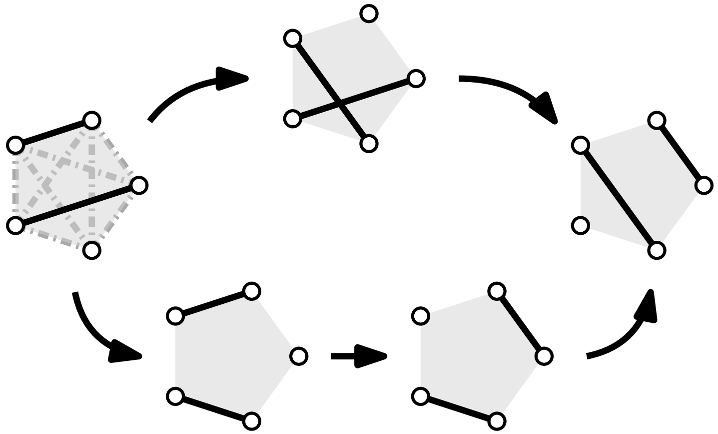
Higher Hardness Results for the Reconfiguration of Odd Matchings

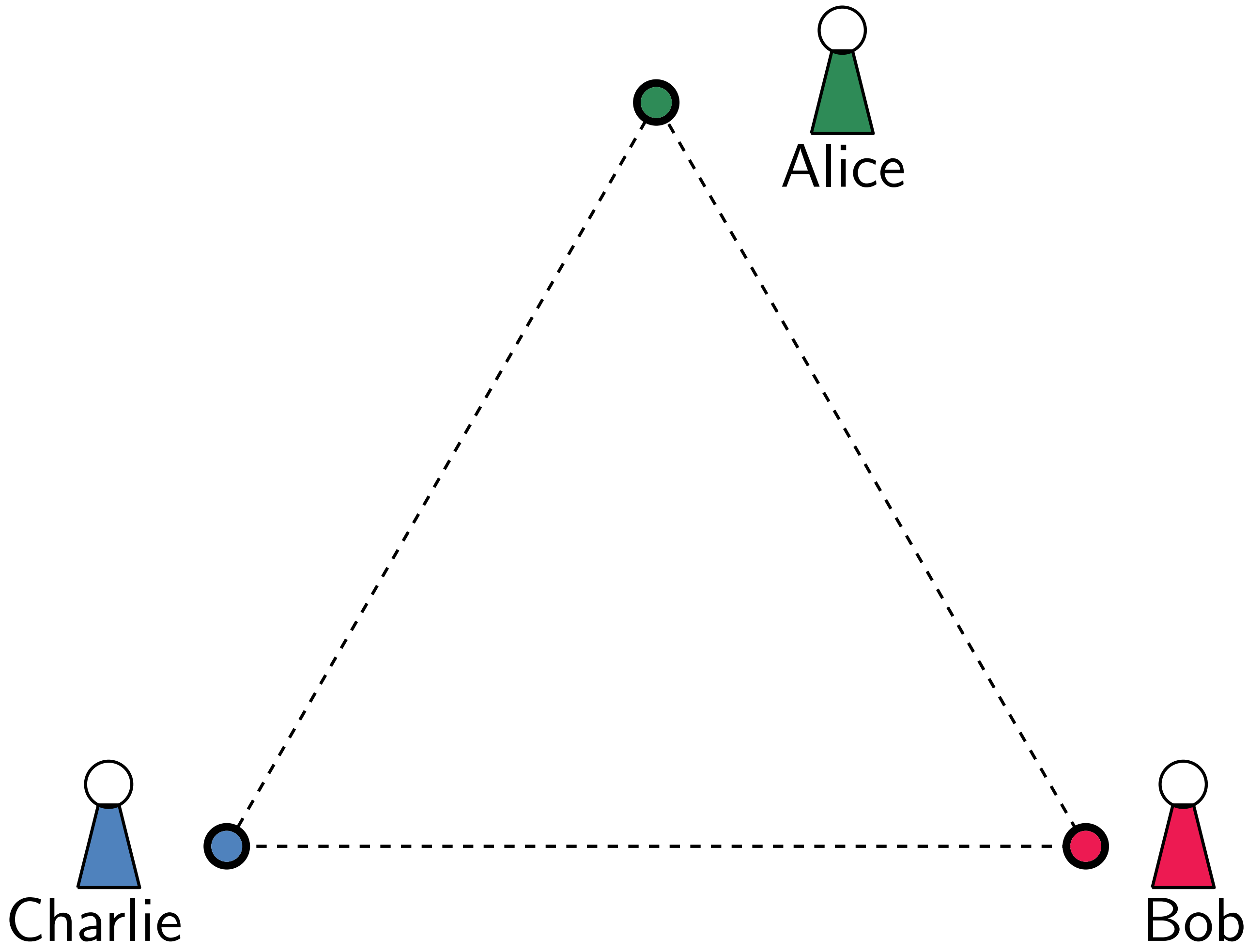
Joseph Dorfer

Graz University of Technology



Austrian Science Fund (FWF) 10.55776/DOC183

Stable Marriage Problem



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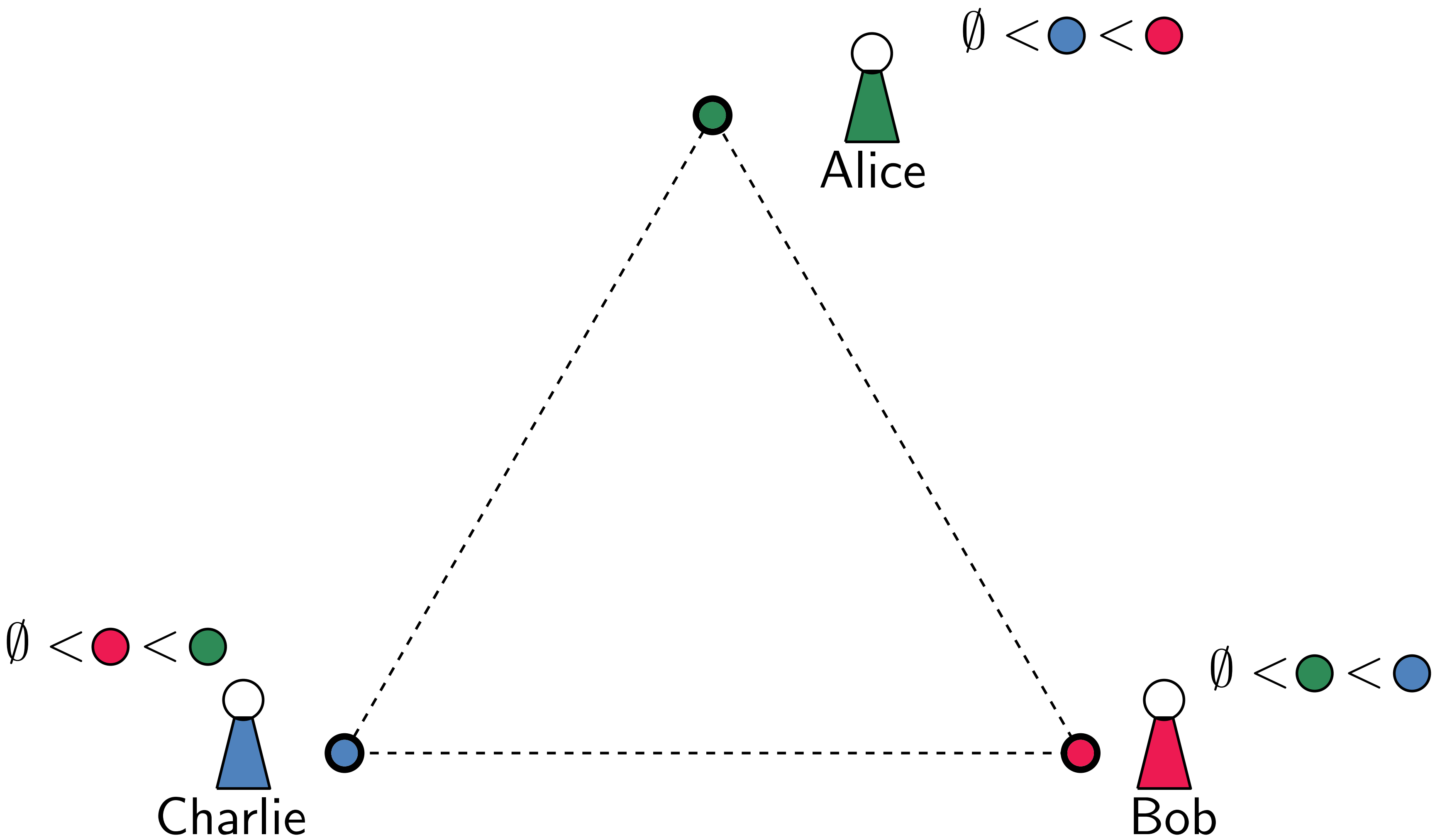

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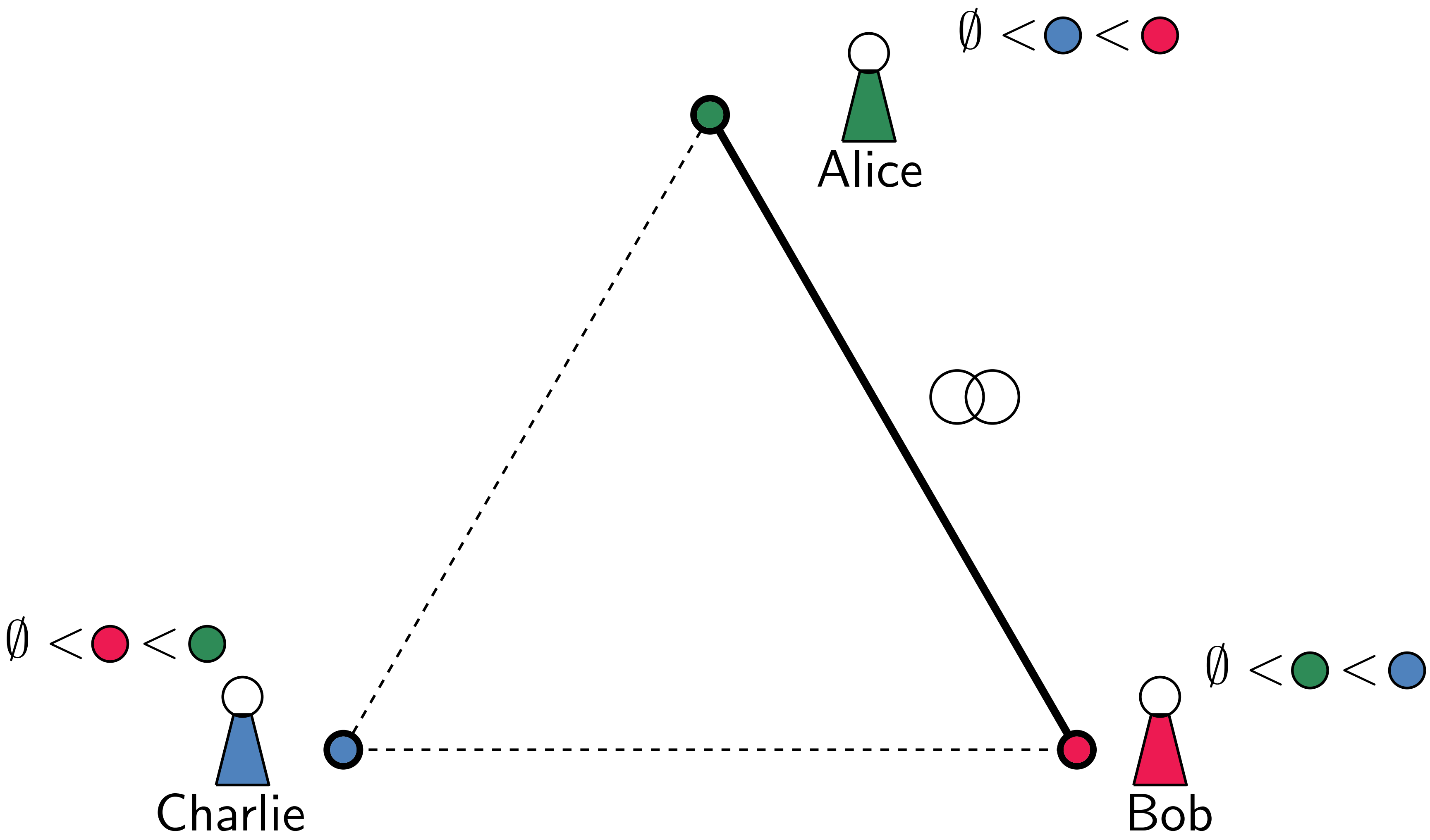

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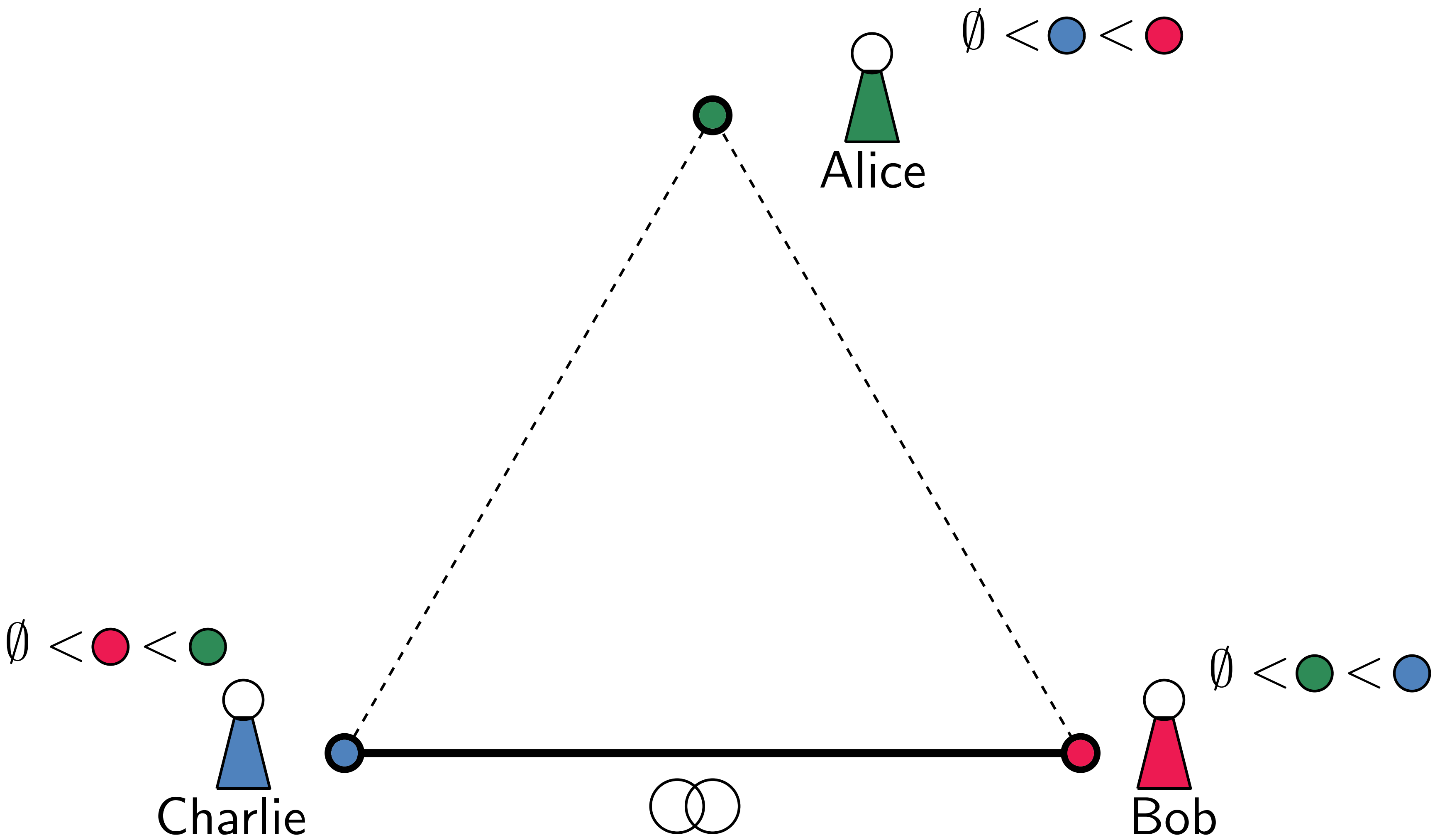

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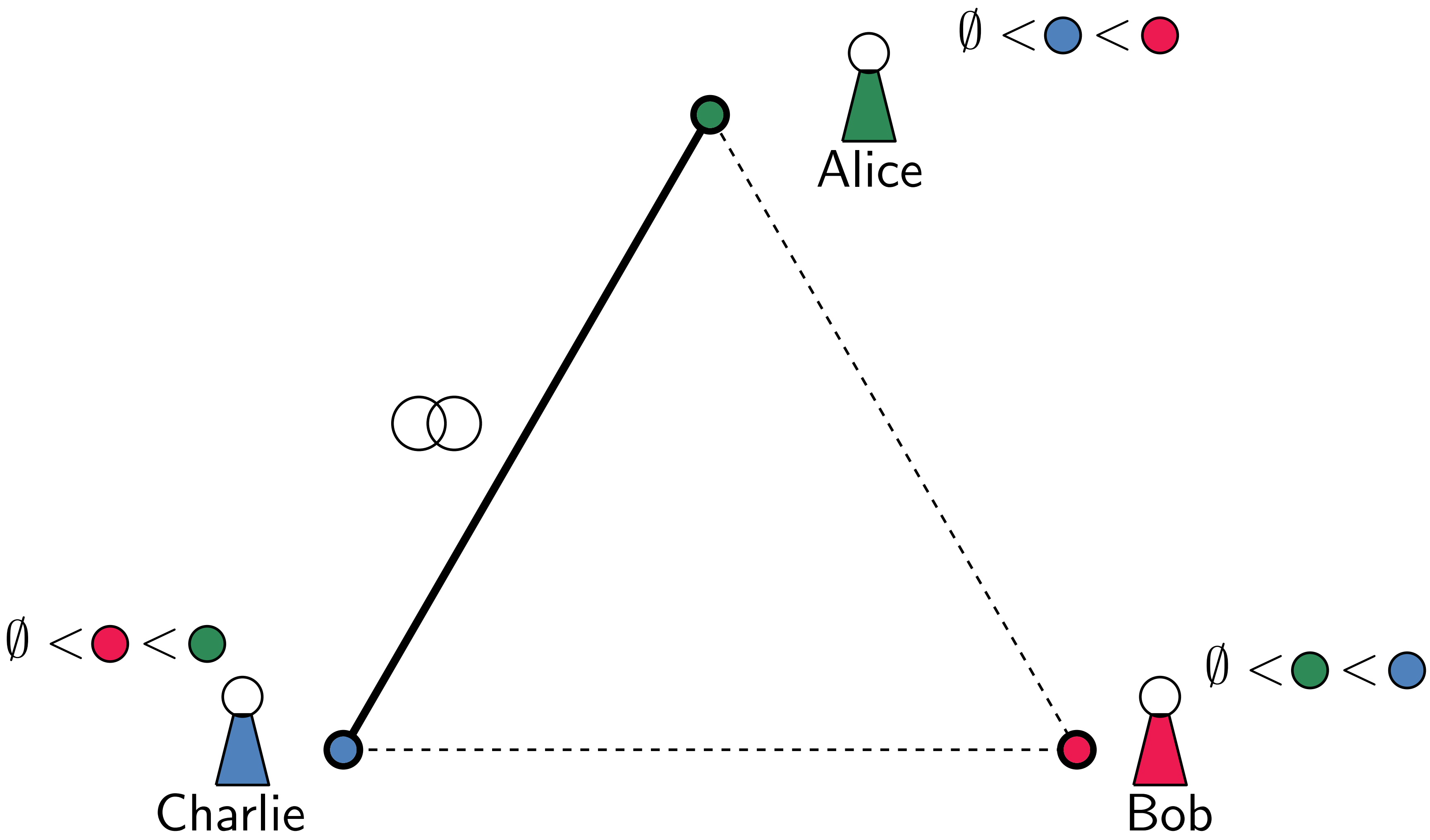

