Homomorphism Indistinguishability, Multiplicity Automata Equivalence, and Polynomial Identity Testing

Marek Černý [1] Tim Seppelt [2]

[1]Universiteit Antwerpen, Belgium
[2]IT-Universitetet i København, Denmark

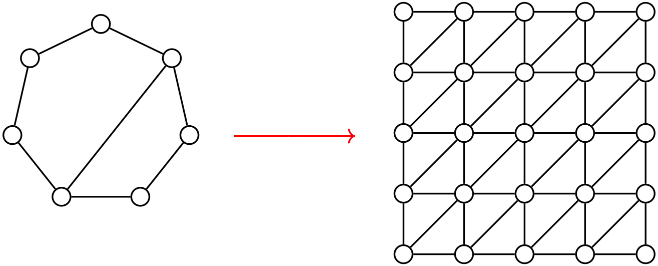
STACS, March 2025

Graph Homomorphisms

Definition

A *(graph) homomorphism* $F \longrightarrow G$ is a function $\varphi : V(F) \to V(G)$ such that

$$uv \in E(F) \implies \varphi(u)\varphi(v) \in E(G).$$



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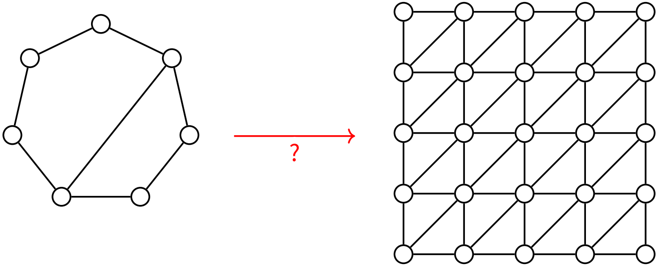
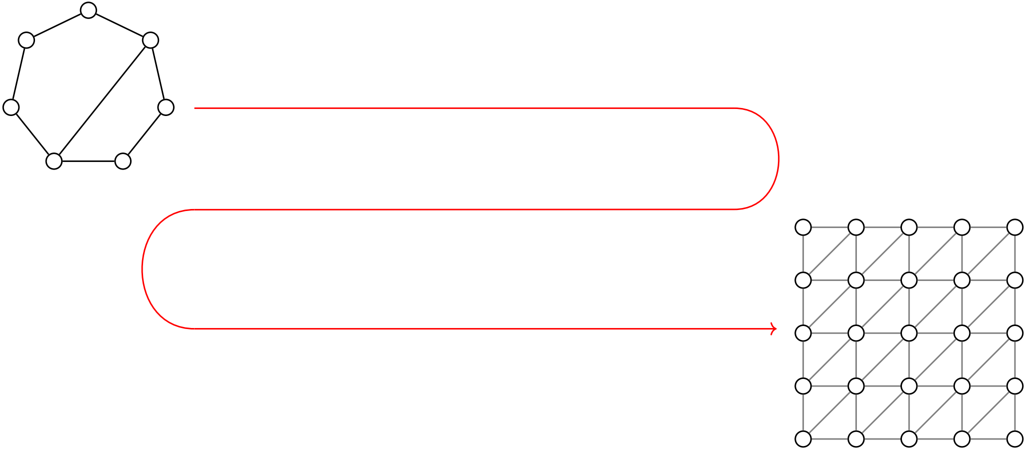


Example of a homomorphism $F \longrightarrow G$.



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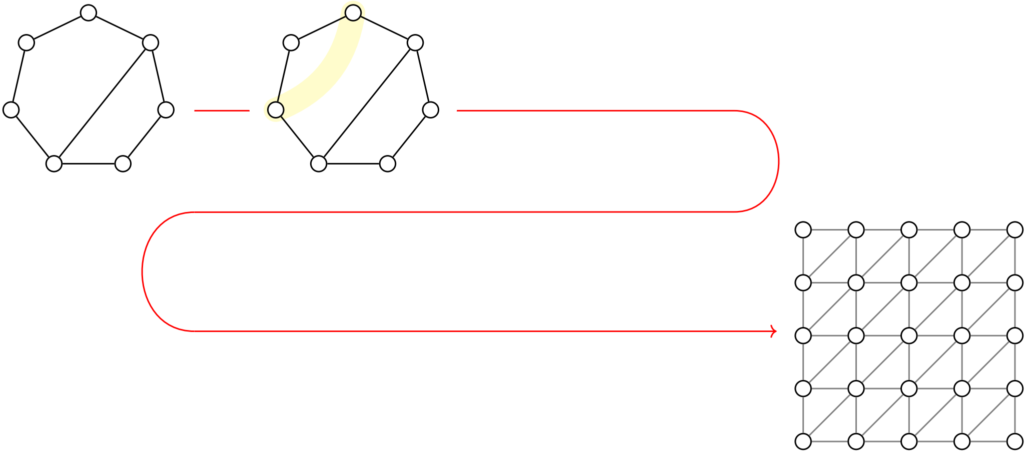

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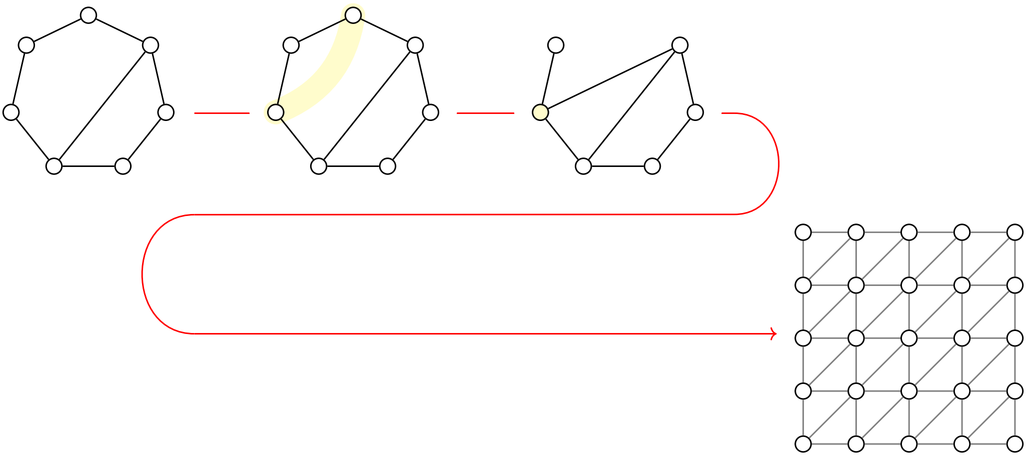

