Generalised Quantifiers based on Rabin-Mostowski Index

Denis Kuperberg, Damian Niwiński, Paweł Parys, Michał Skrzypczak

STACS 2026, Grenoble

UNIVERSITY OF WARSAW

Generalised Quantifiers based on Rabin-Mostowski Index

Denis Kuperberg, Damian Niwiński,
Paweł Parys, Michał Skrzypczak

STACS 2026, Grenoble

University of Warsaw

Setup

Setup

Infinite words w

Setup

Infinite words w

$$\Sigma^\omega \ni w\colon \{0, 1, \ldots\} \to \Sigma$$

Setup

Infinite words w

$$\Sigma^\omega \ni w\colon \{0, 1, \ldots\} \to \Sigma$$

$$w = abaabaaab\cdots$$

Setup

Infinite words w

$$\Sigma^\omega \ni w\colon \{0, 1, \ldots\} \to \Sigma$$

$$w = abaabaaab \cdots$$

Infinite trees t

Setup

Infinite words w

$$\Sigma^\omega \ni w\colon \{0,1,\ldots\} \to \Sigma$$

$$w = abaabaaab\cdots$$

Infinite trees t

$$\mathrm{Tr}_\Sigma \ni t\colon \{\mathtt{L},\mathtt{R}\}^* \to \Sigma$$

Setup

Infinite words w

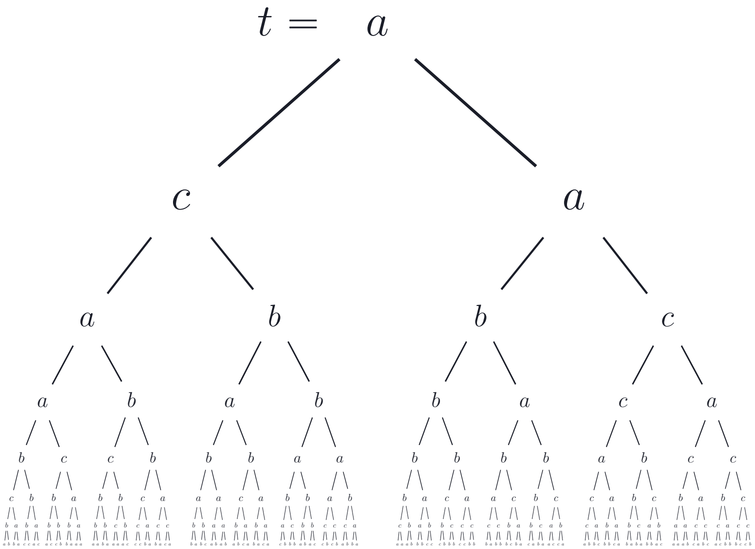
$$\Sigma^\omega \ni w\colon \{0, 1, \ldots\} \to \Sigma$$

$$w = abaabaaab\cdots$$

Infinite trees t

$$\mathrm{Tr}_\Sigma \ni t\colon \{\mathtt{L}, \mathtt{R}\}^* \to \Sigma$$

$t =$



Setup

Infinite words w

$$\Sigma^\omega \ni w\colon \{0,1,\ldots\} \to \Sigma$$

$$w = abaabaaab\cdots$$

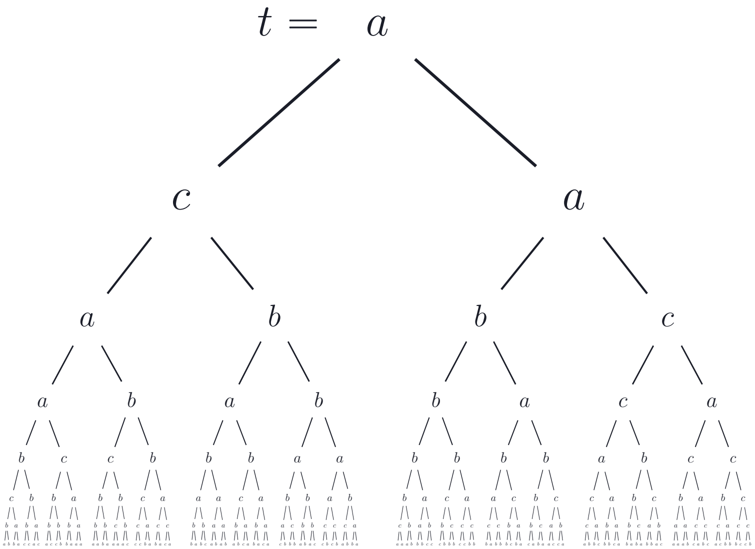
Monadic Second-Order logic

$$\psi ::= \neg\psi \mid \psi \vee \psi \mid \exists x.\ \psi \mid \exists X.\ \psi \mid \ldots$$

Infinite trees t

$$\mathrm{Tr}_\Sigma \ni t\colon \{\mathtt{L},\mathtt{R}\}^* \to \Sigma$$

$t =$



Setup

Infinite words w

$$\Sigma^\omega \ni w\colon \{0, 1, \ldots\} \to \Sigma$$

$$w = abaabaaab\cdots$$

Monadic Second-Order logic

$$\psi ::= \neg\psi \mid \psi \vee \psi \mid \exists x.\ \psi \mid \exists X.\ \psi \mid \ldots$$