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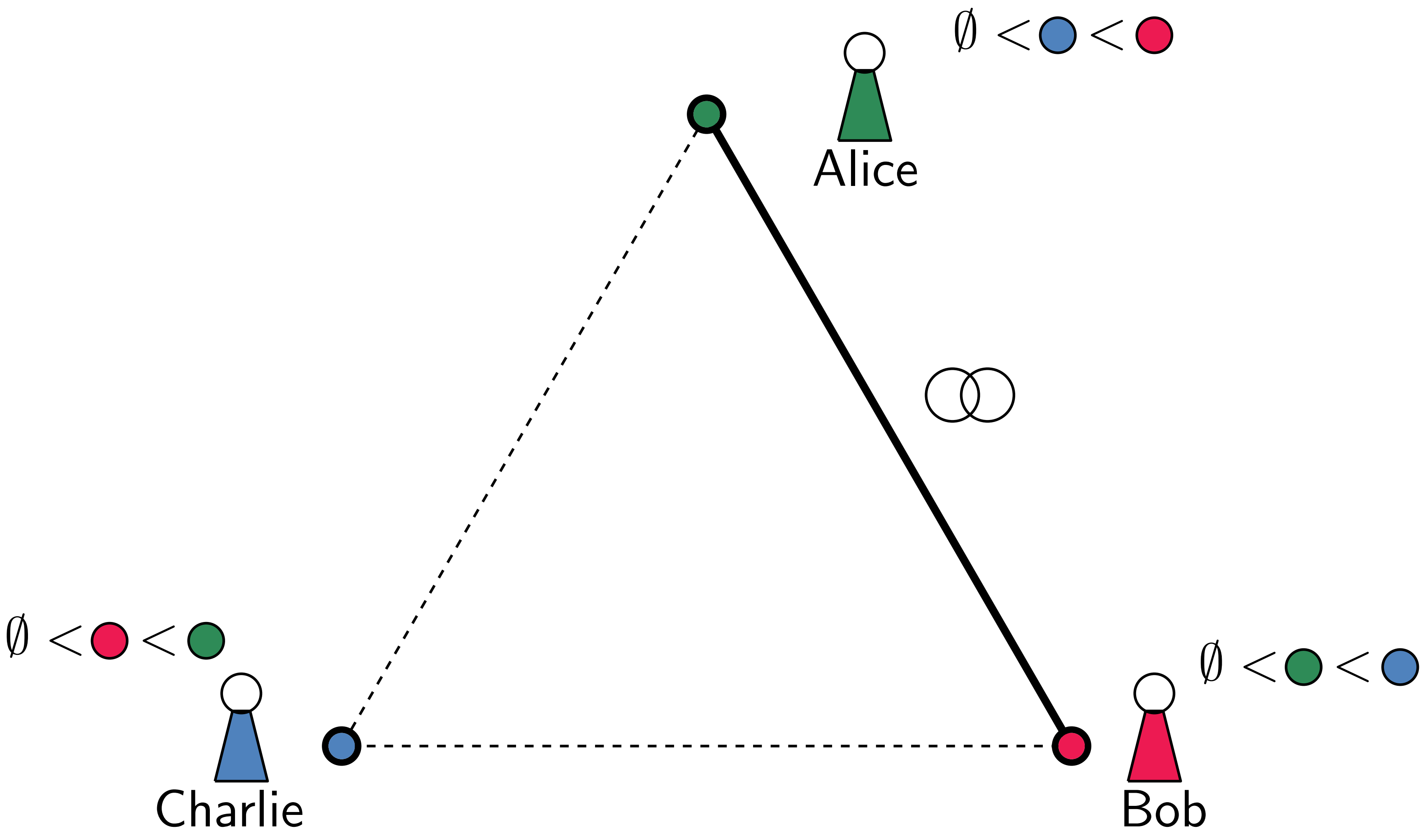
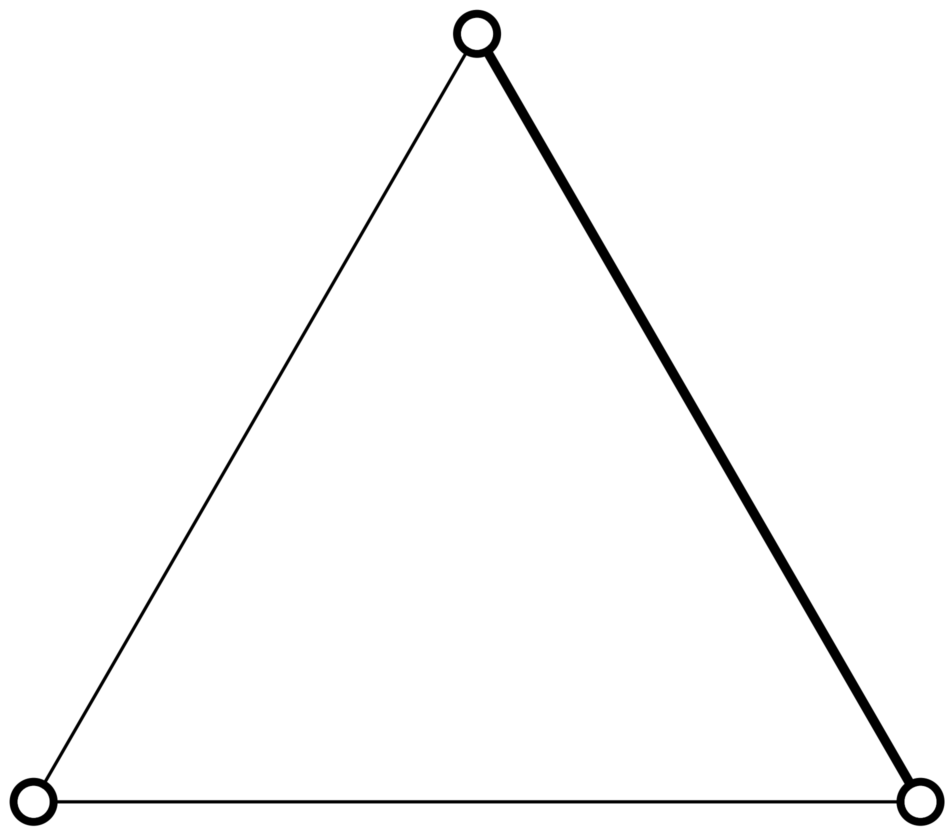


Odd matchings and flips

Odd matching of graphs: All vertices except one are matched



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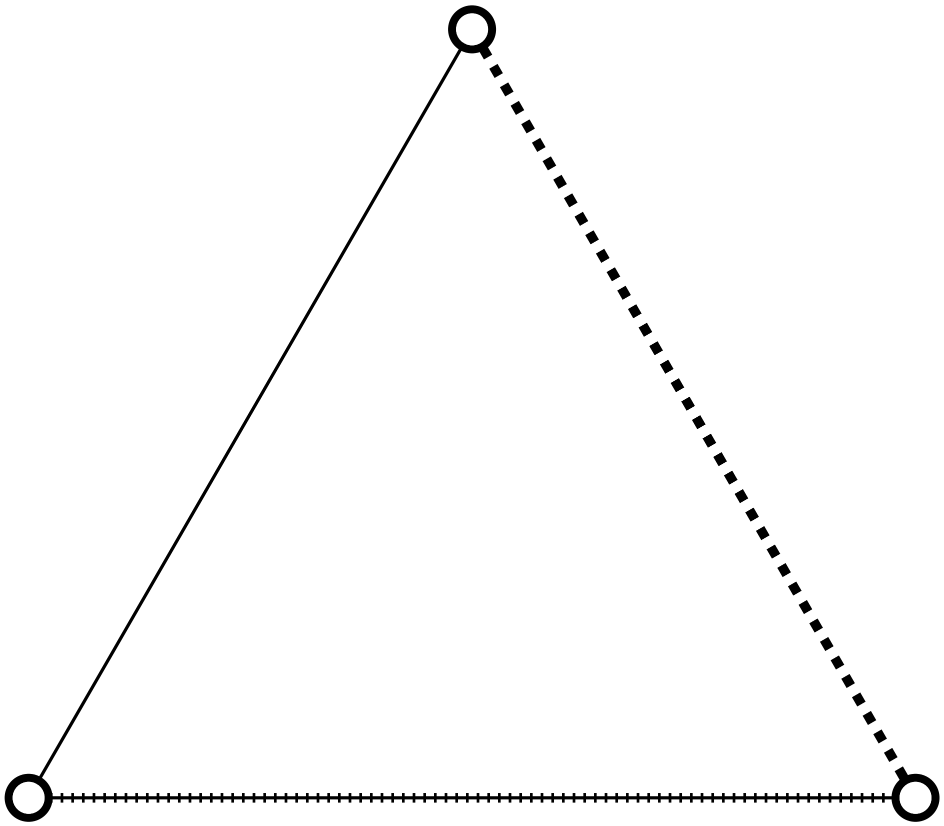

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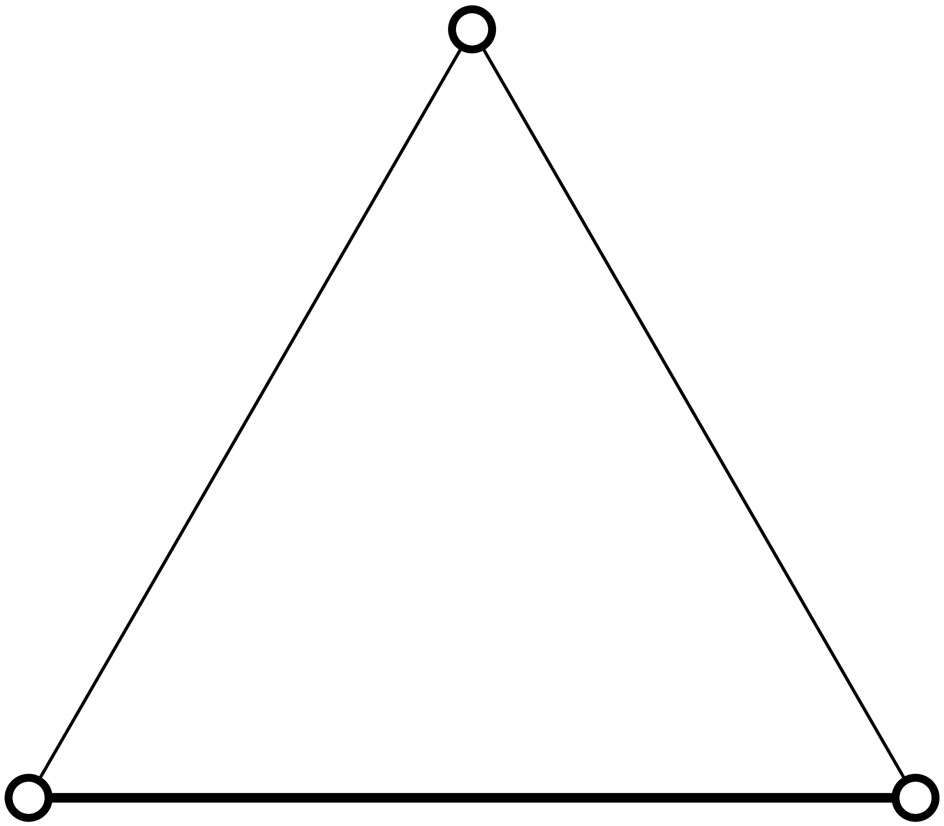

Flipping - Central Questions

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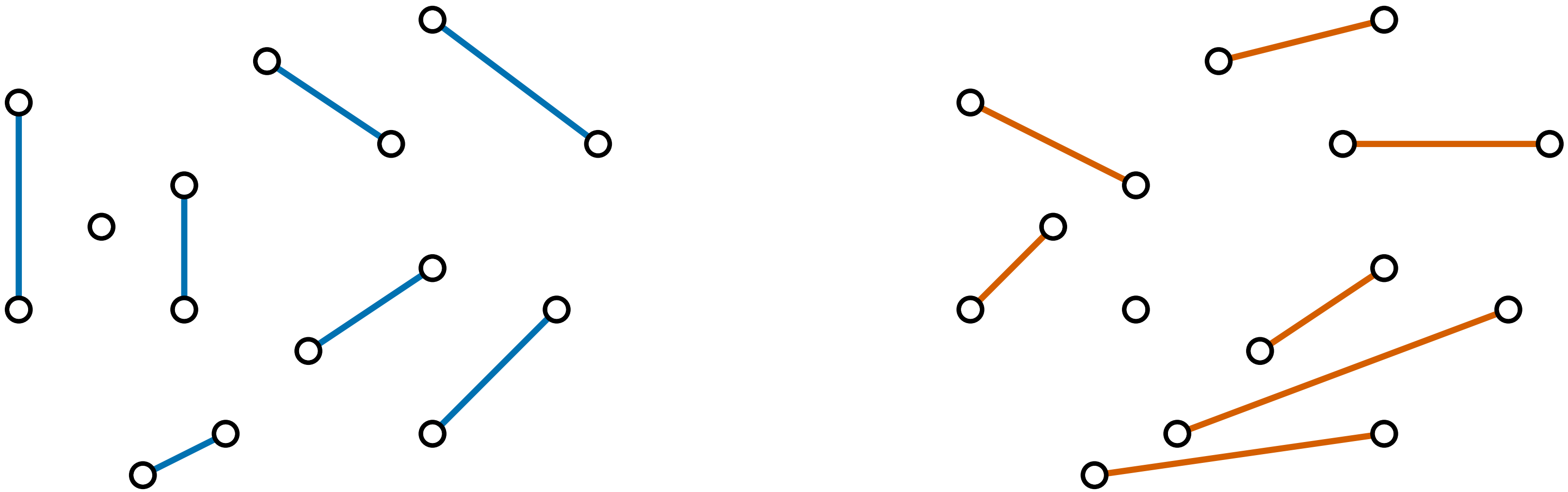
Complexity: Two specific configurations: How many flips needed? How to compute flip sequence?

Union of matchings



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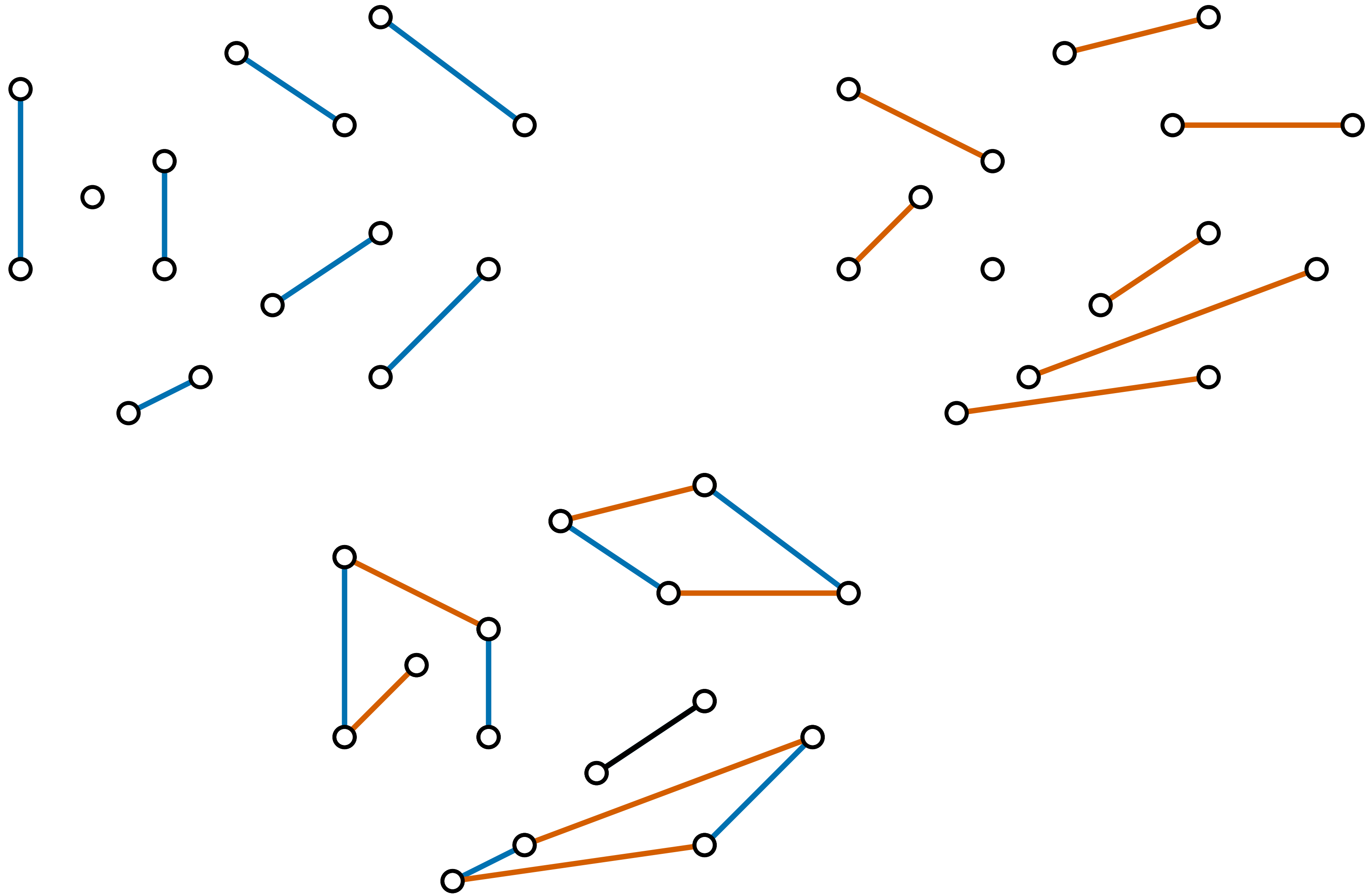

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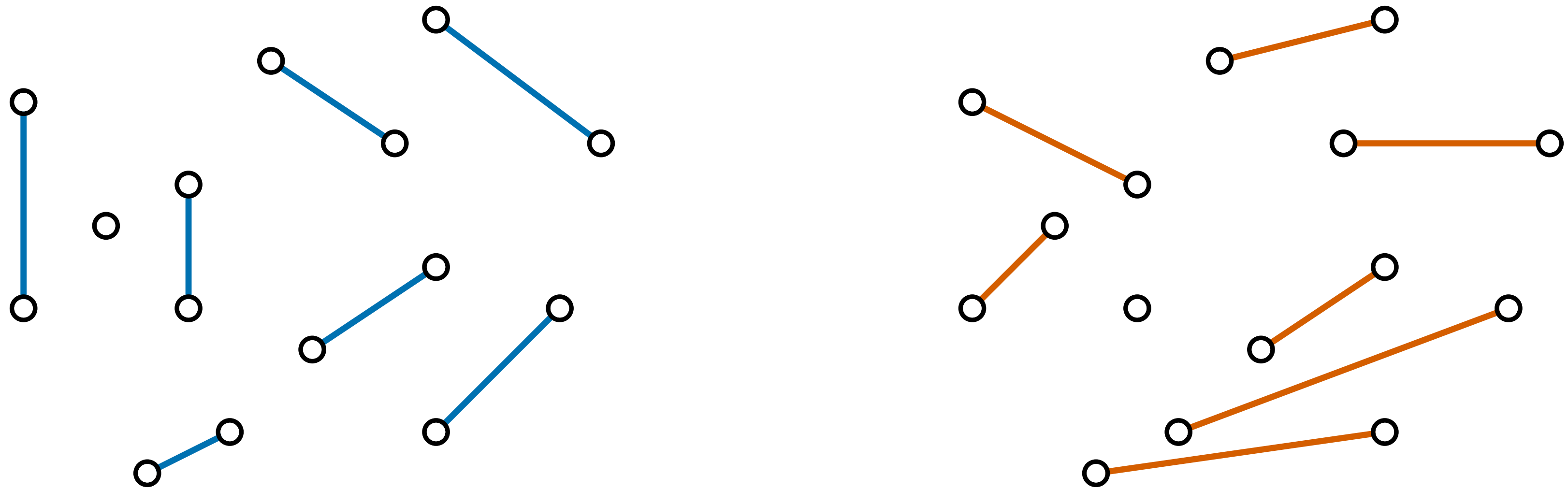
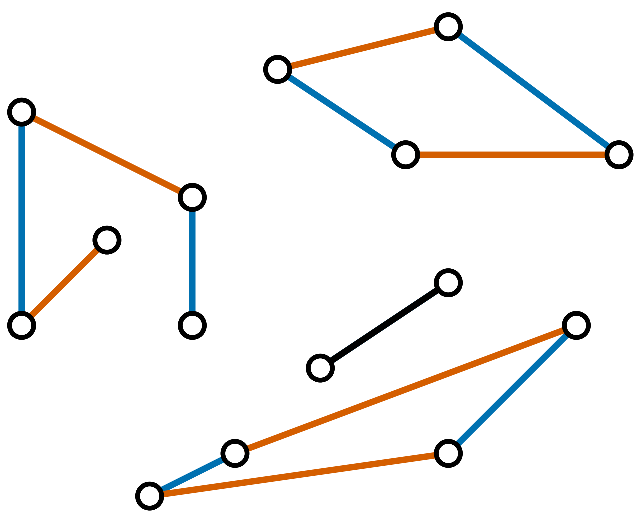