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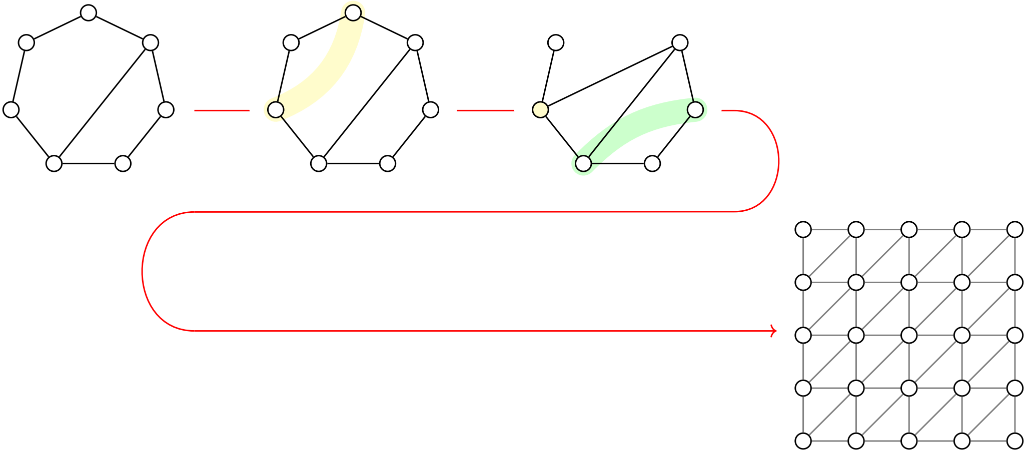

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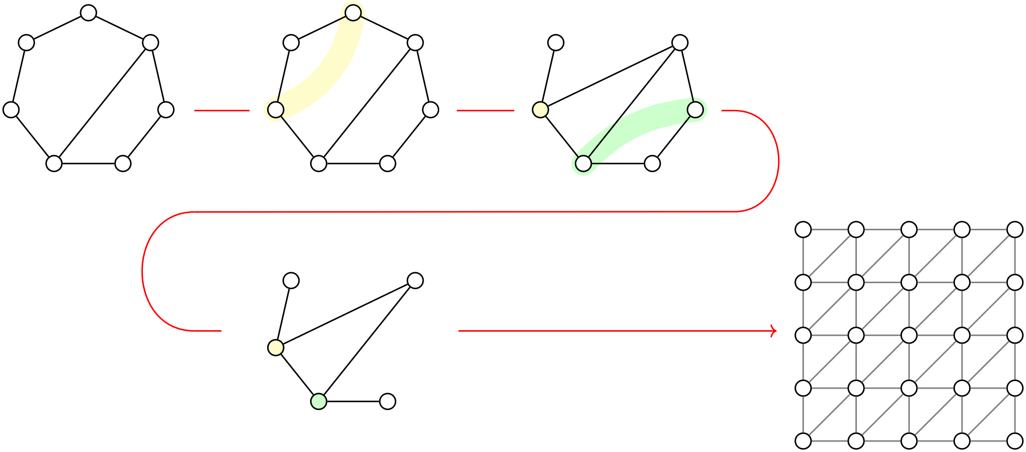

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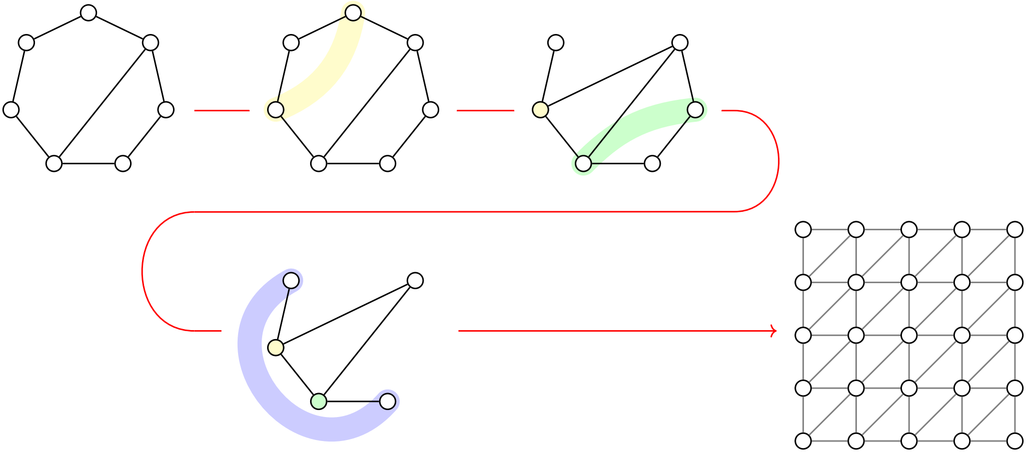

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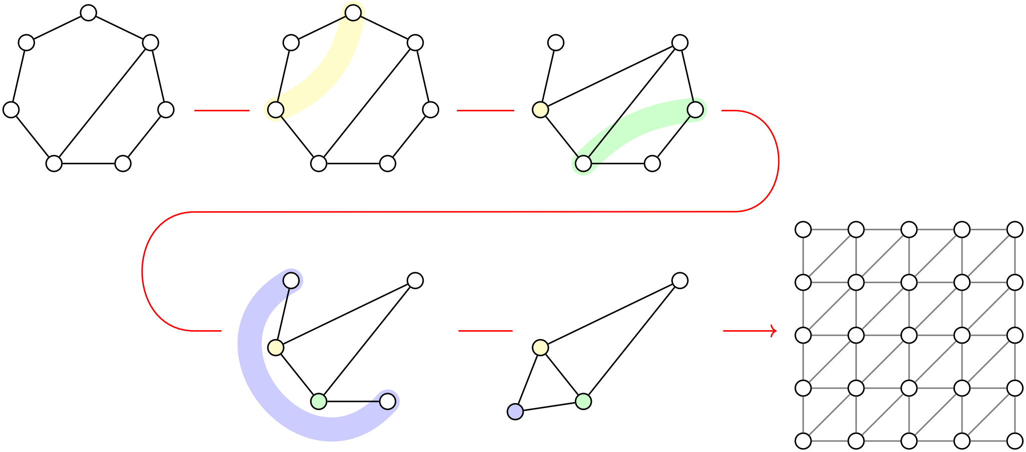

