a^r b (a^m b)^j a^s].$

This is the longest part of the proof.

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Conclusion

We have proved that every constant-free word equation on three variables has a parametric solution with at most three numerical parameters and with nesting level at most two. This is a more exact version of the old result of Hmelevskii.

Several interesting questions remain, for example:

- Is the result optimal?
- What is the complexity of finding a parametric solution?