Alternating path connecting isolated vertices



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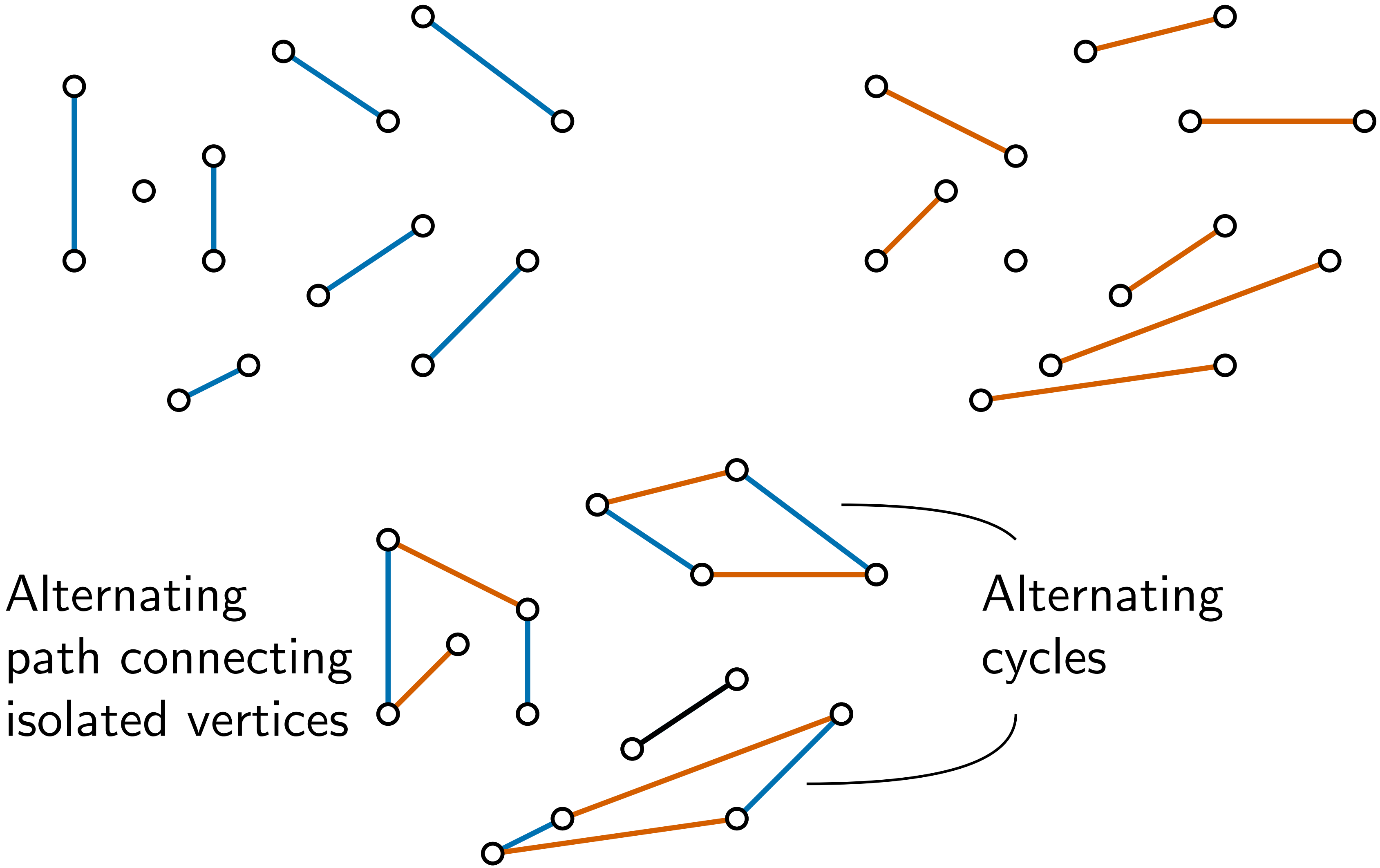

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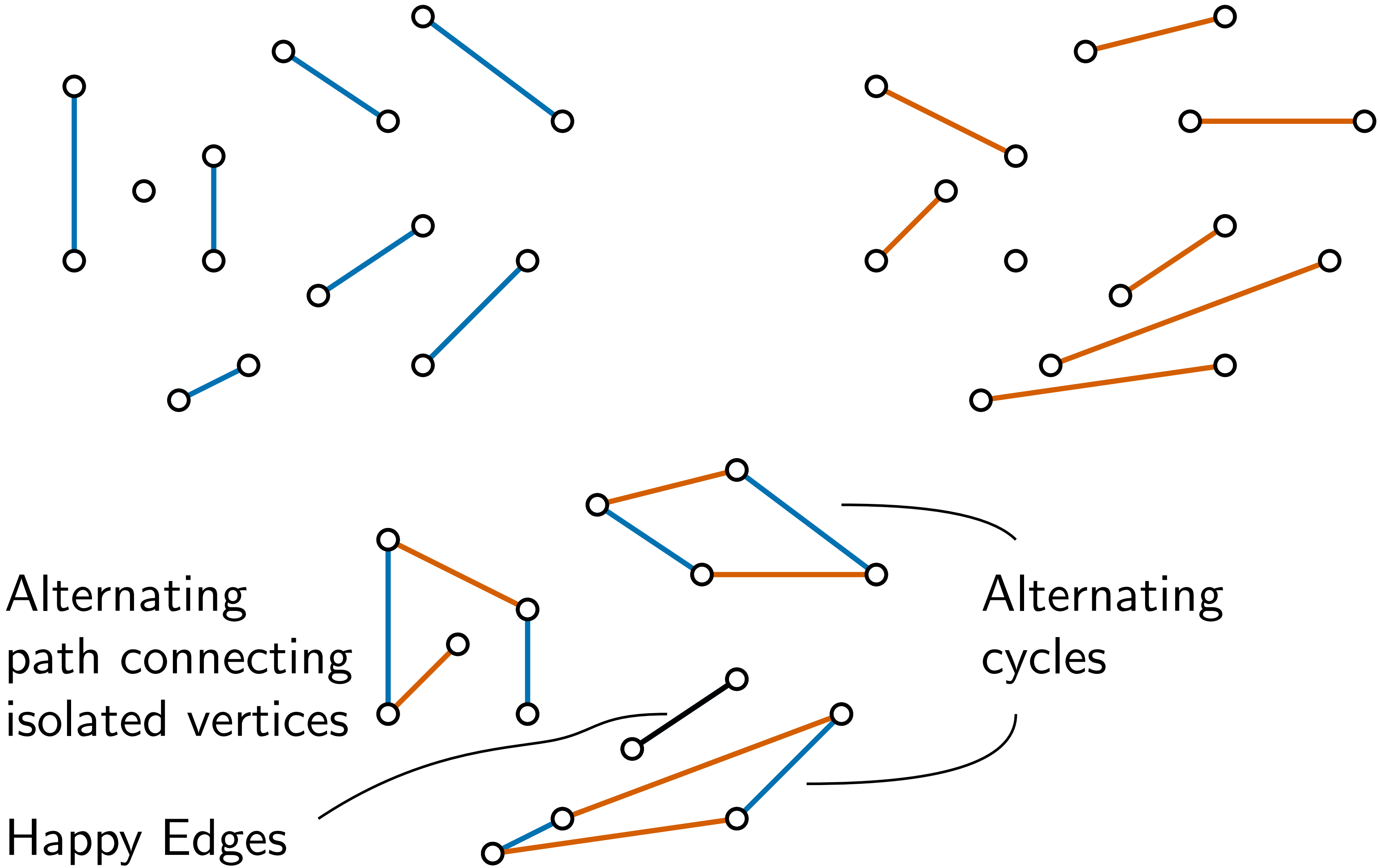

Union of Matchings

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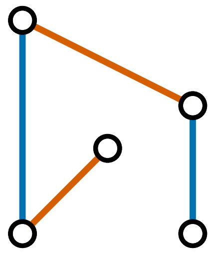

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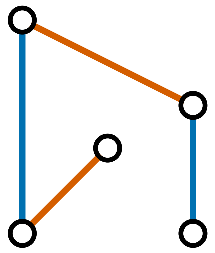
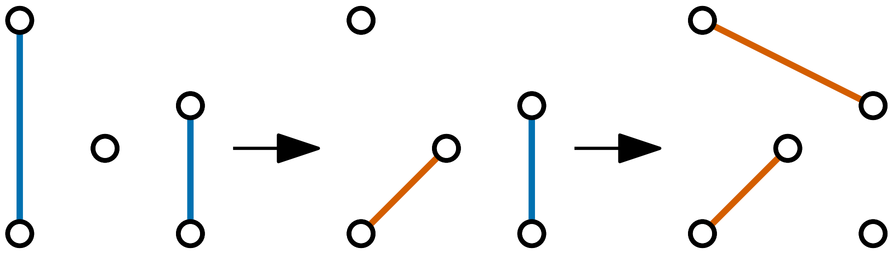
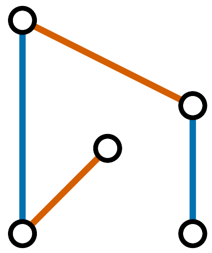


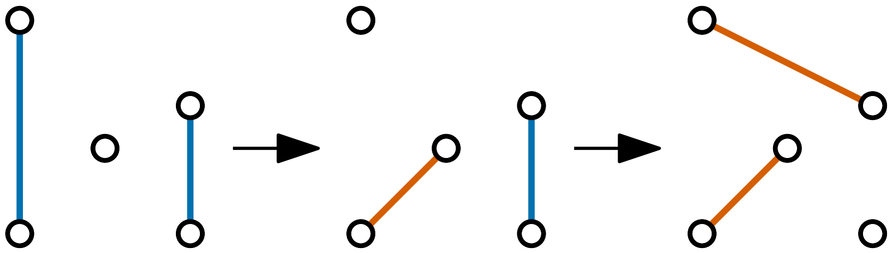


One flip per edge in one matching

Union of Matchings

Alternating paths





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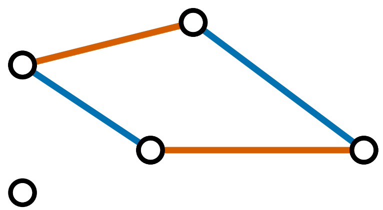

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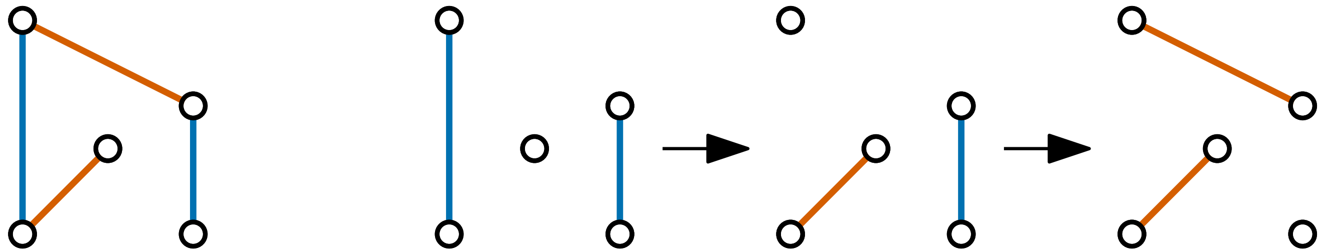

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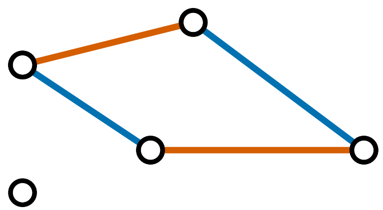

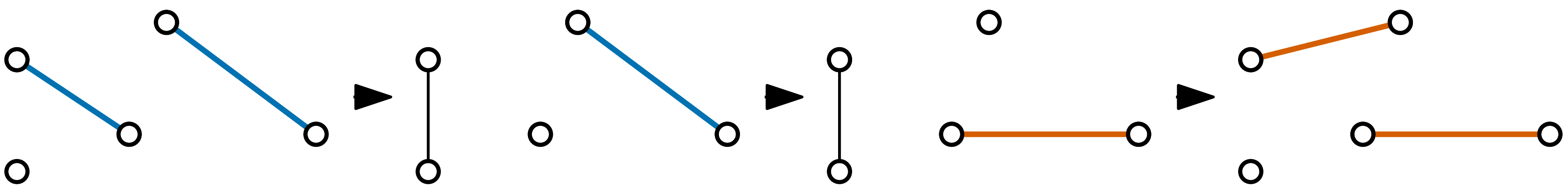

One flip more than the number of edges

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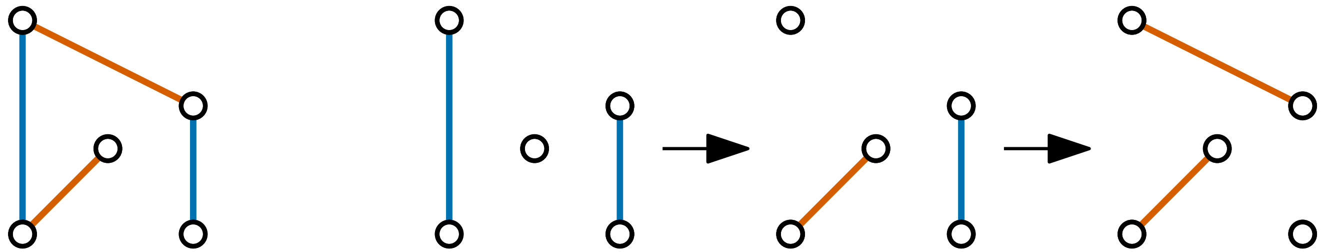

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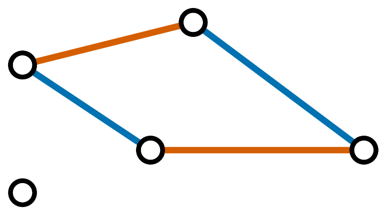
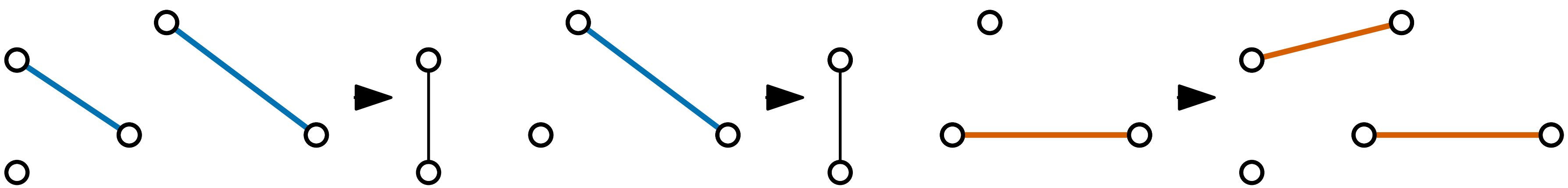


Happy Edges: No flips at all or at least two



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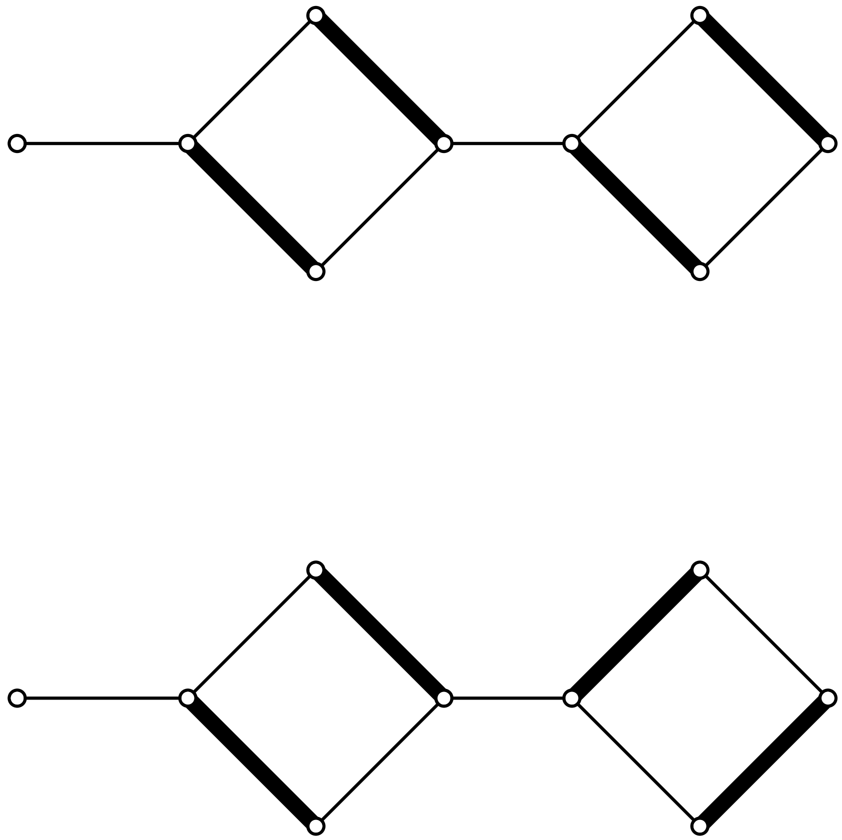

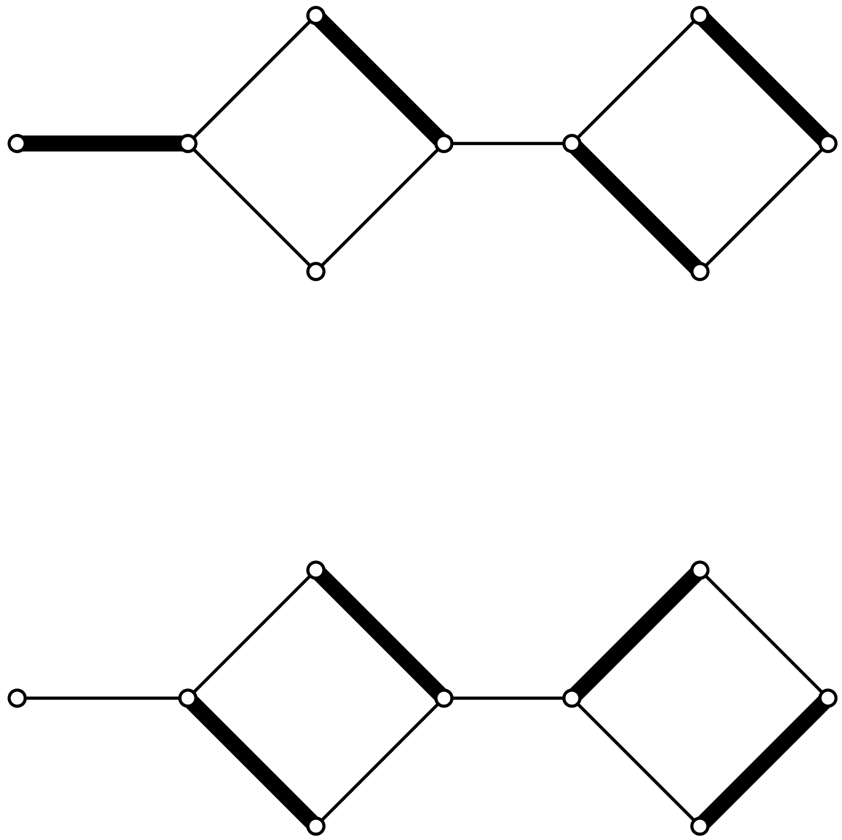

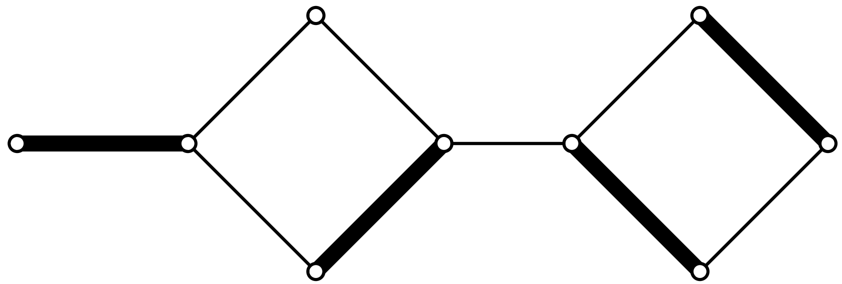

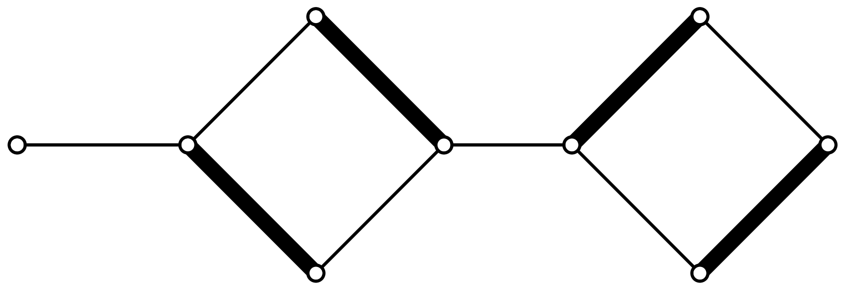

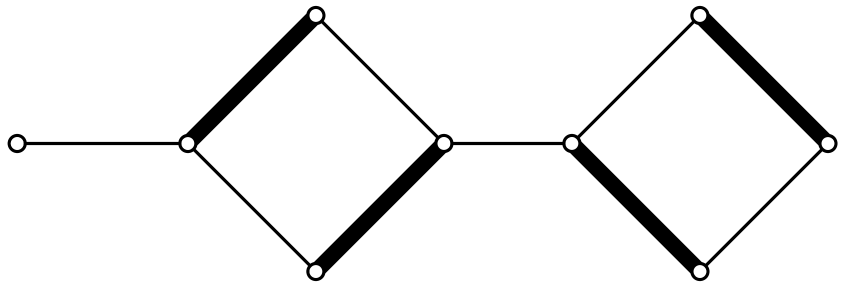

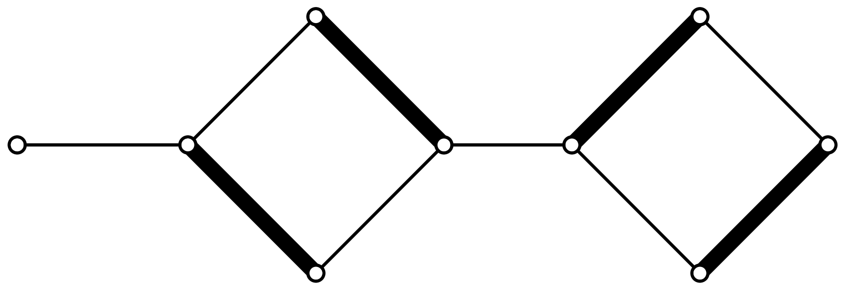

Connectedness	Diameter	Complexity

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Connectedness is polynomial time checkable:
Different Algorithms/Characterizations:

With arbitrary number (≥ 1) of isolated vertices:
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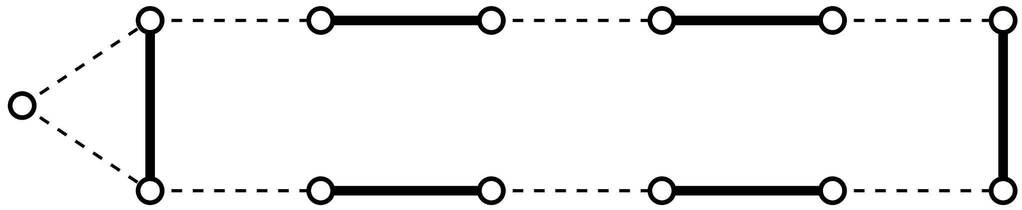
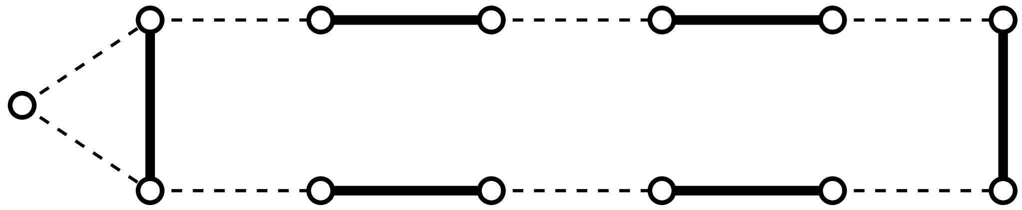
Flip Sequences and $\exists y_1, \ldots, y_l$





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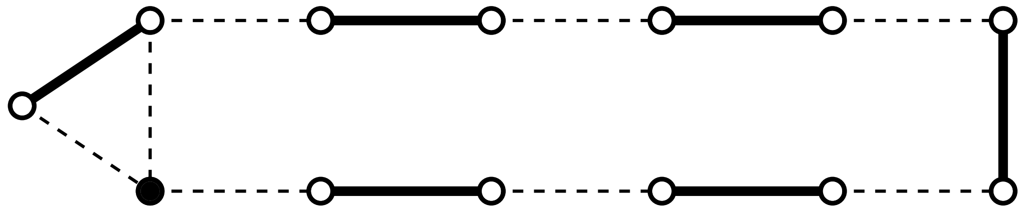

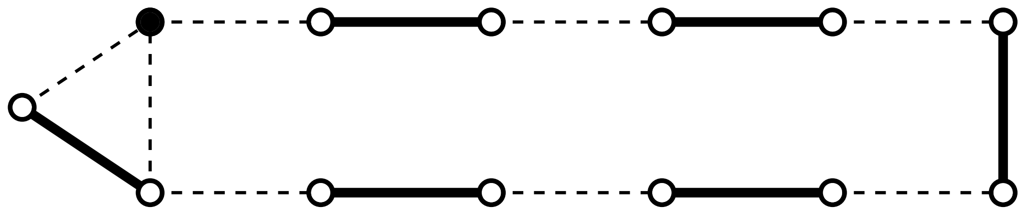

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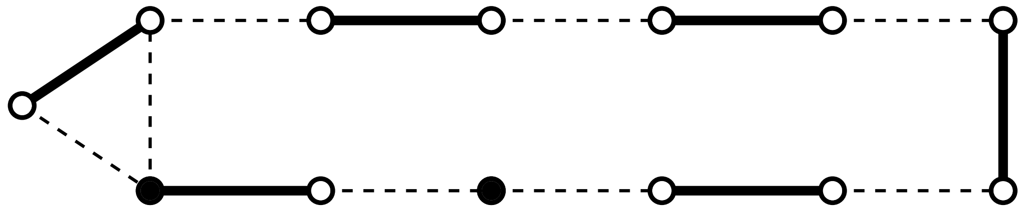

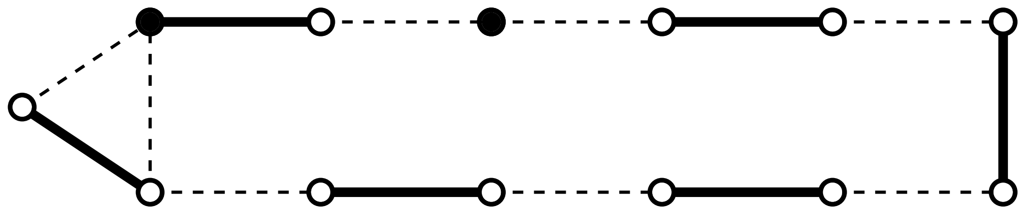

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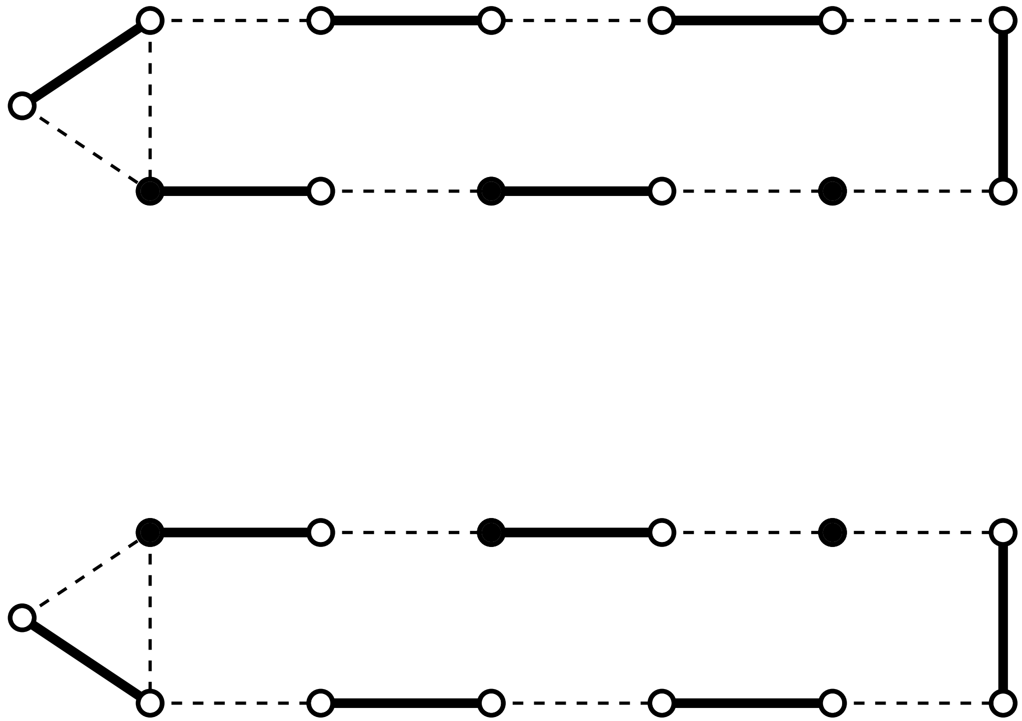

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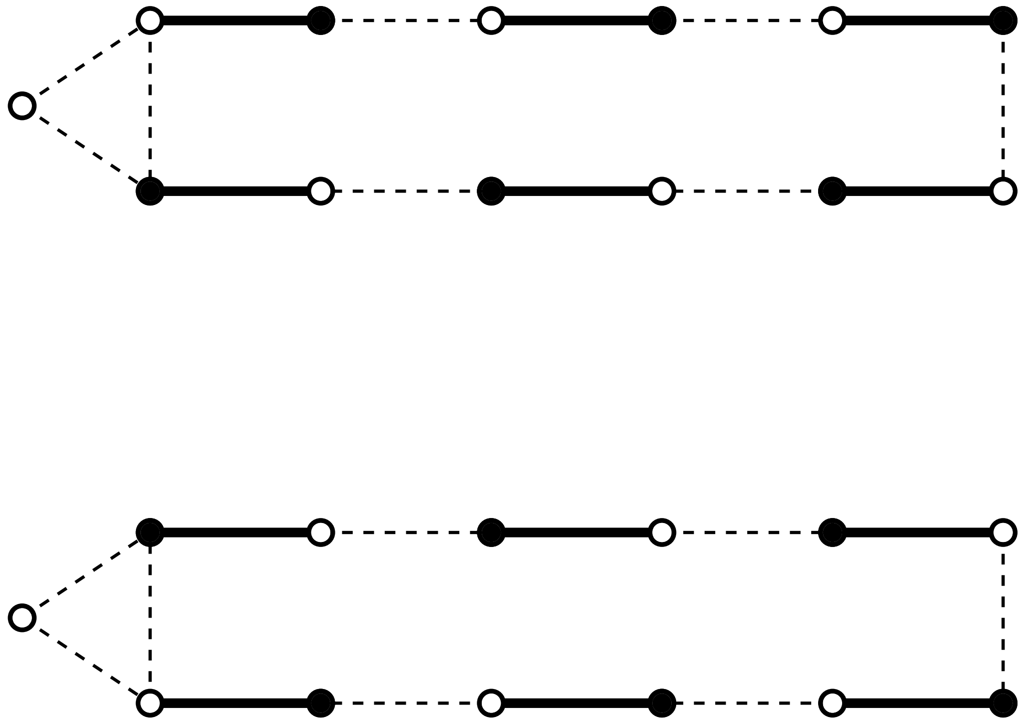

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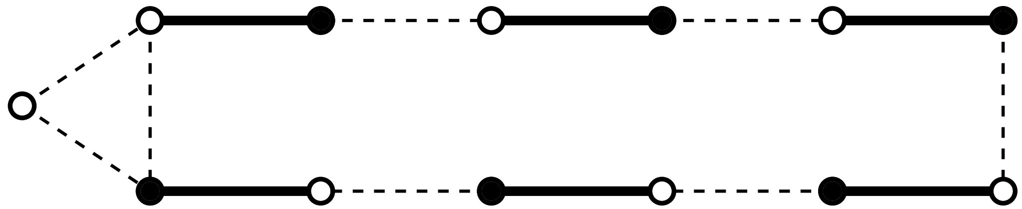

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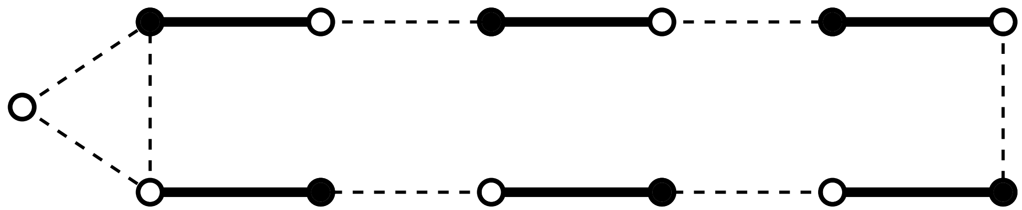
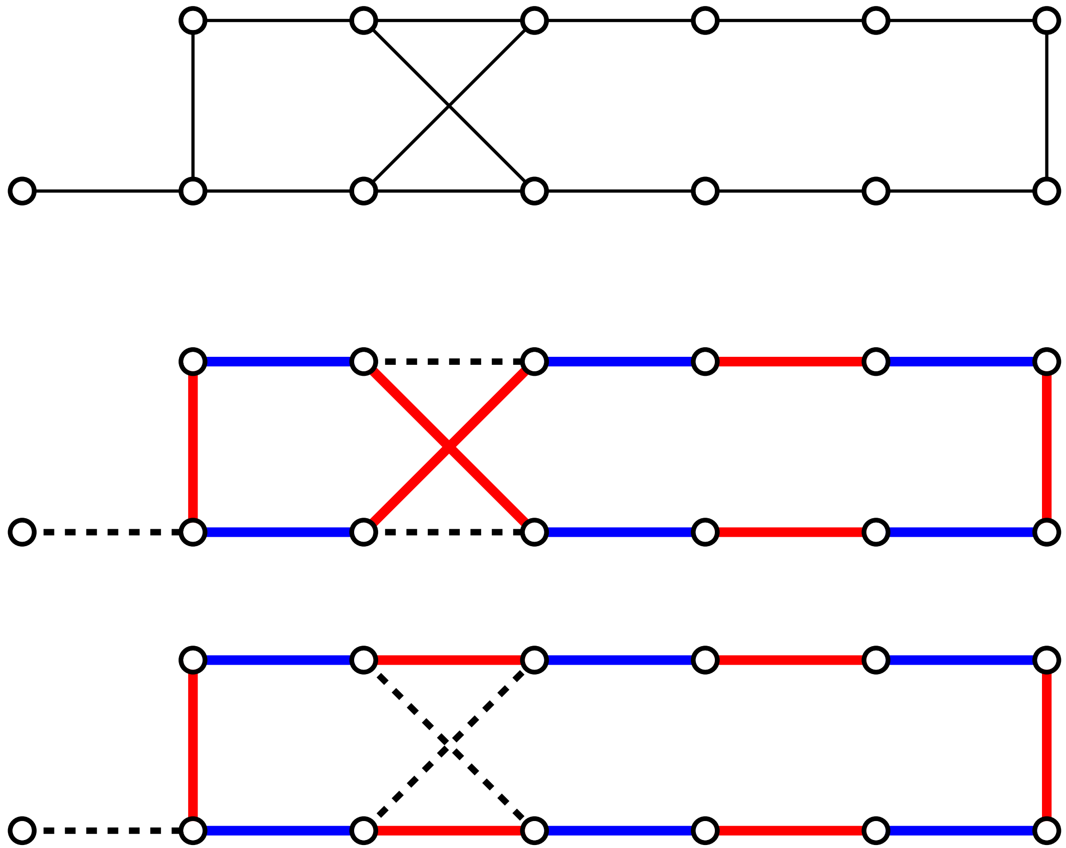


y_i=FALSE

Encodes binary Decisions

Pairs of Matchings and $\forall x_1, \ldots, x_k$



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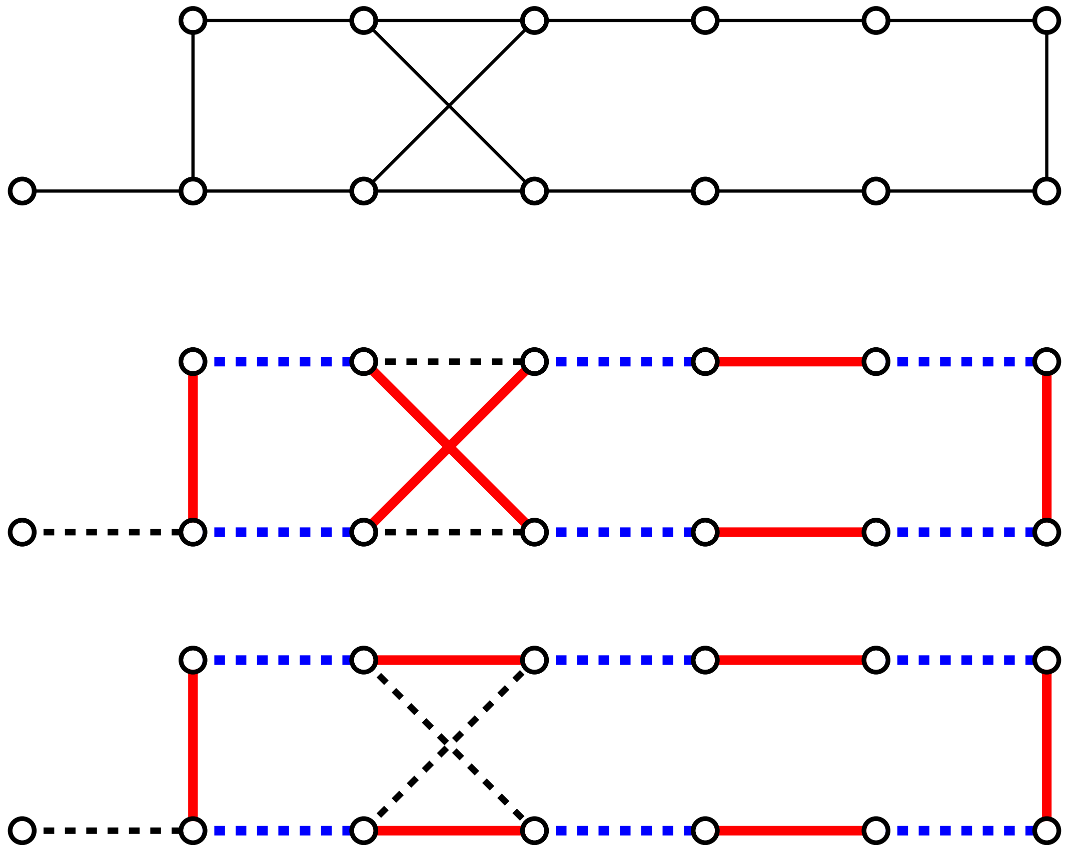