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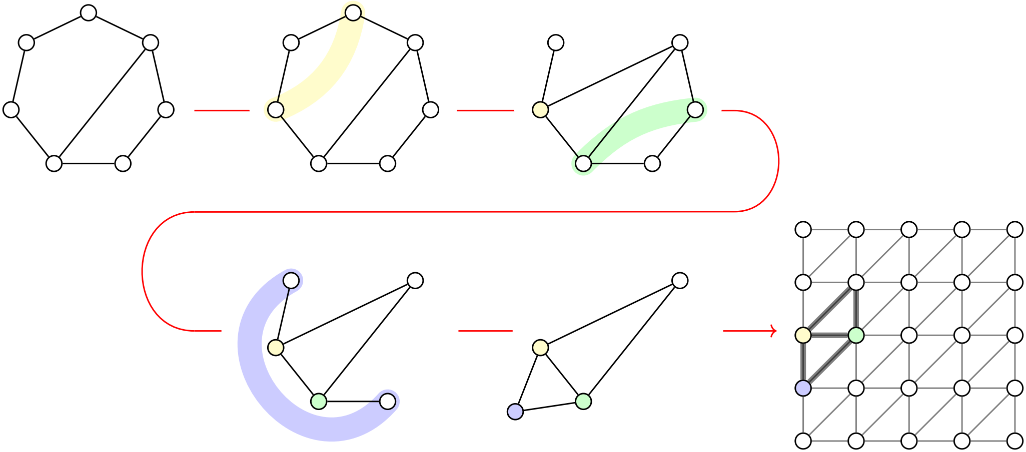
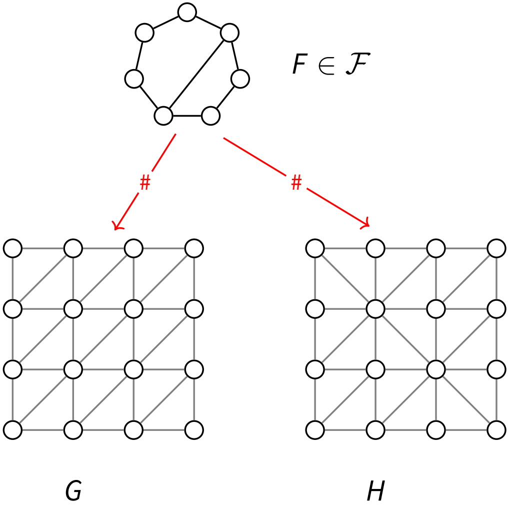


Homomorphism Indistinguishability over $\mathcal{F}$

$\hom(F, G) \dots$ the number of homomorphisms $F \longrightarrow G$.



Definition ($G \equiv_{\mathcal{F}} H$)

$$\hom(F, G) = \hom(F, H) \quad \forall F \in \mathcal{F}.$$

Problem (HomInd($\mathcal{F}$))

Input: *Graphs G and H.*
Decide: $G \equiv_{\mathcal{F}} H$.

Connections of HomInd($\mathcal{F}$) to Other Problems

Graph class $\mathcal{F}$ Problem $G \equiv_{\mathcal{F}} H$

See the HomInd Zoo: tseppelt.github.io/homind-database .

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Circles	Cospectrality of adjacency matrices	folklore

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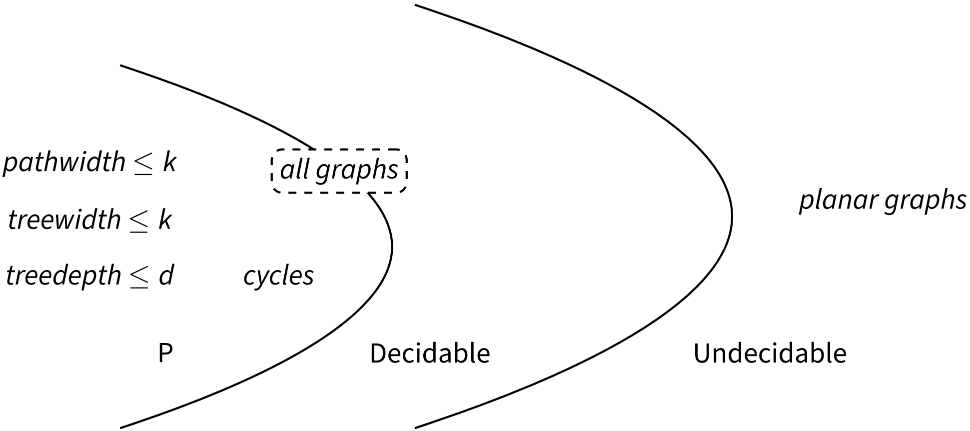
Complexity of HomInd($\mathcal{F}$)



Results: Complexity under logspace many-one reductions ($\Longrightarrow$)

HomInd($\mathcal{P}$)
for recognisable $\mathcal{P}$
of bounded **path**width

HomInd($\mathcal{T}$)
for recognisable $\mathcal{T}$
of bounded **tree**width

L NL $\mathsf{C_{=}L}$ $\mathsf{L^{C_{=}L}}$ $\mathsf{L^{\#L}}$ DET $\mathsf{NC^2}$ NC P PIT coRP

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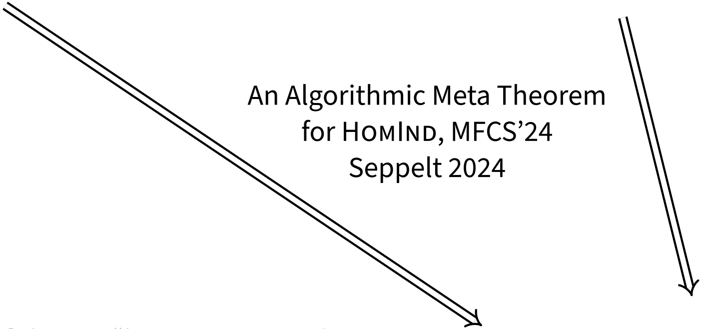

An Algorithmic Meta Theorem
for HomInd, MFCS'24
Seppelt 2024

L NL $C_=L$ $L^{C_=L}$ $L^{\#L}$ DET NC^2 NC P PIT coRP

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Multiplicity **word**
automata equivalence

Multiplicity **tree**
automata equivalence

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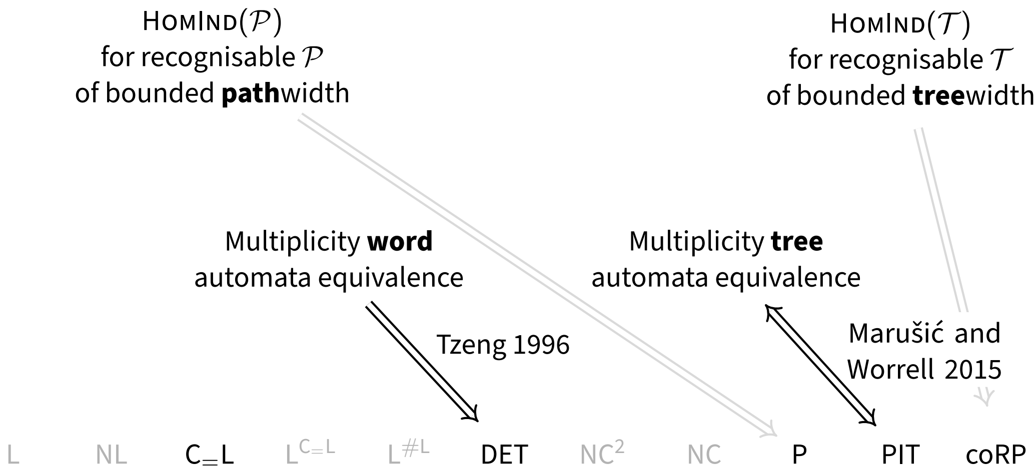

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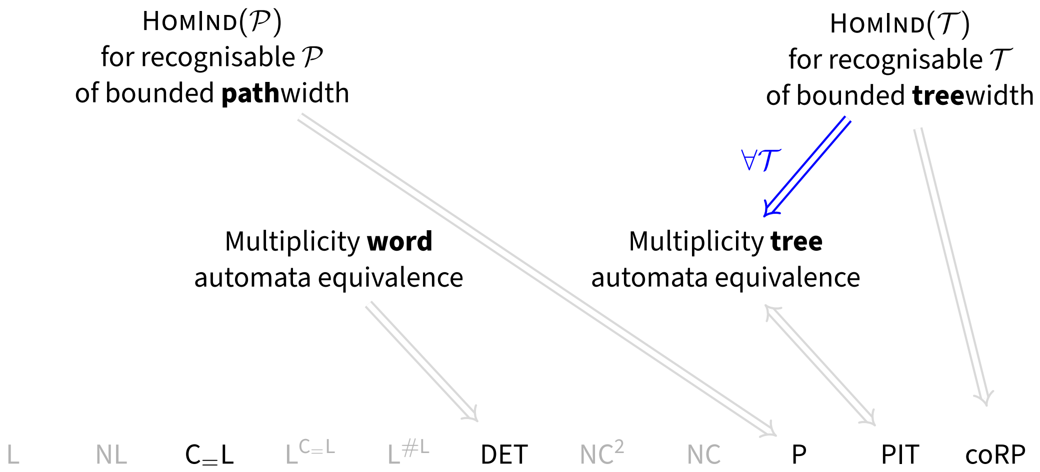

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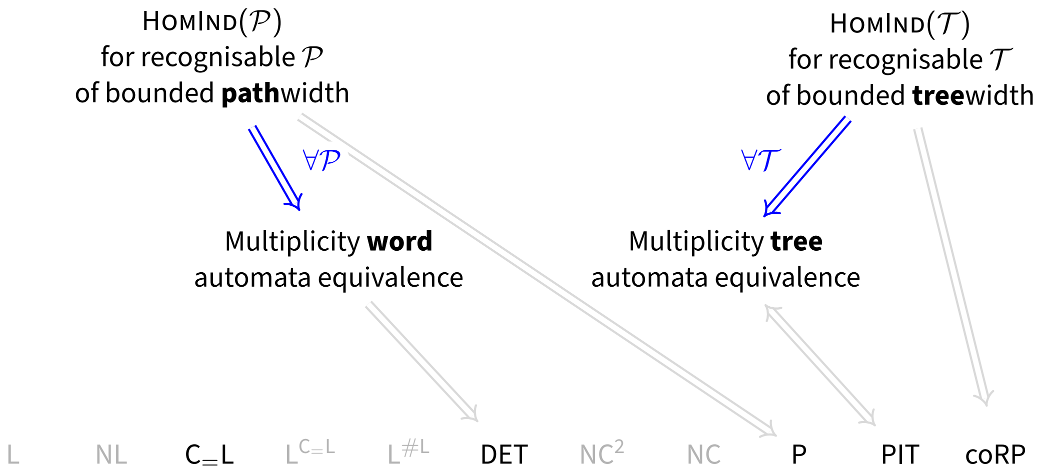

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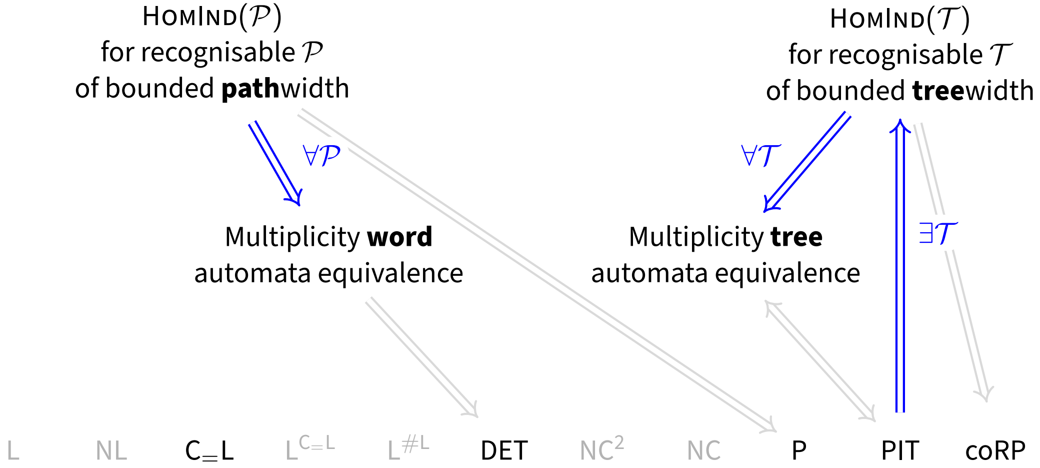