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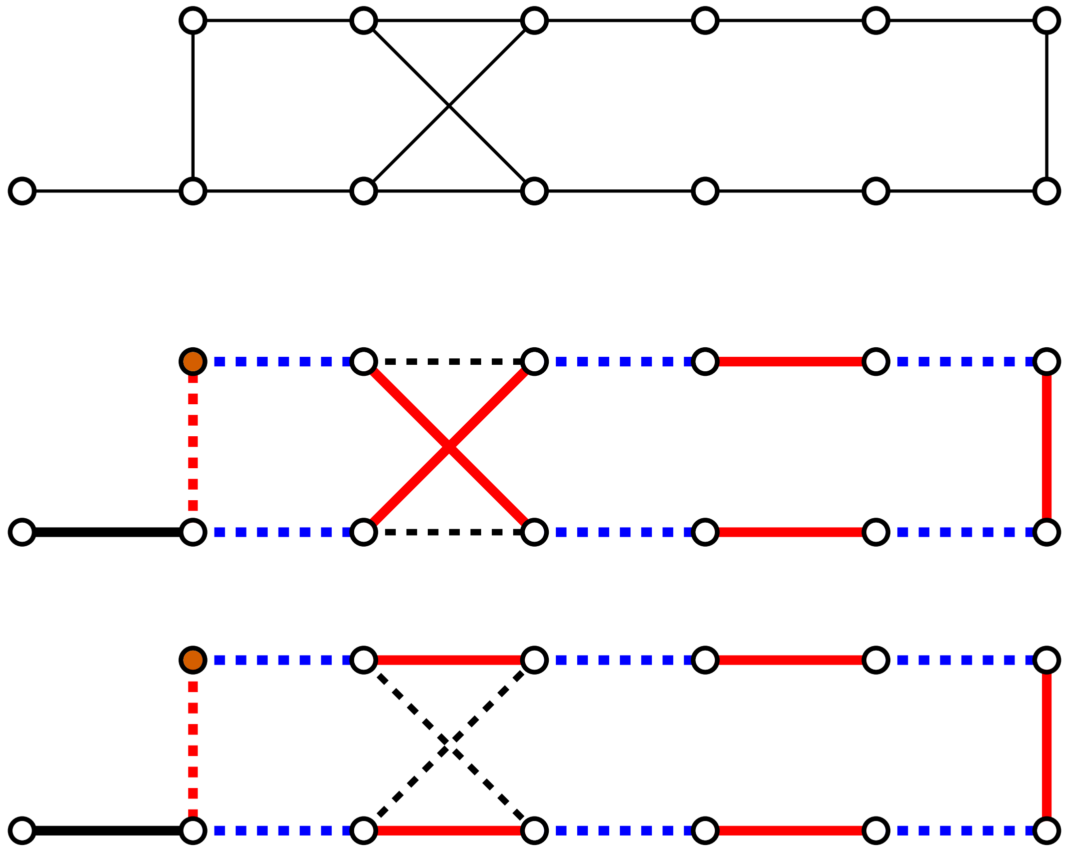

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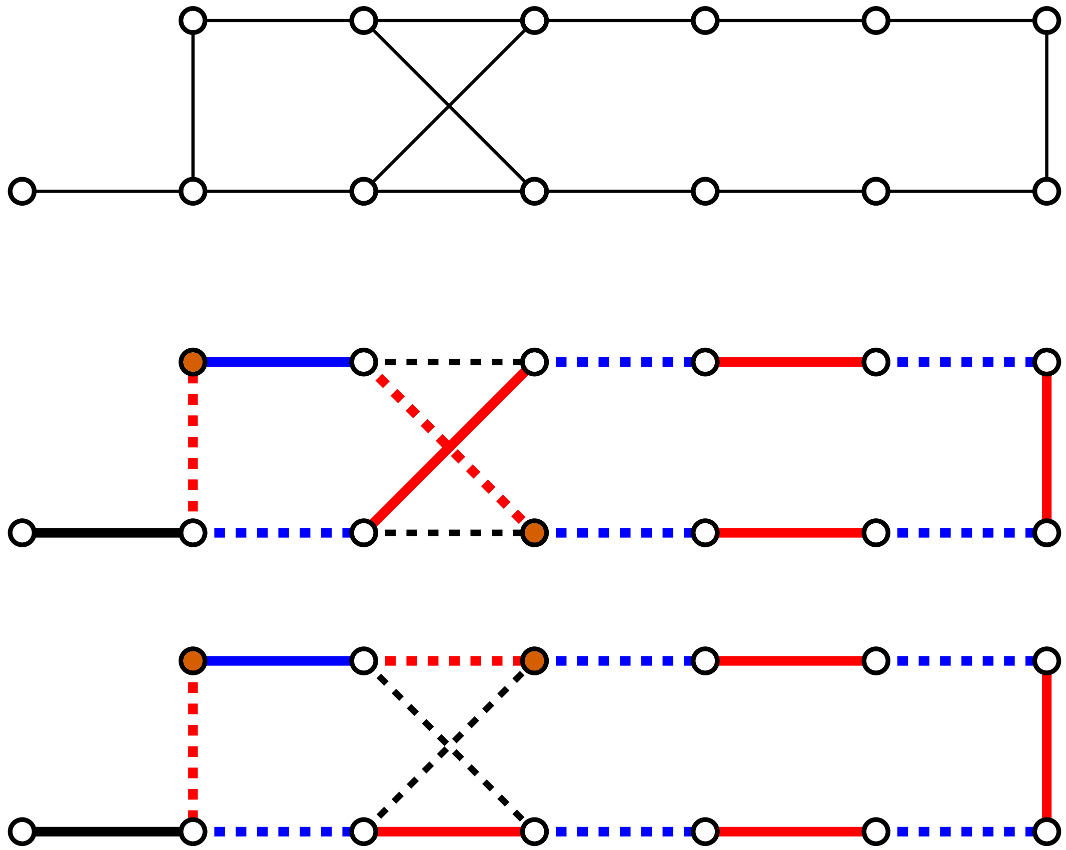

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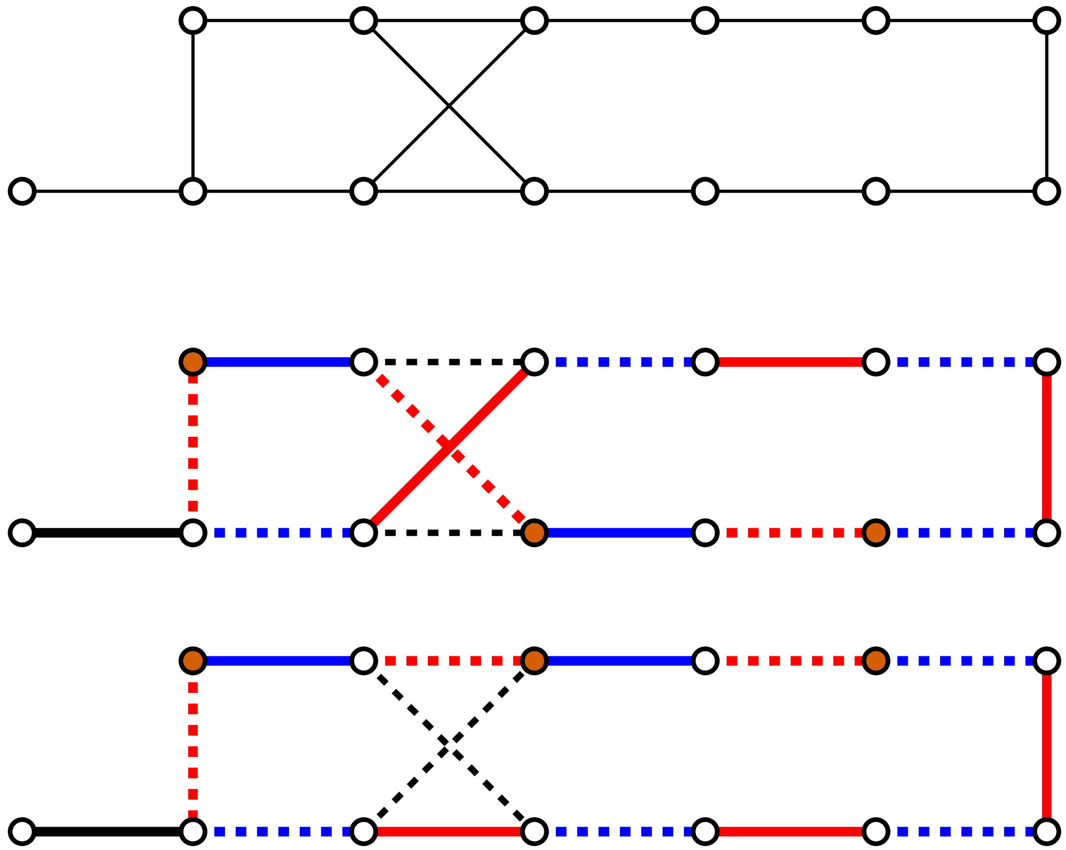

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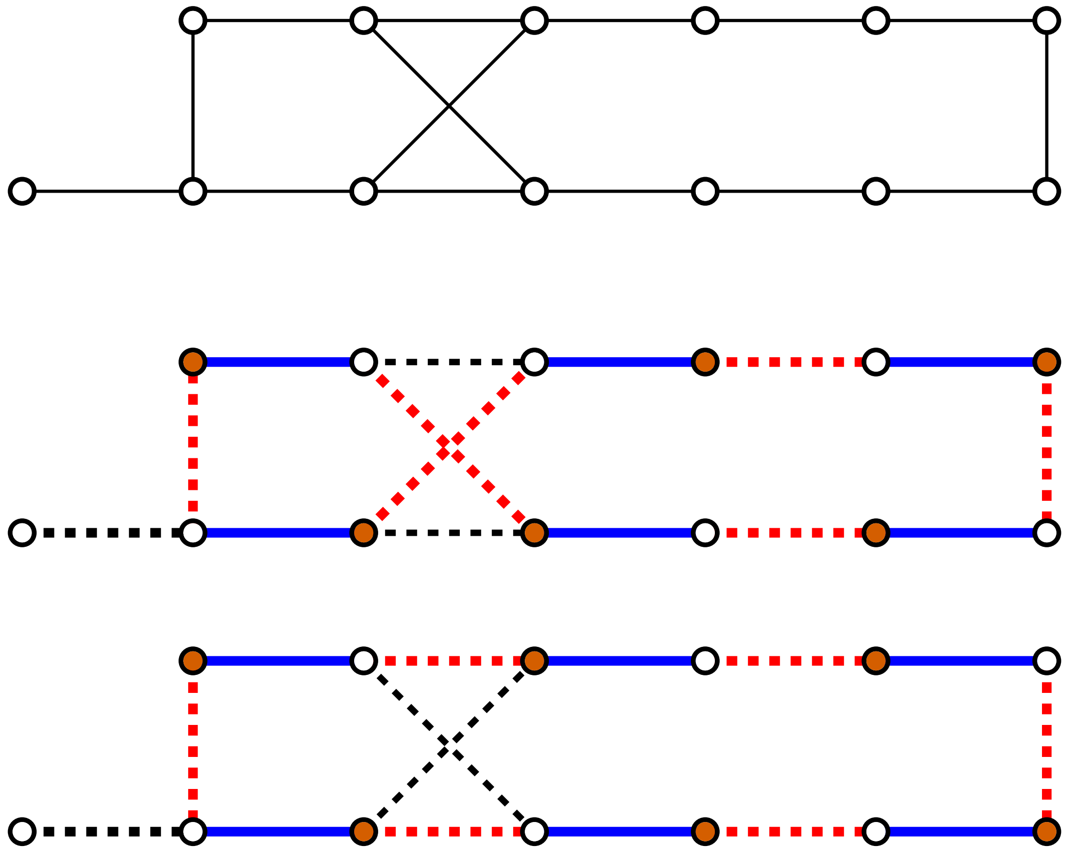

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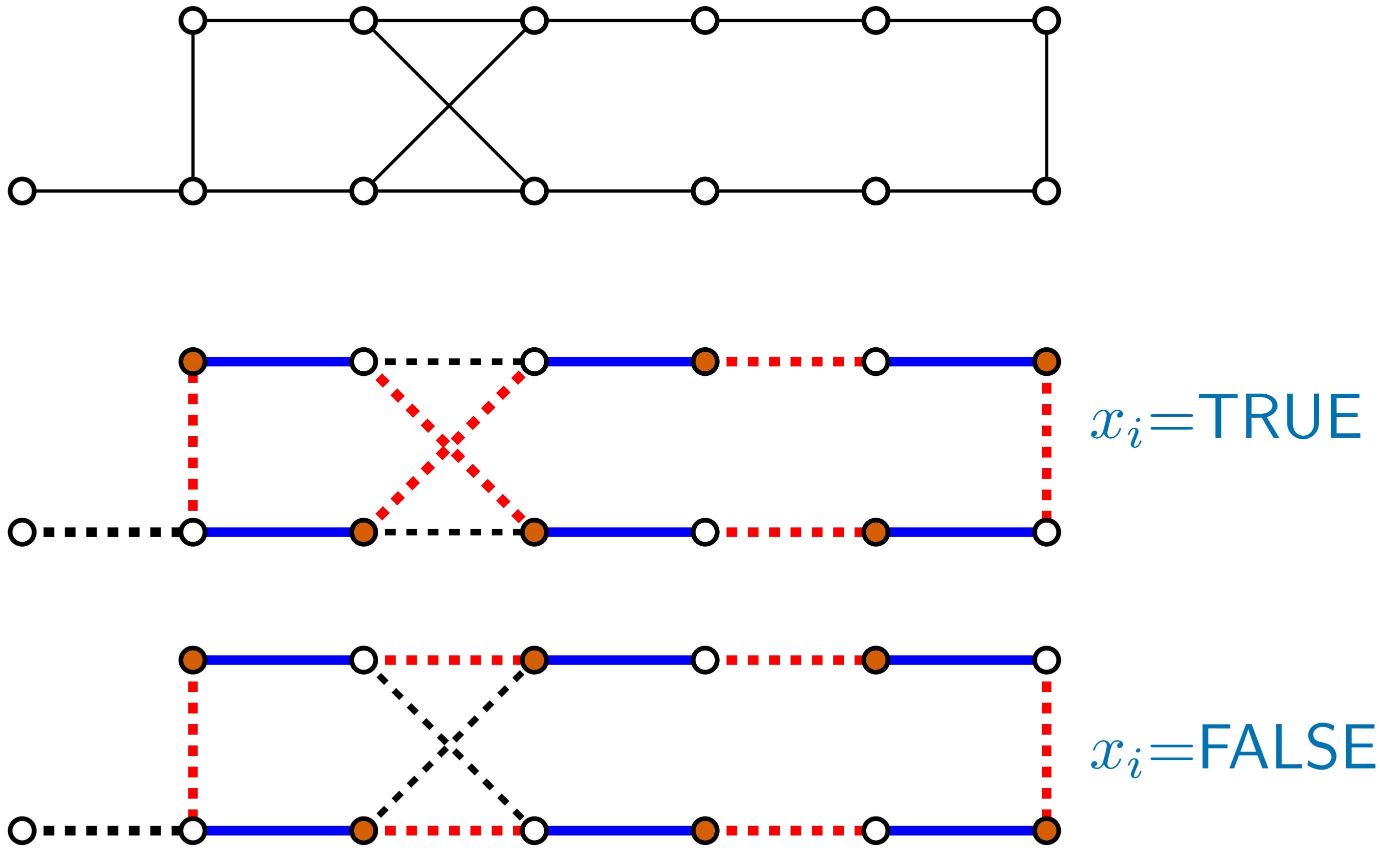

Encodes binary decision

The Reduction

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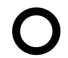
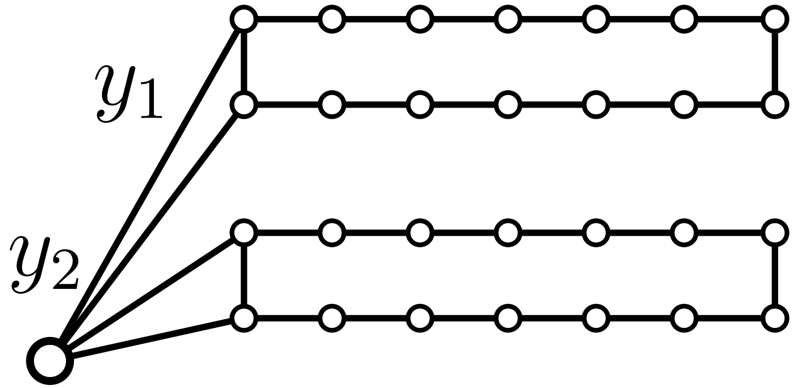
Start with a single vertex