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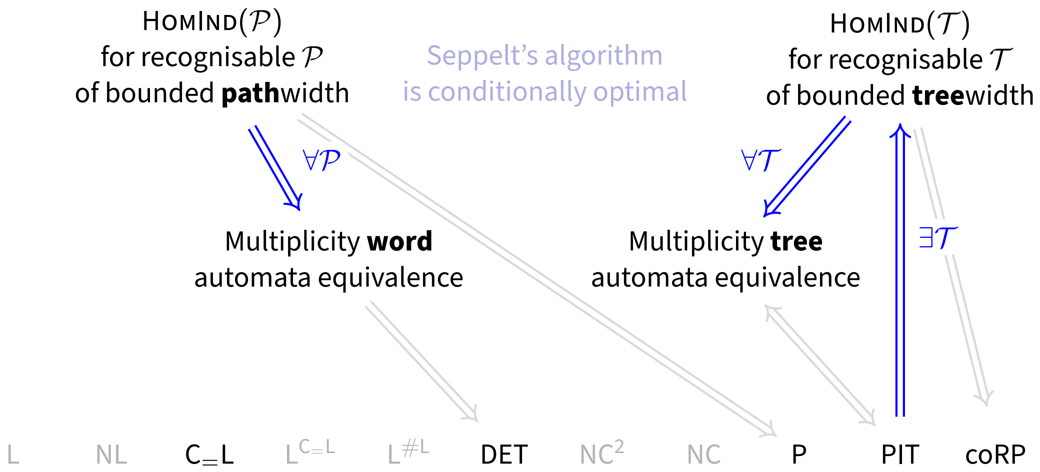

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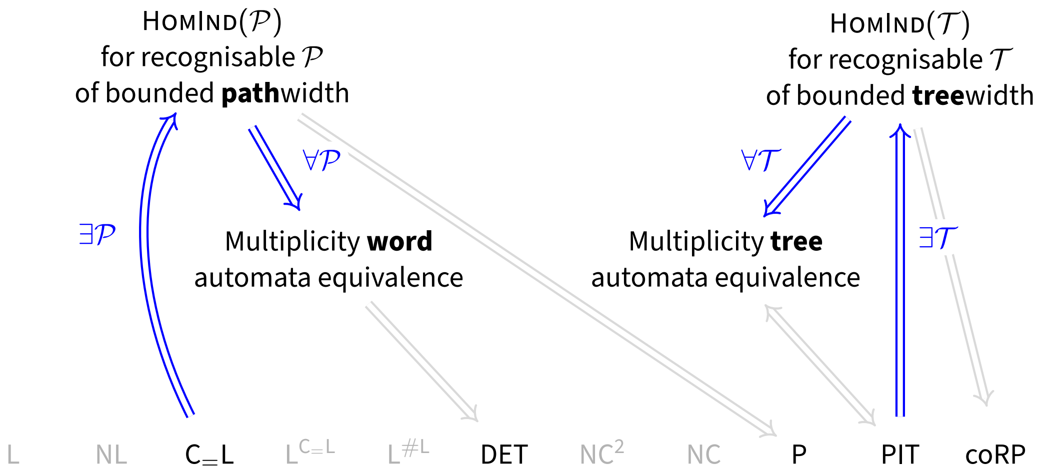

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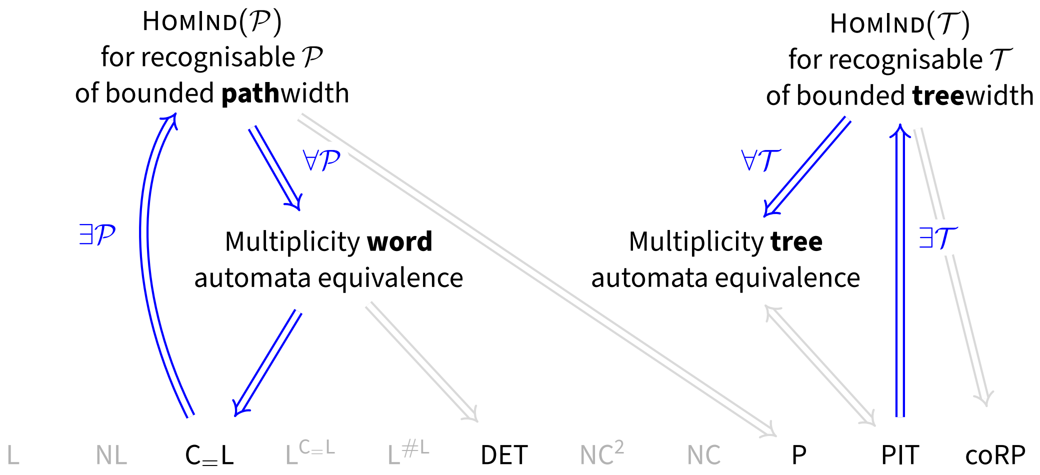

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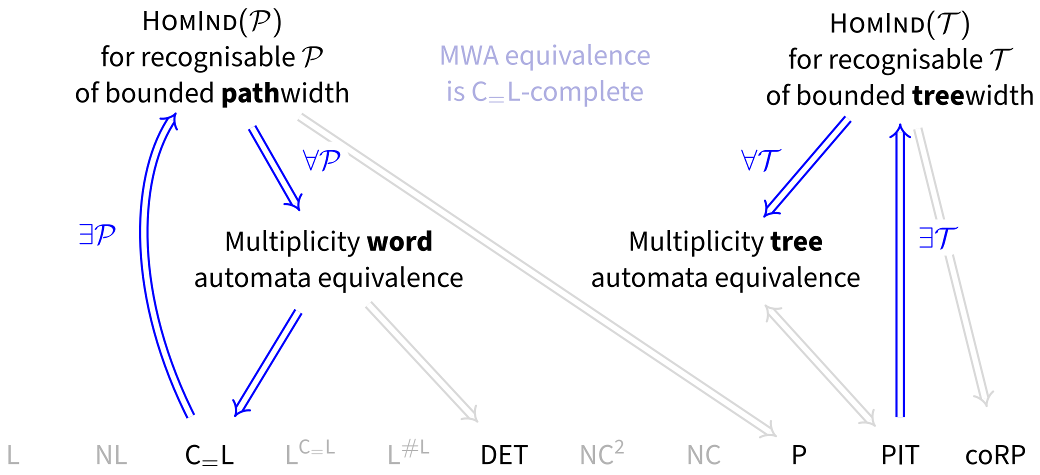

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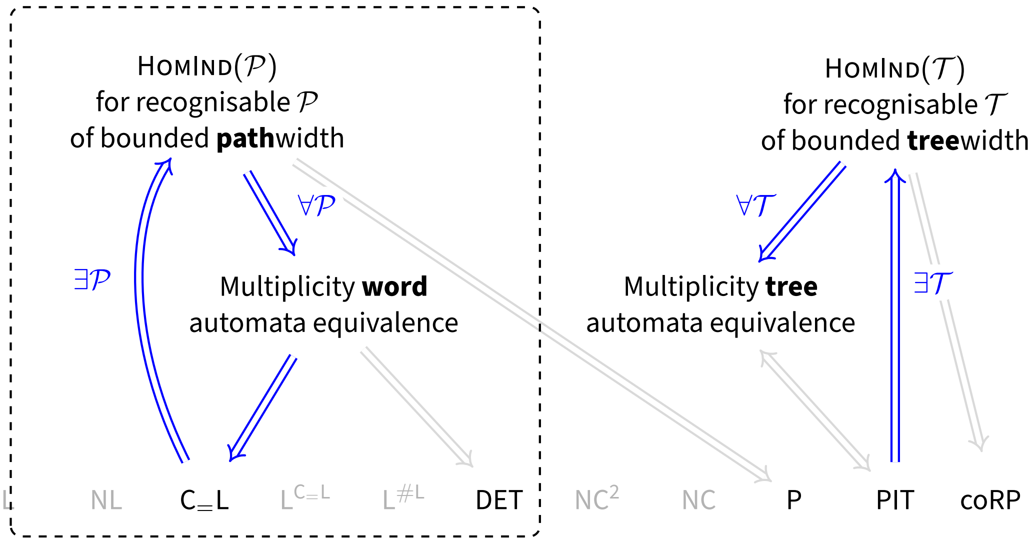


Recognisability is CMSO_2-Definability

- Monadic Second-Order logic with modular counting predicates
- Examples of CMSO_2-definable graph classes:
  - treewidth $\leq k$, pathwidth $\leq k$, planar graphs, bounded degree, …
  - minor-closed classes (using the Theorem of Robertson and Seymour 2004)

Definition (Recognisability, *informal*)

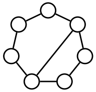
A property of a graph is *recognisable* if it can be recognised by a finite-state tree automaton.

Theorem

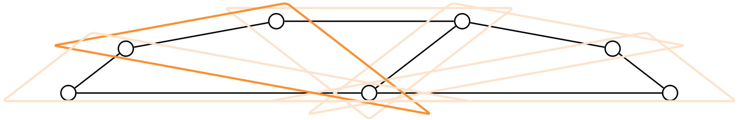
For every class of graphs $\mathcal{F}$ of bounded treewidth holds

$$\begin{aligned} \mathsf{CMSO}_2\text{-}\textit{definable} &\implies \textit{recognisable} && \textit{(Courcelle 1990)}, \\ \mathsf{CMSO}_2\text{-}\textit{definable} &\impliedby \textit{recognisable} && \textit{(Bojańczyk and Pilipczuk 2016)}. \end{aligned}$$

Recognising $\mathcal{P}$ of Bounded Pathwidth



graph F



a path decomposition (P, β) of F of width $k = 3 - 1 = 2$

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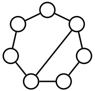

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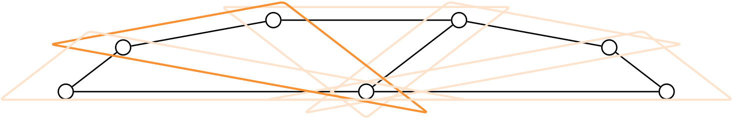

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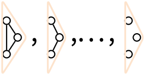
- Alphabet of bags Σ. For $k = 2$, $\Sigma = \{$  $\}$.

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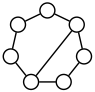

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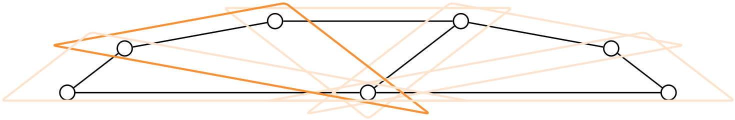

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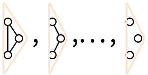
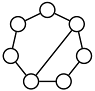
- Alphabet of bags Σ. For $k = 2$, $\Sigma = \{$  $\}$.

$\mathcal{A}_{\mathcal{P}}$ ($\mathcal{P}$ membership)

Recognising $\mathcal{P}$ of Bounded Pathwidth



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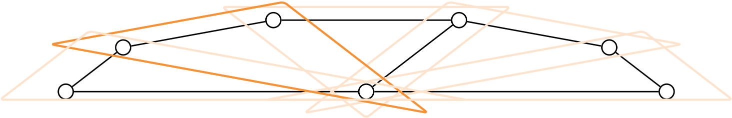

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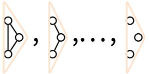
- Alphabet of bags Σ. For $k = 2$, $\Sigma = \{$  $\}$.

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$\mathcal{A}_G$ (counting homomorphisms to G)

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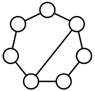

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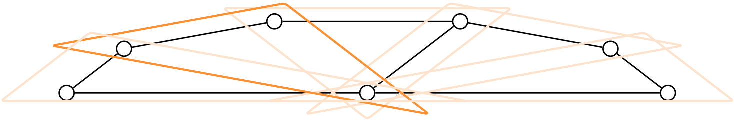

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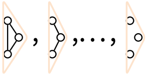
- Alphabet of bags Σ. For $k = 2$, $\Sigma = \{$  $\}$.

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- *Recognises* $\mathcal{P}$ as a *regular language* $\Sigma^* \to \{0, 1\}$.
- $[\![\mathcal{A}_\mathcal{P}]\!]((P, \beta)) = \mathbf{1}_{\{F \in \mathcal{P}\}}$
- Bojańczyk and Pilipczuk 2016; Courcelle 1990

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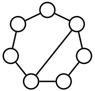

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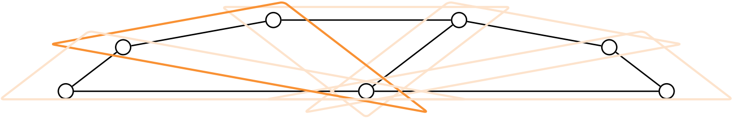

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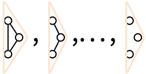
- Alphabet of bags Σ. For $k = 2$, $\Sigma = \{$  $\}$.