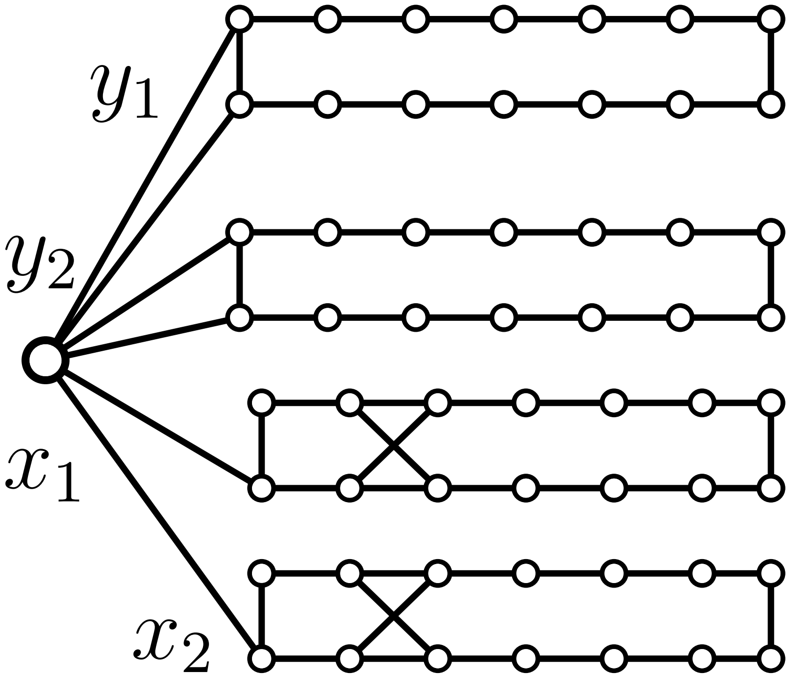
The Reduction

Add one gadget for each $\exists y$ variable



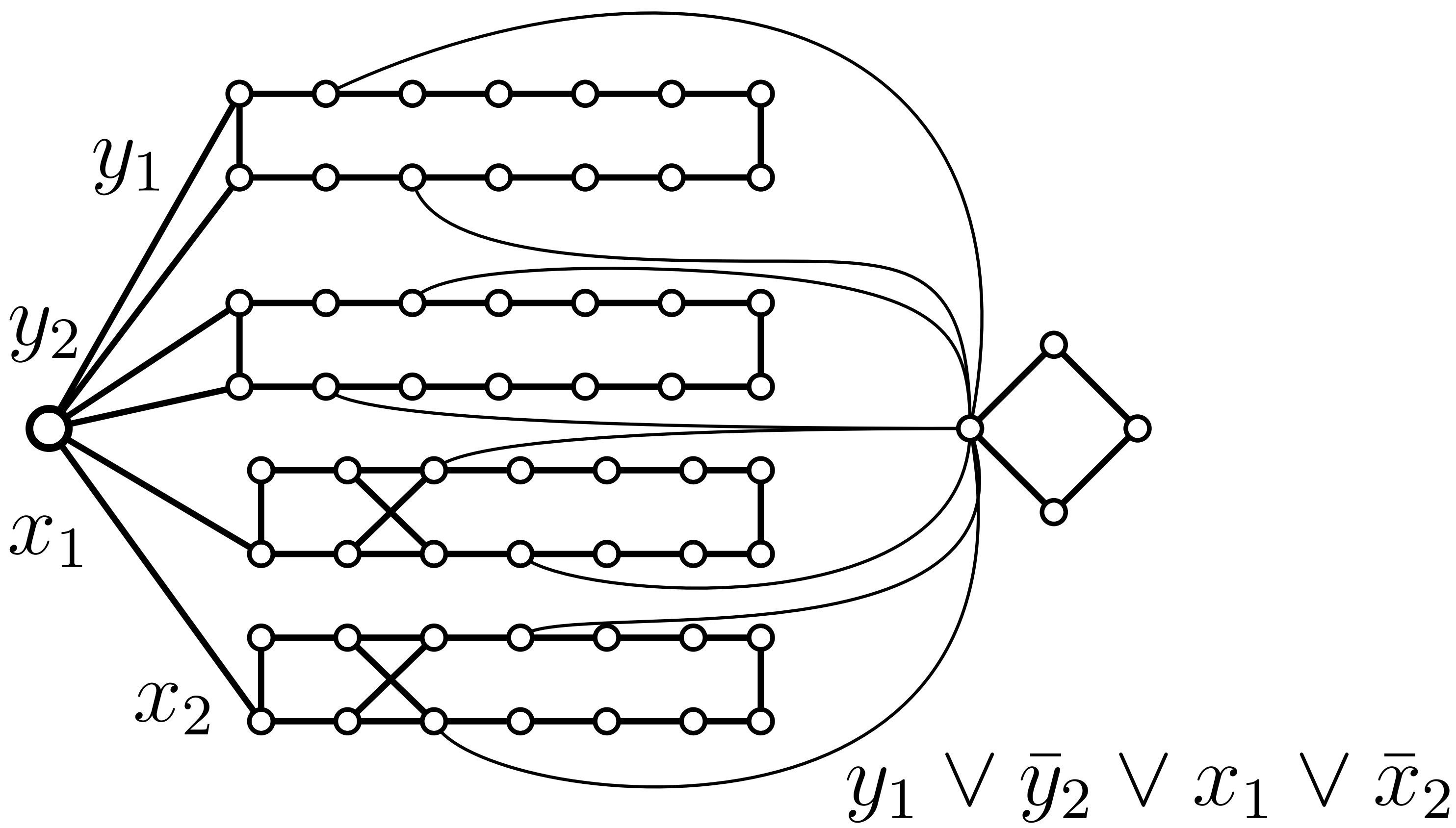
The Reduction

Add a gadget for each $\forall x$ variable



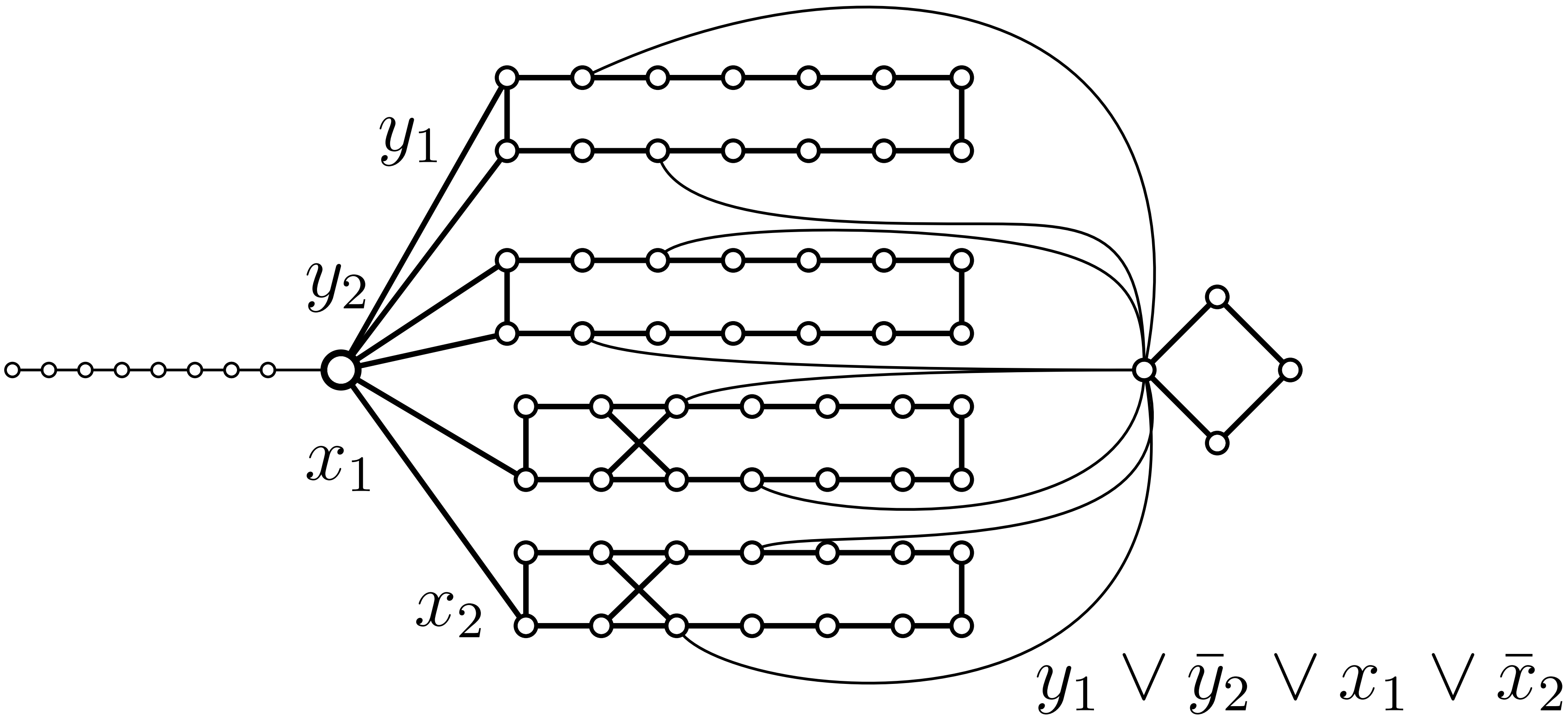
The Reduction

Add one gadget for every clause, and connect accordingly

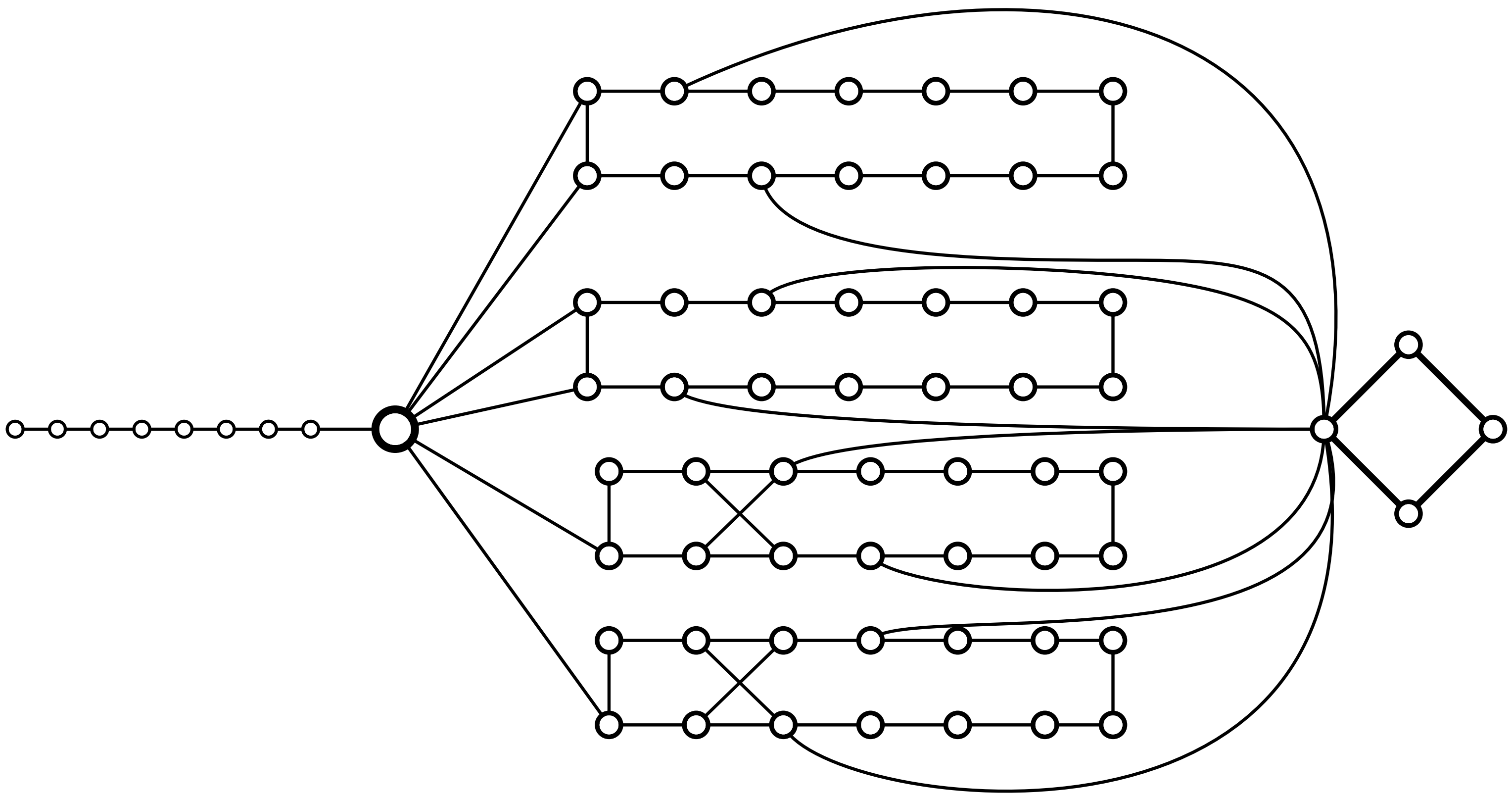


The Reduction

Add a tail

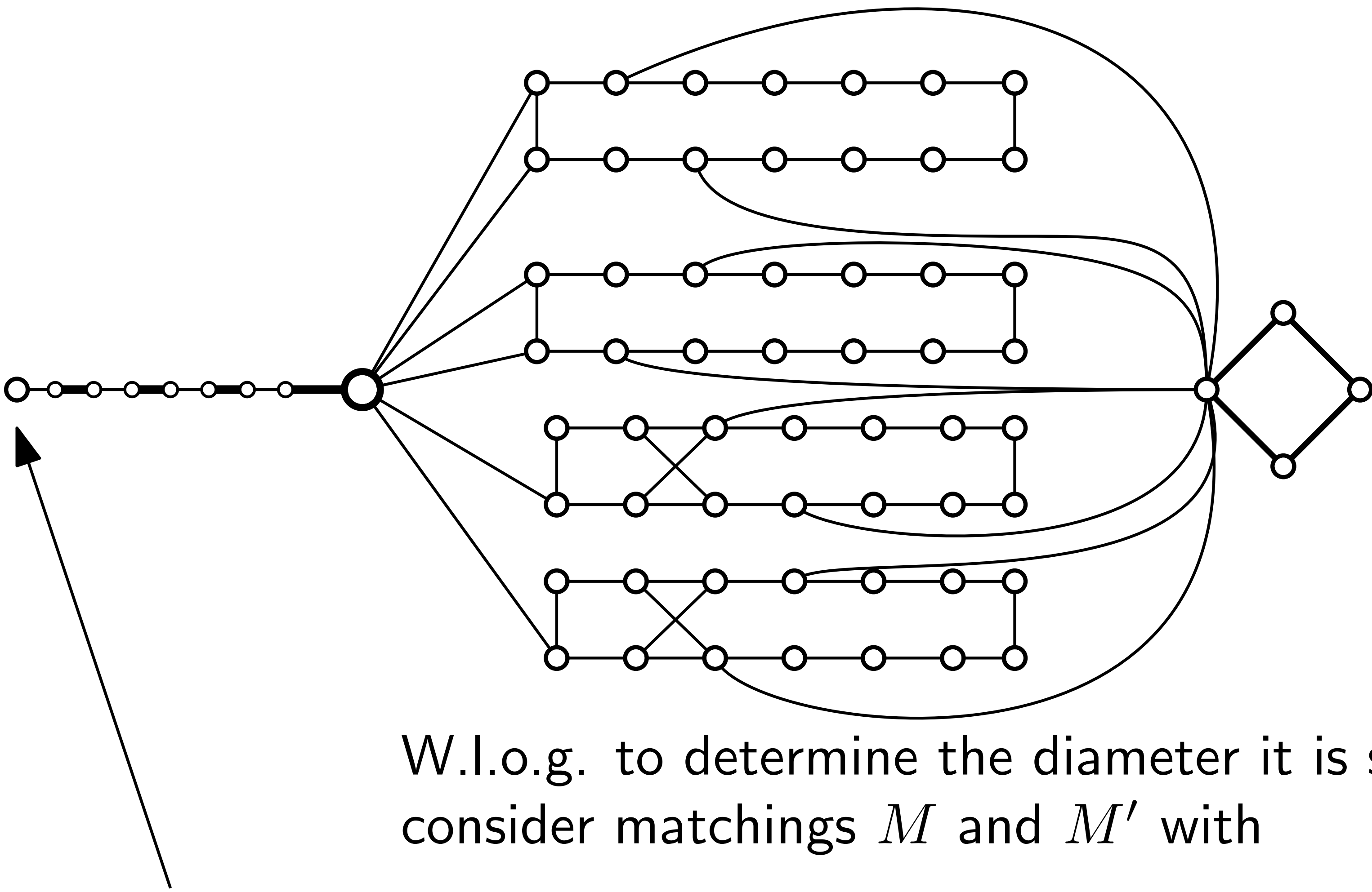


A Technical Lemma



W.l.o.g. to determine the diameter it is sufficient to consider matchings M and M' with

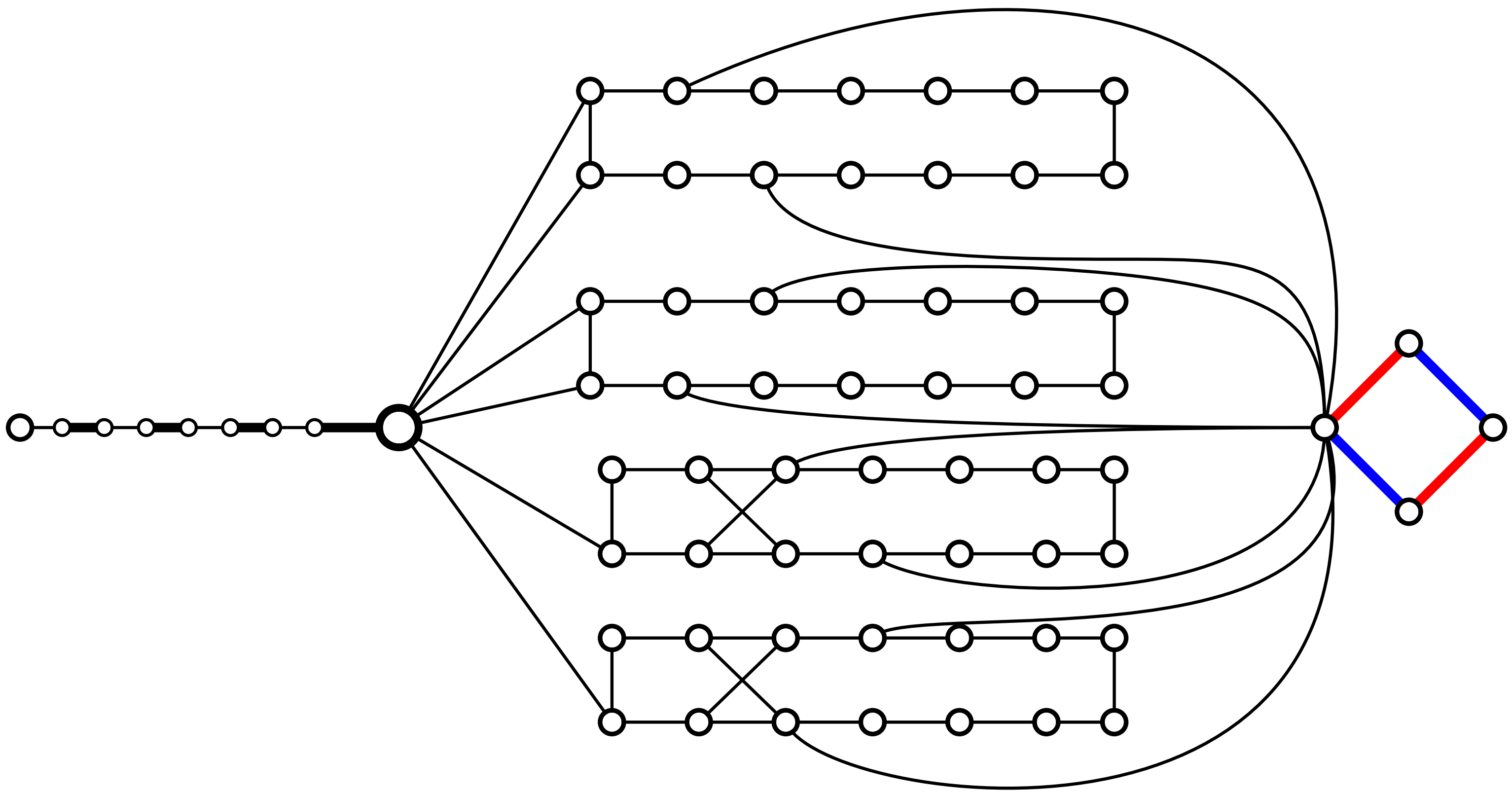
A Technical Lemma



W.l.o.g. to determine the diameter it is sufficient to consider matchings M and M' with

The isolated vertex here

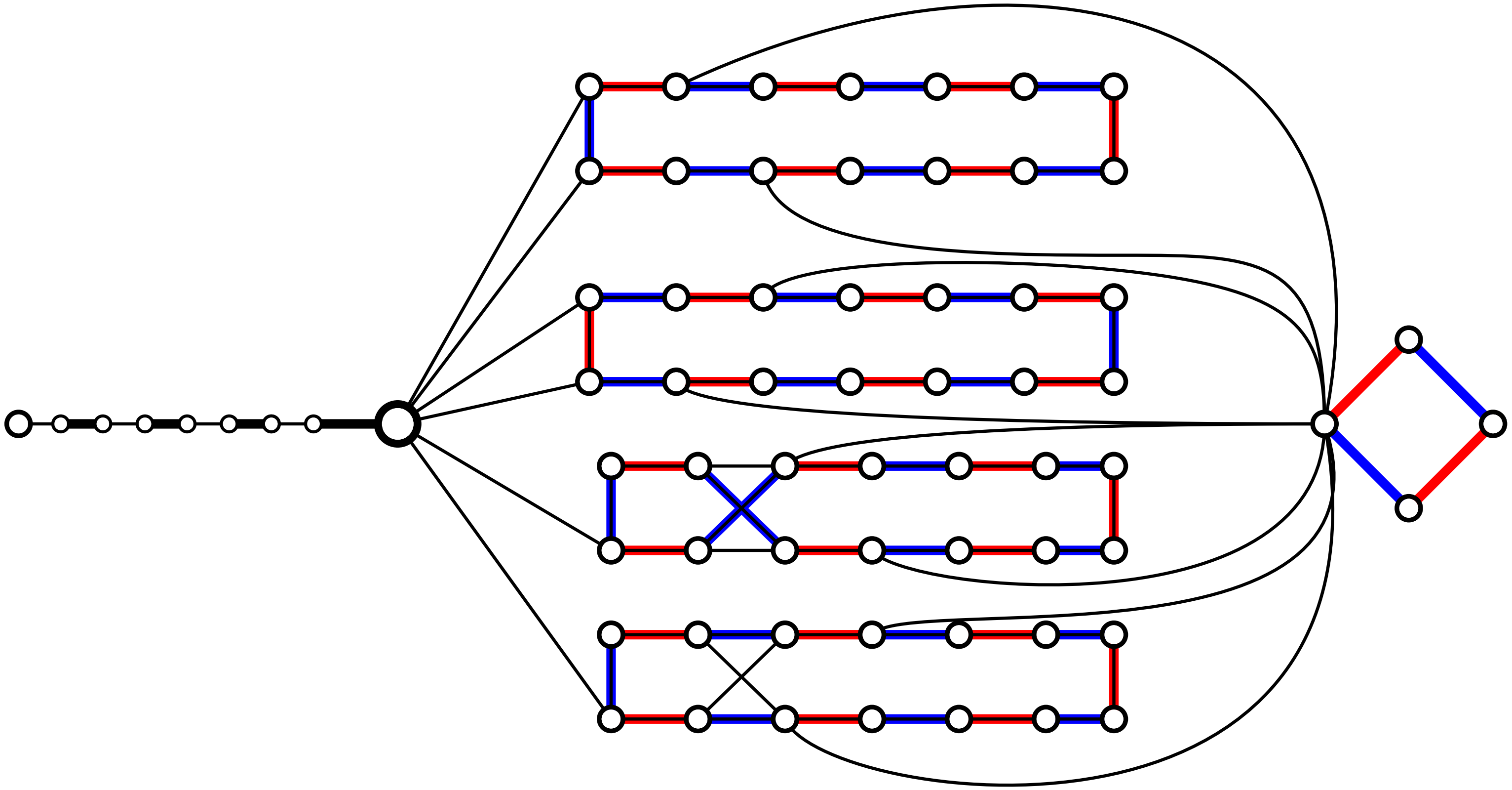
A Technical Lemma



W.l.o.g. to determine the diameter it is sufficient to consider matchings M and M' with