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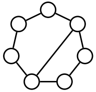
$\mathcal{A}_G$ (counting homomorphisms to G)

- *Recognises* $\hom(F, G)$ for each F of pathwidth $\leq k$ as a *rational series* over $\Sigma^* \to \mathbb{Q}$.
- $[\![\mathcal{A}_G]\!]((P, \beta)) = \hom(F, G)$
- Seppelt 2024; Grohe, Rattan, and Seppelt 2025 (Homomorphism tensors)

Recognising $\mathcal{P}$ of Bounded Pathwidth (cont.)



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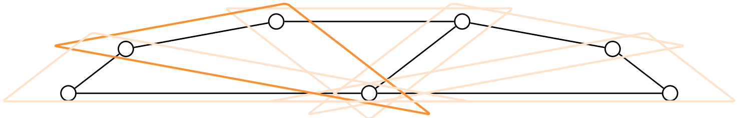

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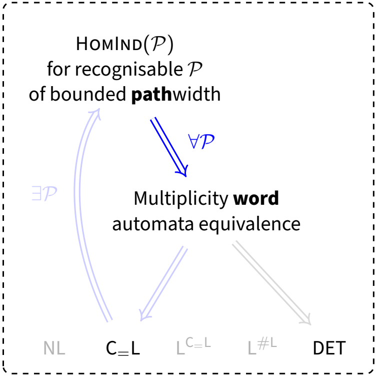

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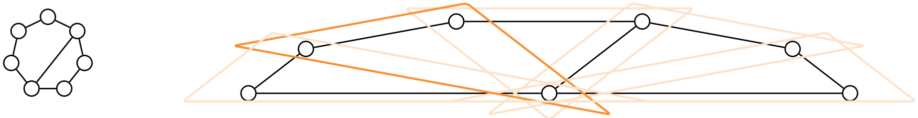
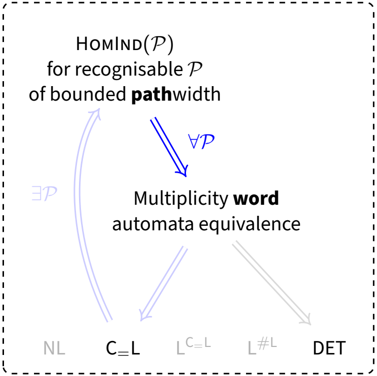
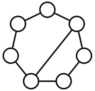


graph F | a path decomposition (P, β) of F of width $k = 3 - 1 = 2$

- *Product* $\mathcal{A}_{\mathcal{P}} \otimes \mathcal{A}_G$ of a DFA and an MWA.



Recognising $\mathcal{P}$ of Bounded Pathwidth (cont.)



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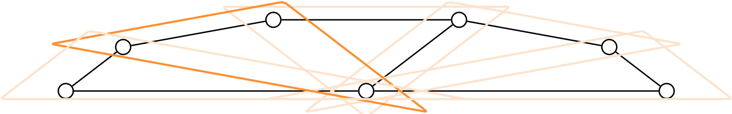
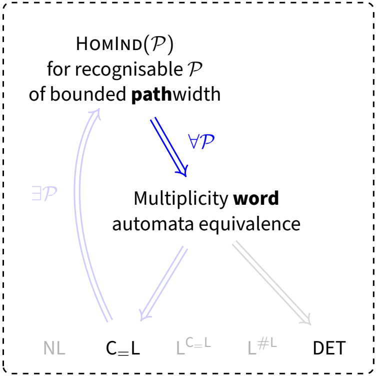


a path decomposition (P, β) of F of width $k = 3 - 1 = 2$

- *Product* $\mathcal{A}_{\mathcal{P}} \otimes \mathcal{A}_G$ of a DFA and an MWA.
- $[\![\mathcal{A}_{\mathcal{P}} \otimes \mathcal{A}_G]\!] : \Sigma^* \to \mathbb{Q}$.



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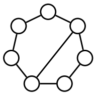

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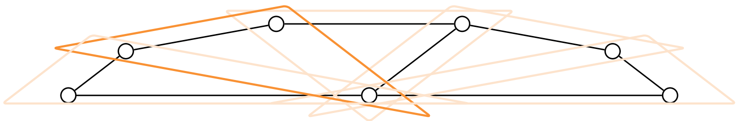

a path decomposition (P, β) of F of width $k = 3 - 1 = 2$

- *Product* $\mathcal{A}_{\mathcal{P}} \otimes \mathcal{A}_G$ of a DFA and an MWA.
- $[\![\mathcal{A}_{\mathcal{P}} \otimes \mathcal{A}_G]\!] \colon \Sigma^* \to \mathbb{Q}$.
- $[\![\mathcal{A}_{\mathcal{P}} \otimes \mathcal{A}_G]\!]((P, \beta)) = \mathbf{1}_{\{F \in \mathcal{P}\}} \cdot \hom(F, G)$

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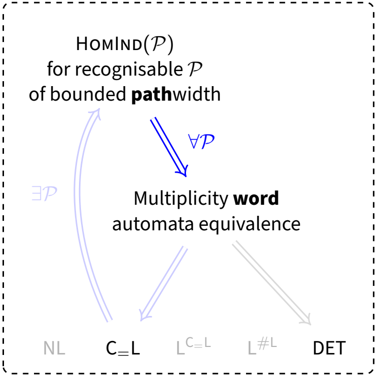
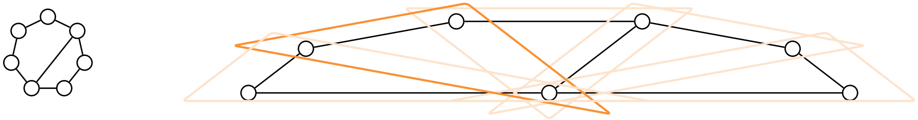


Recognising $\mathcal{P}$ of Bounded Pathwidth (cont.)