Alternating cycles in all clause gadgets

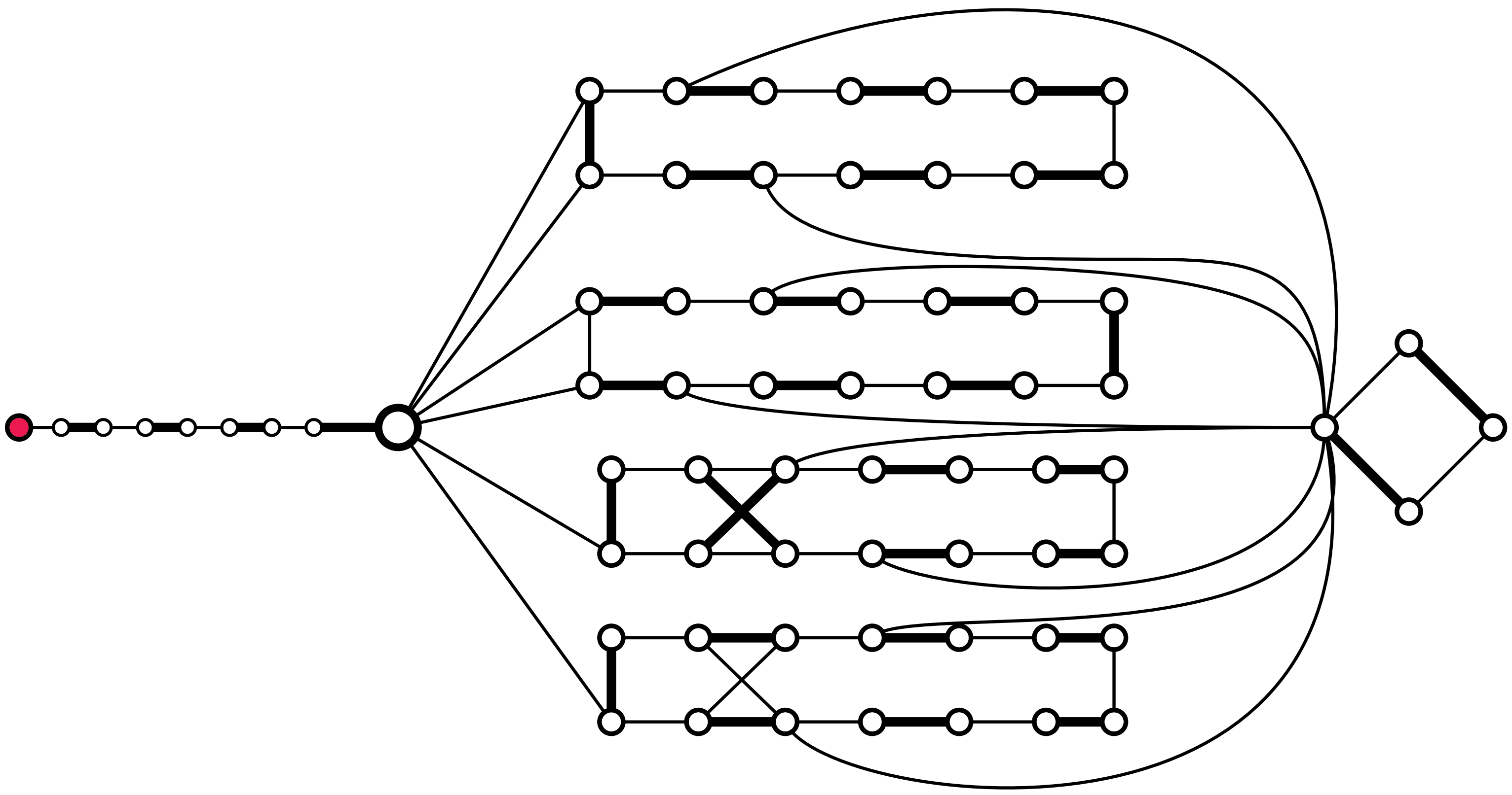
A Technical Lemma



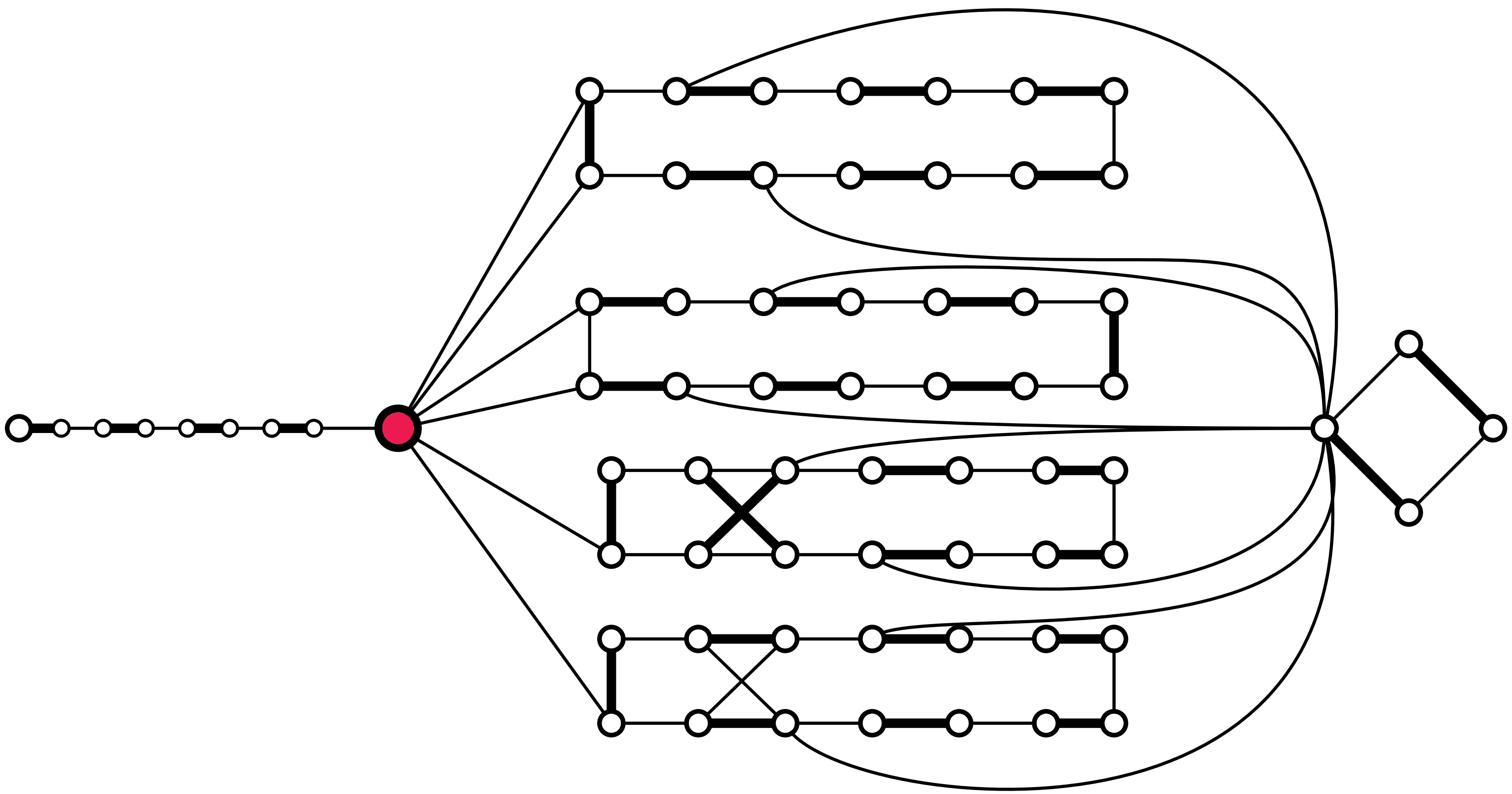
W.l.o.g. to determine the diameter it is sufficient to consider matchings M and M' with

Alternating cycles in all variable gadgets

Flip Sequences

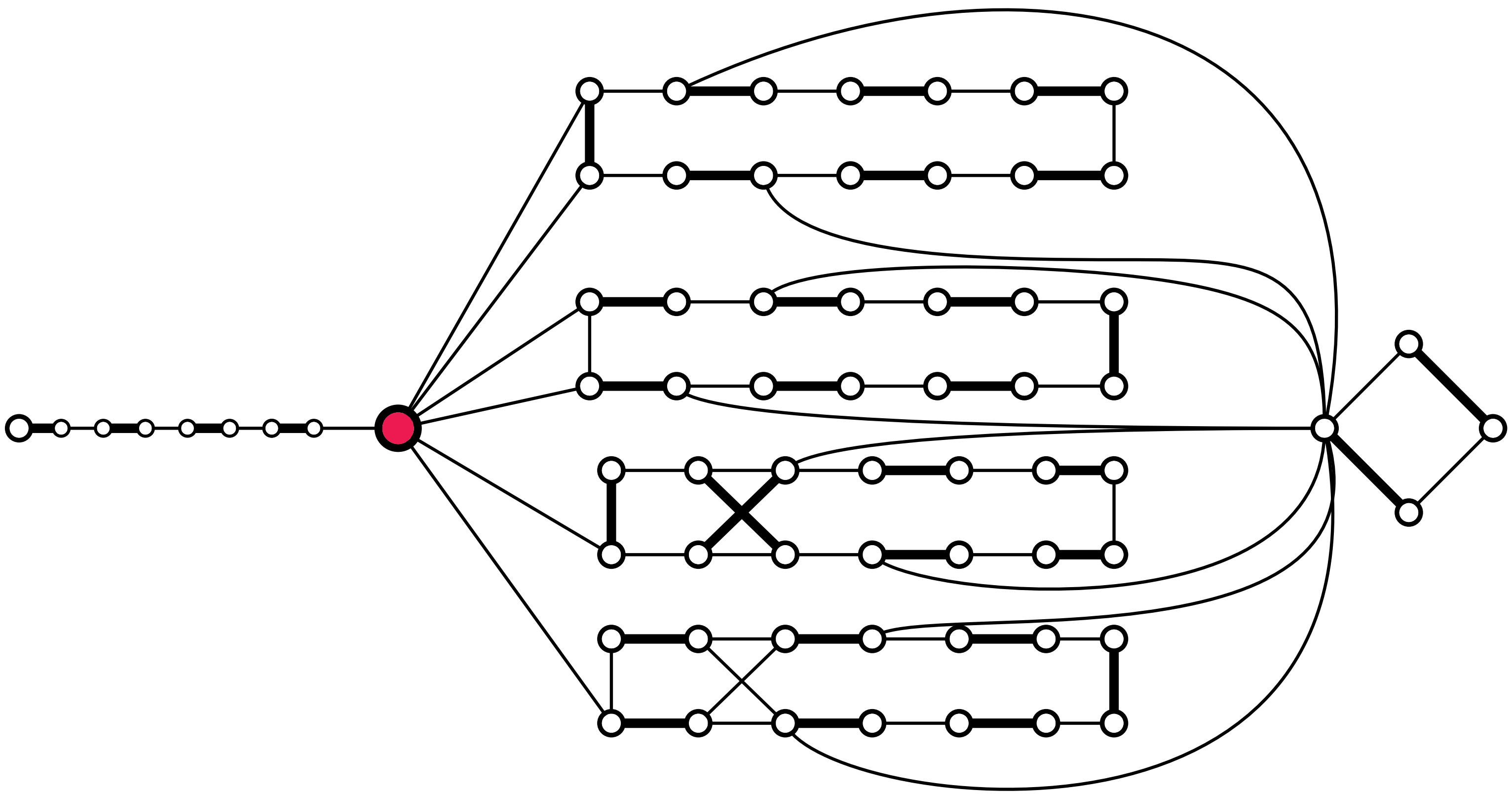


Flip Sequences



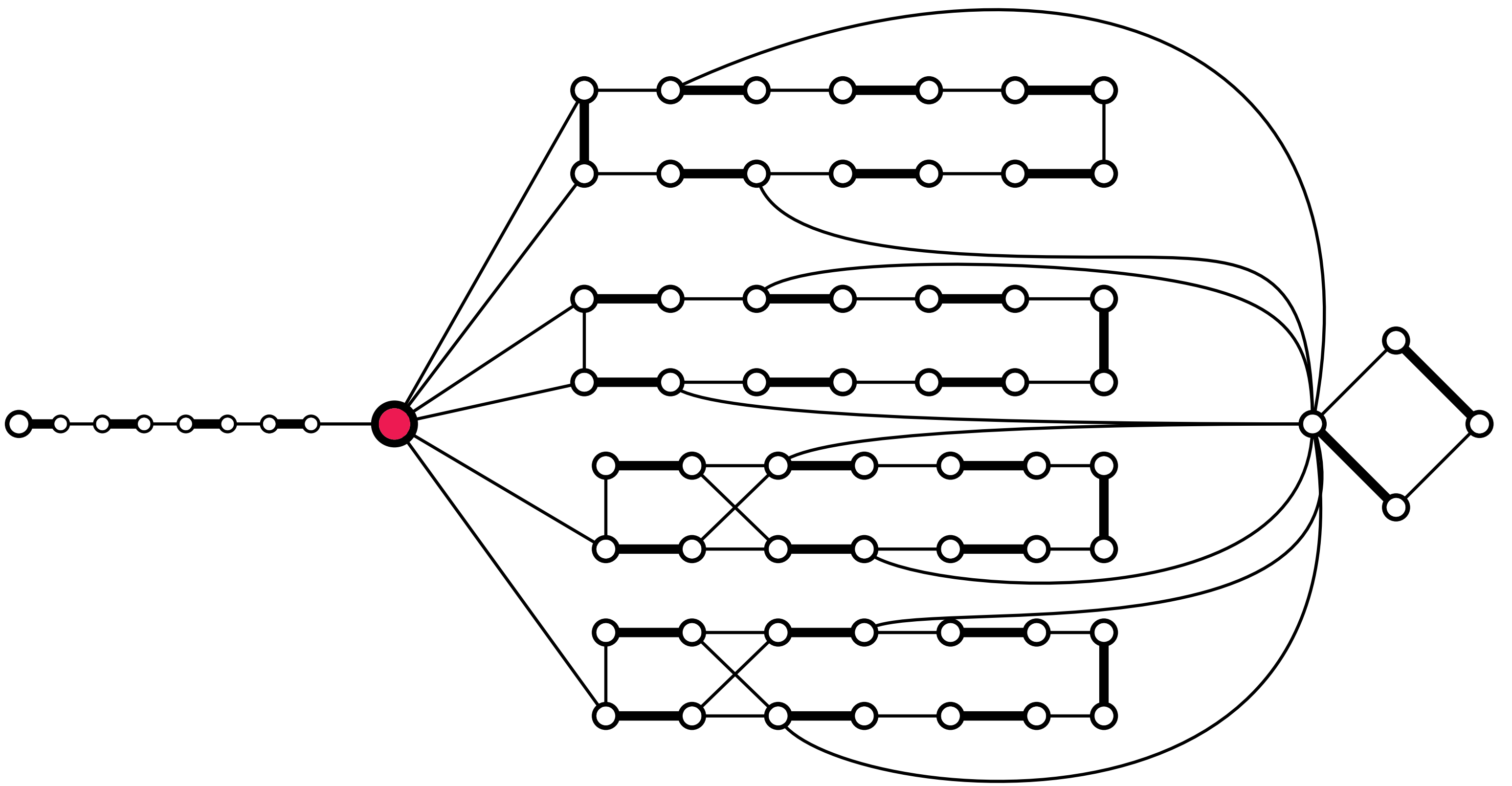
Step 1) Move isolated vertex

Flip Sequences



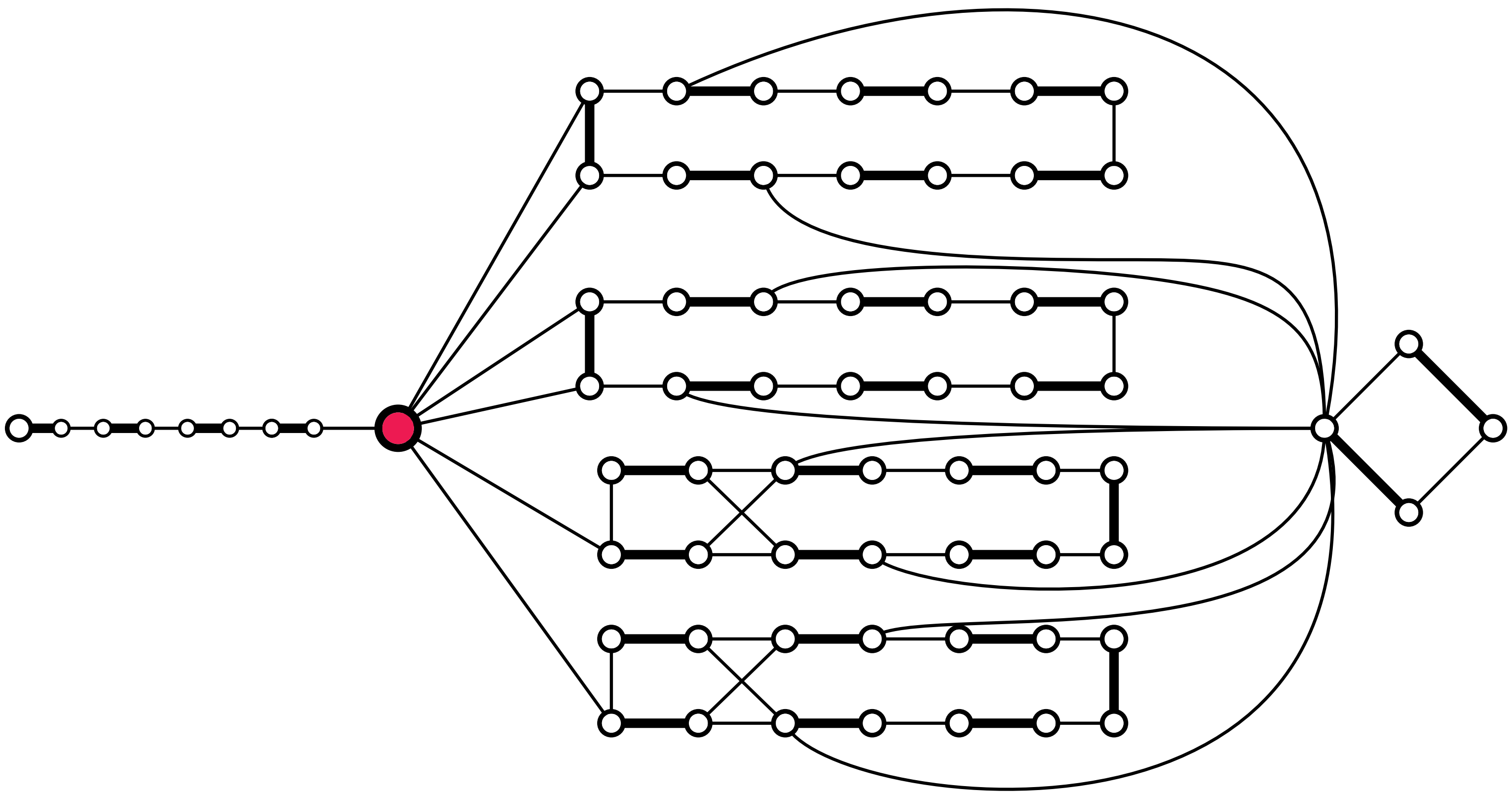
Step 2) Flip cycles in ∀ gadgets

Flip Sequences



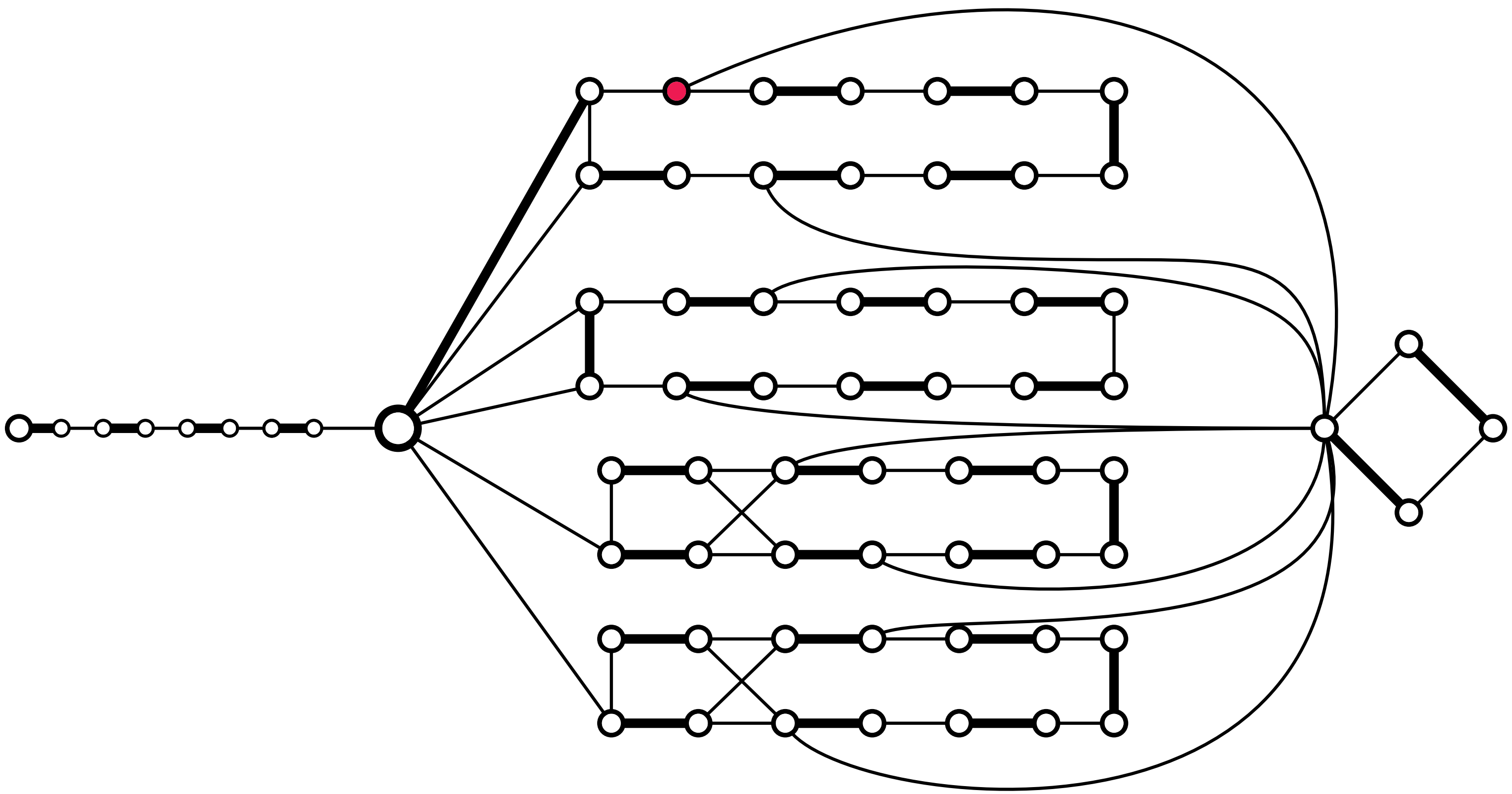
Step 2) Flip cycles in ∀ gadgets
If the isolated vertex is placed next to a clause gadget, switch the clause gadget. But here it isn't

Flip Sequences



Step 3) Switch the cycles of ∃ gadgets

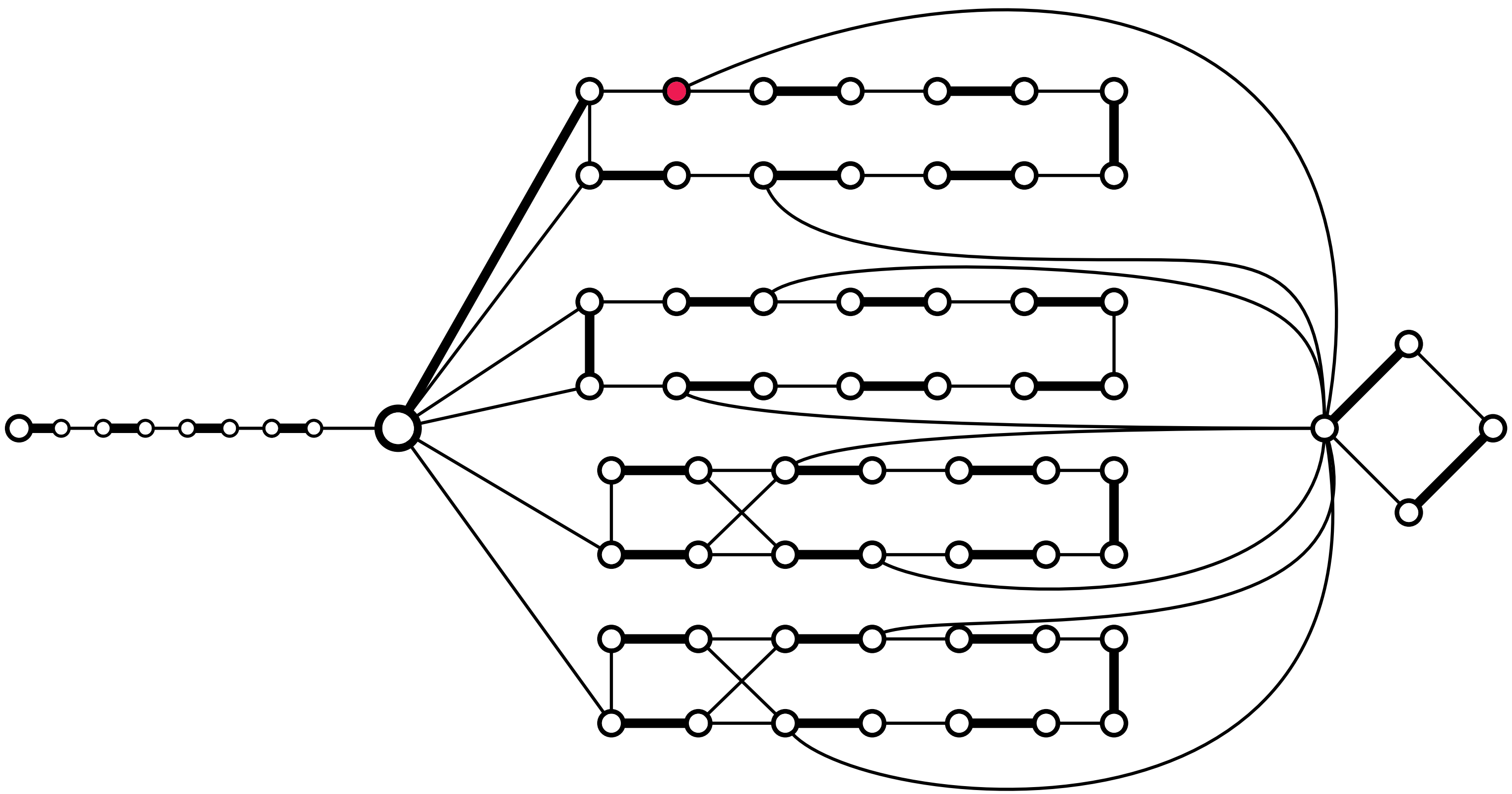
Flip Sequences



Step 3) Switch the cycles of ∃ gadgets

If the isolated vertex lies next to a clause gadget...

Flip Sequences

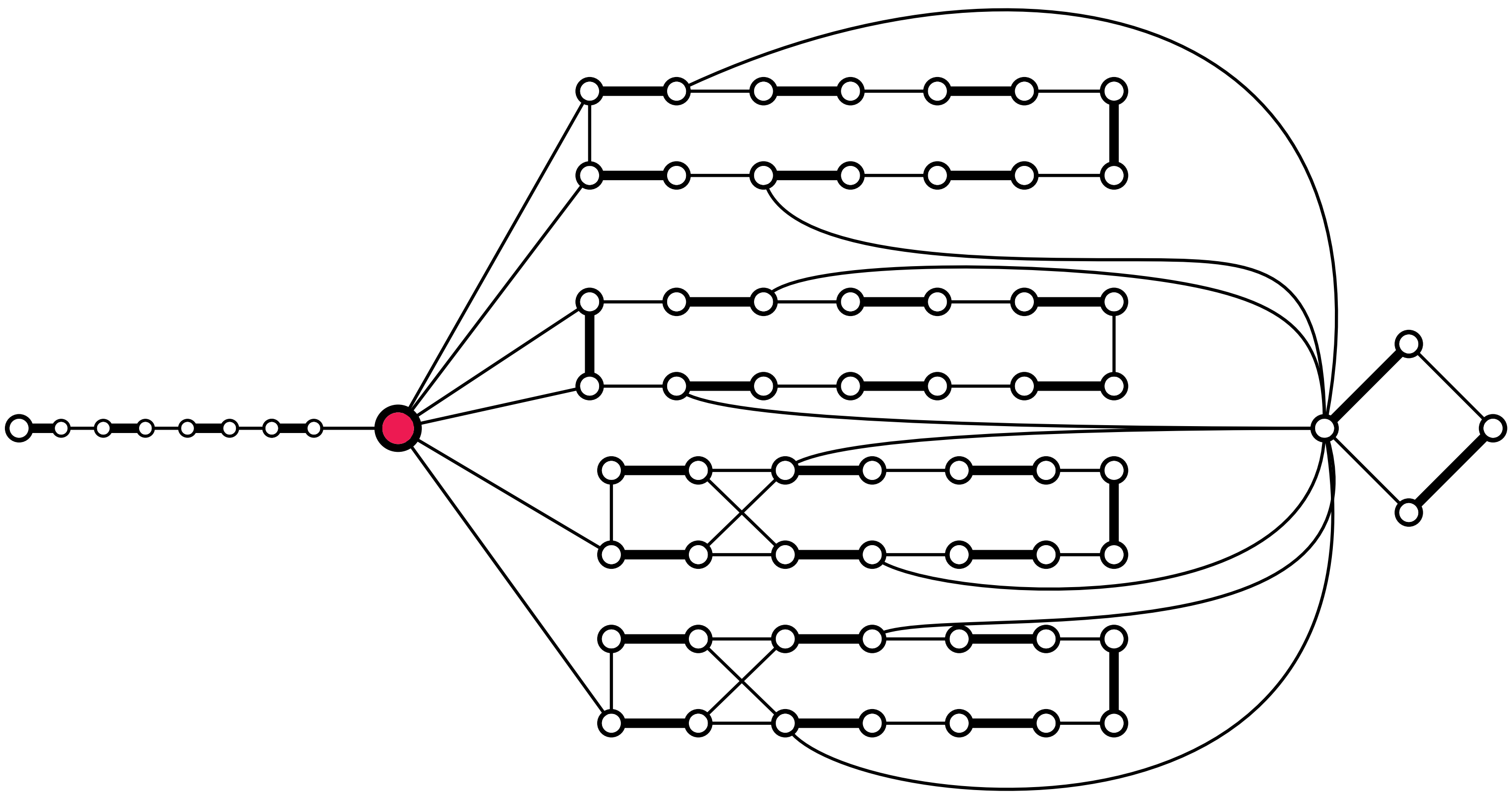


Step 3) Switch the cycles of ∃ gadgets

If the isolated vertex lies next to a clause gadget...

...switch the clause gadget

Flip Sequences

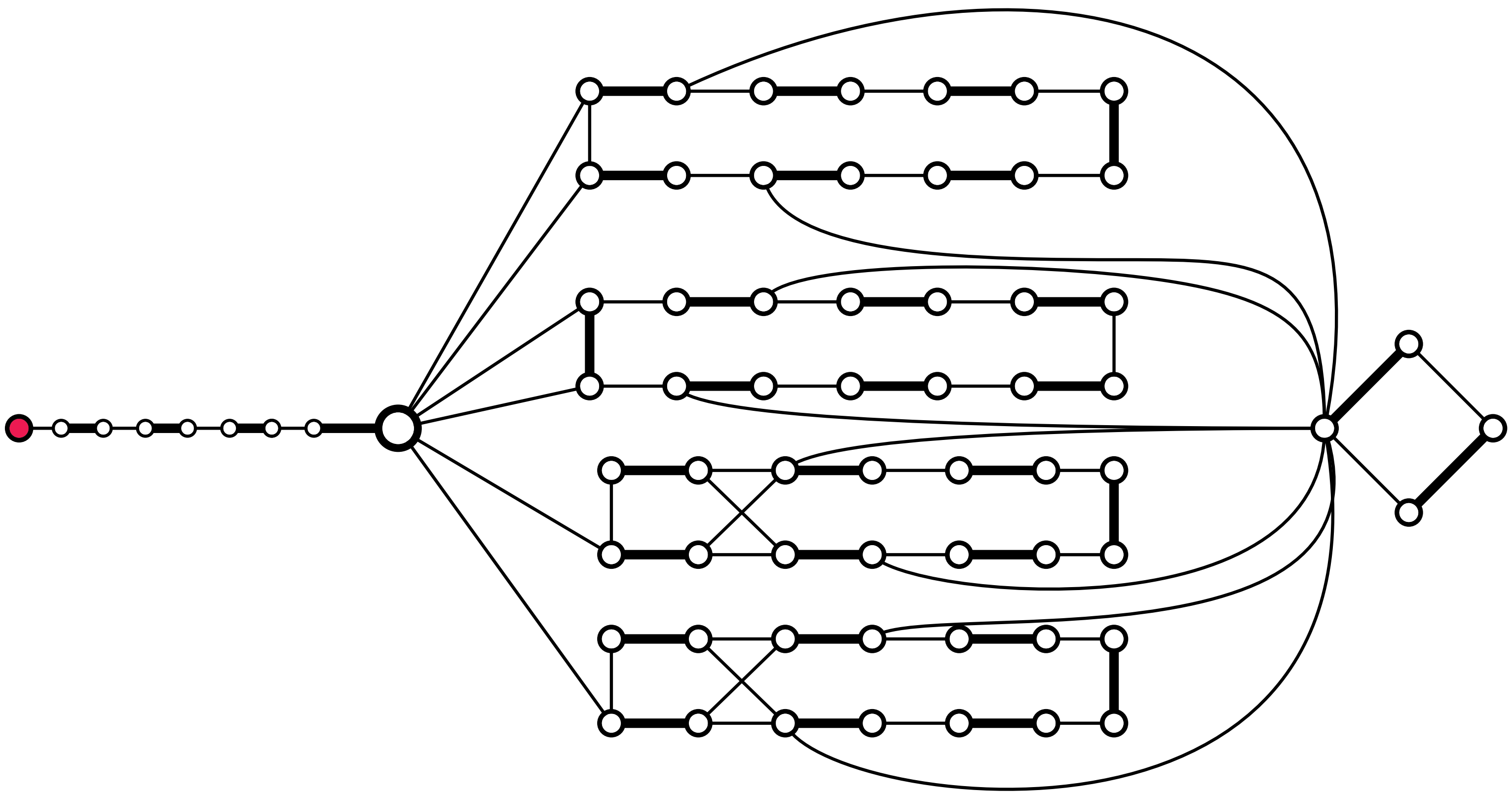


Step 3) Switch the cycles of ∃ gadgets

If the isolated vertex lies next to a clause gadget...

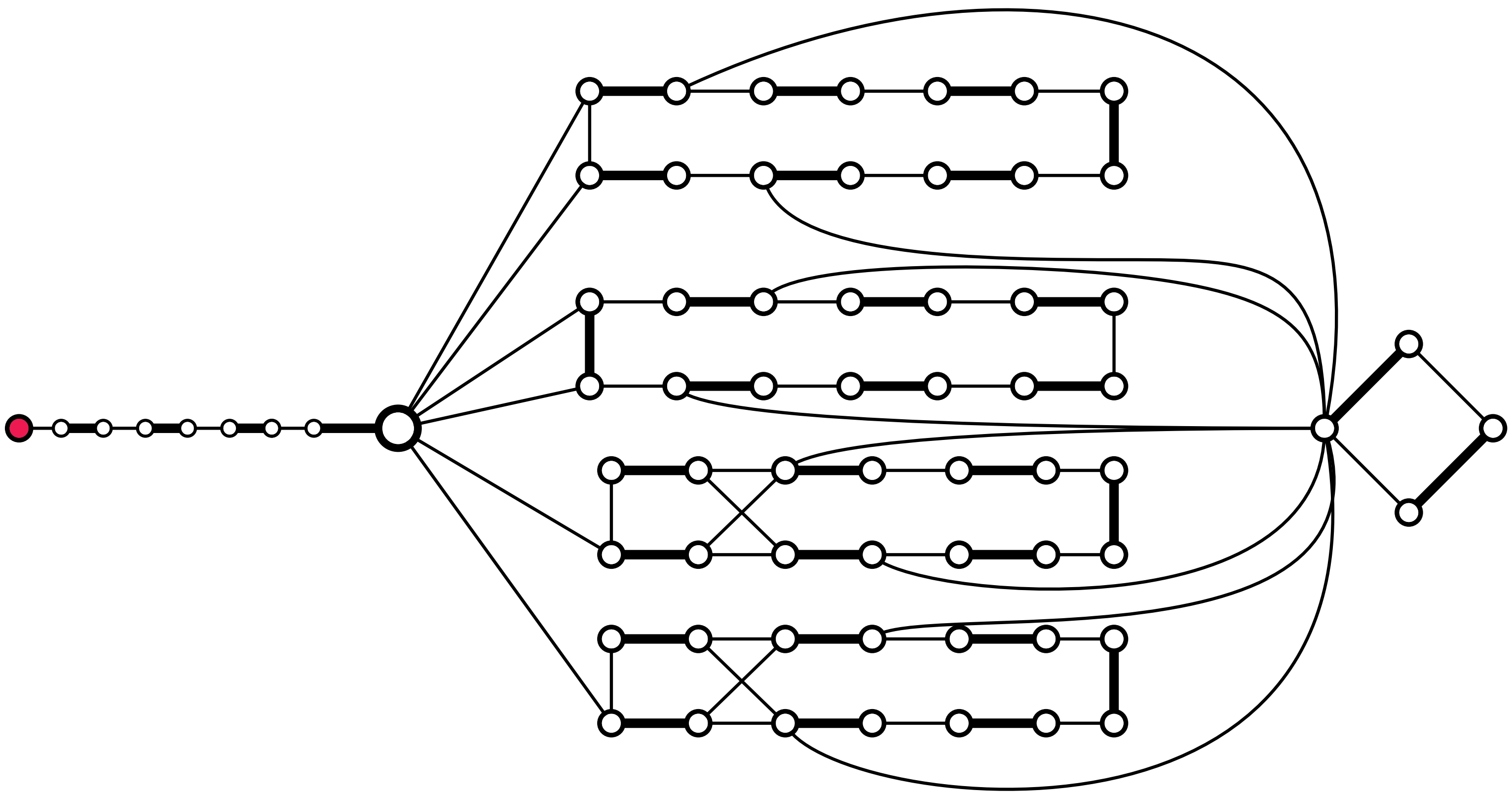
...switch the clause gadget

Flip Sequences



Step 4) Move the isolated vertex back

Flip Sequences



Step 4) Move the isolated vertex back

All clause gadgets get switched for any pair M, M'
$\Leftrightarrow \phi$ is an accepted $\forall\exists$ SAT-instance

Next step

Problem Description Given a graph G and an integer k. Is the radius of the flip graph of odd matchings of G at most k.

Diameter: $\min_M \max_{M'} d(M, M')$

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This is Σ_3^p-complete
NEW RESULT

Reduce from ∃∀∃-SAT

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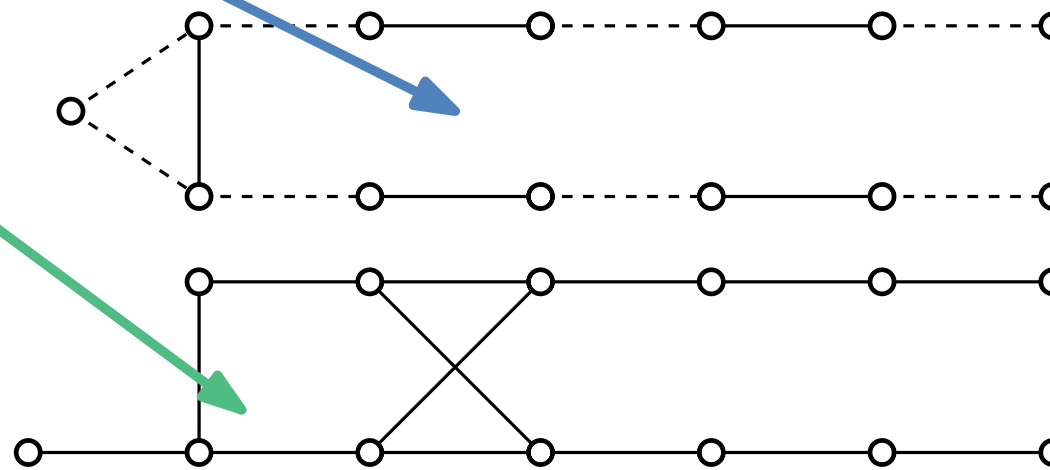
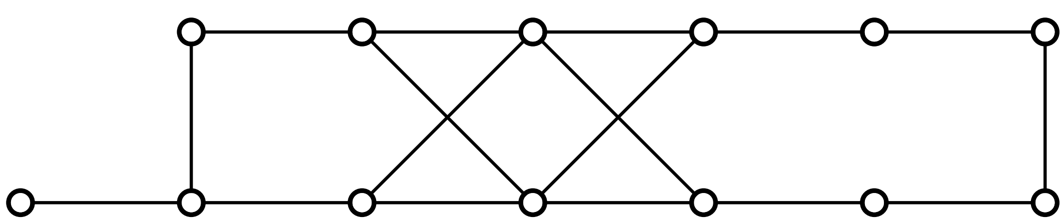
This is Σ_3^p-complete
NEW RESULT

Reduce from ∃∀∃-SAT
Uses an additional gadget





Thanks for your attention