- *Product* $\mathcal{A}_\mathcal{P} \otimes \mathcal{A}_G$ of a DFA and an MWA.
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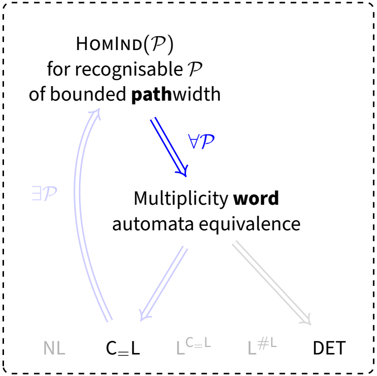
- HOMIND($\mathcal{P}$) as MWA equivalence (proof idea):

$$G \equiv_\mathcal{P} H \iff [\![\mathcal{A}_\mathcal{P} \otimes \mathcal{A}_G]\!] = [\![\mathcal{A}_\mathcal{P} \otimes \mathcal{A}_H]\!].$$



Multiplicity Word Automata (MWA)

Definition

Multiplicity word automaton is $\mathcal{A} = (S, \Sigma, M, \alpha, \eta)$

- states S, and alphabet Σ (both finite)
- $\alpha \in \mathbb{Q}^{1\times S}$ (initial v.), $\eta \in \mathbb{Q}^{S}$ (final v.),
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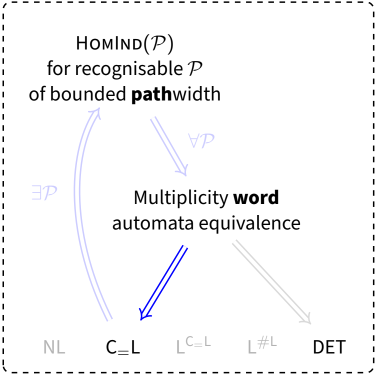

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- Exact Counting Logspace $\quad L \subseteq NL \subseteq C_=L \subseteq L^{C_=L} \subseteq L^{\#L} \subseteq DET \subseteq NC^2 \subseteq NC \subseteq P$.

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$C_=L$-completeness of HomInd($\mathcal{P}$) for recognisable $\mathcal{P}$ of bounded pathwidth
(Proof idea)

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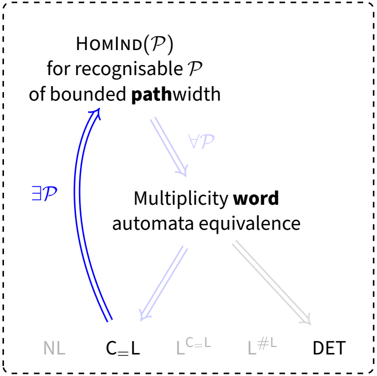

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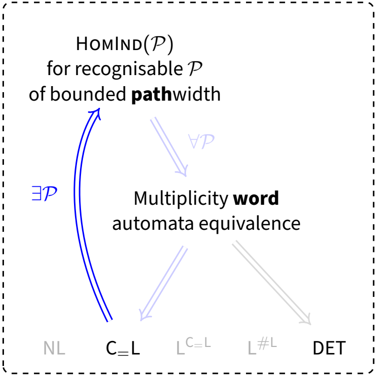

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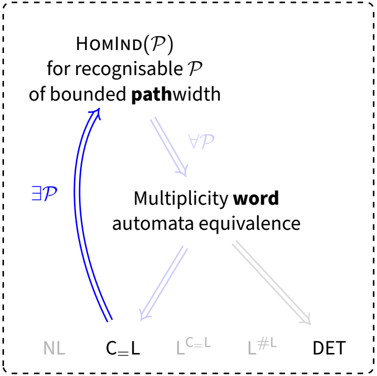

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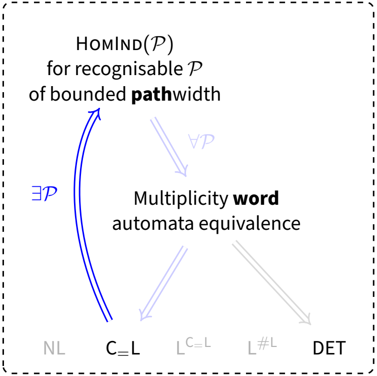

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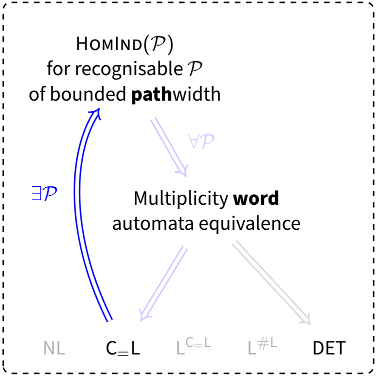

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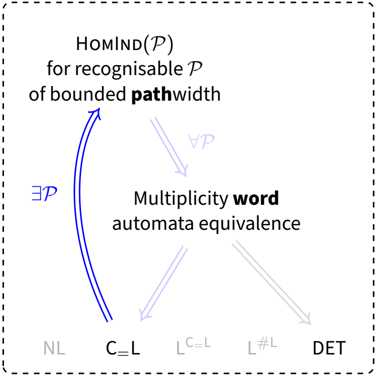

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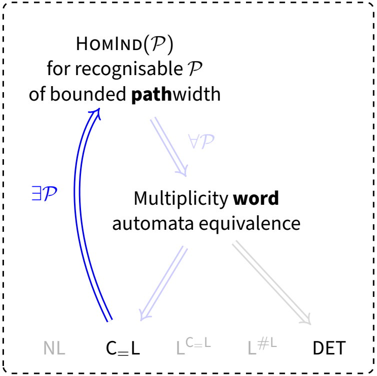

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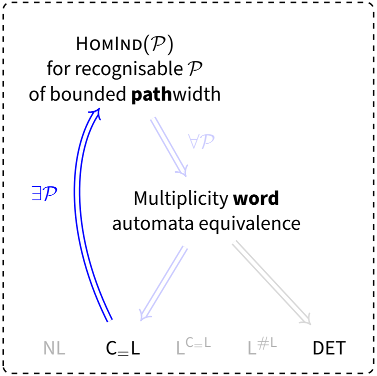

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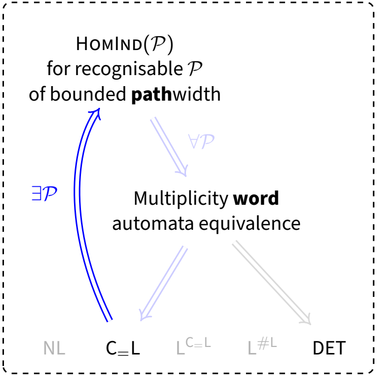
  - Directed variant $\vec{G} \equiv_{\vec{\mathcal{C}}} \vec{H}$.
- Use (other) gadgets to construct equivalent undirected case for $\mathcal{P}$ of bounded pathwidth.



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