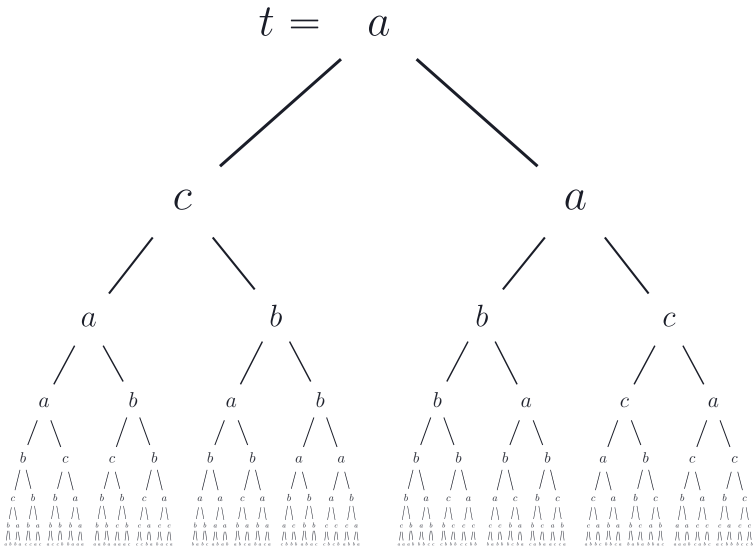
Theorem (J.R. Büchi [1961])

Satisfiability of **MSO** over **infinite words** is **decidable**.

Infinite trees t

$$\mathrm{Tr}_\Sigma \ni t\colon \{\mathtt{L}, \mathtt{R}\}^* \to \Sigma$$

$t =$



Setup

Infinite words w

$$\Sigma^\omega \ni w \colon \{0, 1, \ldots\} \to \Sigma$$

$$w = abaabaaab \cdots$$

Monadic Second-Order logic

$$\psi ::= \neg\psi \mid \psi \vee \psi \mid \exists x.\ \psi \mid \exists X.\ \psi \mid \ldots$$

Theorem (J.R. Büchi [1961])

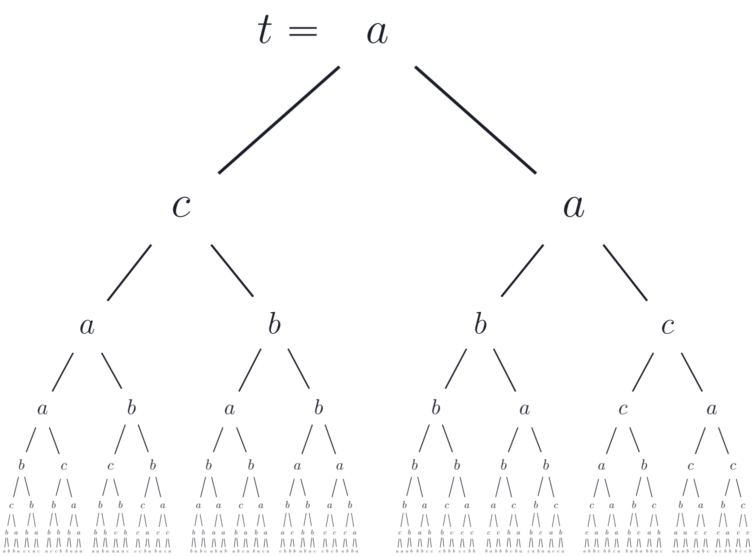
Satisfiability of **MSO** over **infinite words** is **decidable**.

Theorem (M.O. Rabin [1969])

Satisfiability of **MSO** over **infinite trees** is **decidable**.

Infinite trees t

$$\mathrm{Tr}_\Sigma \ni t \colon \{\mathtt{L}, \mathtt{R}\}^* \to \Sigma$$



Setup

Infinite words w

$$\Sigma^\omega \ni w \colon \{0, 1, \ldots\} \to \Sigma$$

$$w = abaabaaab \cdots$$

Monadic Second-Order logic

$$\psi ::= \neg\psi \mid \psi \vee \psi \mid \exists x.\ \psi \mid \exists X.\ \psi \mid \ldots$$

Theorem (J.R. Büchi [1961])

Satisfiability of **MSO** over **infinite words** is **decidable**.

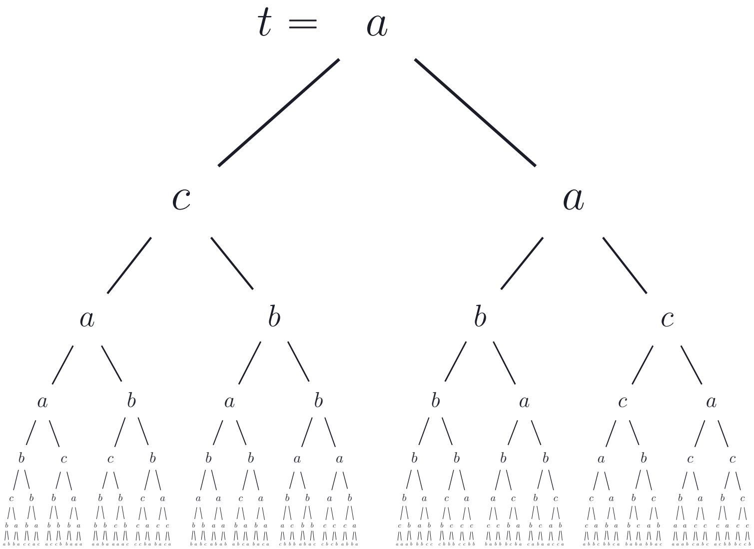
Theorem (M.O. Rabin [1969])

Satisfiability of **MSO** over **infinite trees** is **decidable**.

Infinite trees t

$$\mathrm{Tr}_\Sigma \ni t \colon \{\mathtt{L}, \mathtt{R}\}^* \to \Sigma$$

$t =$



Modern proofs:

Setup

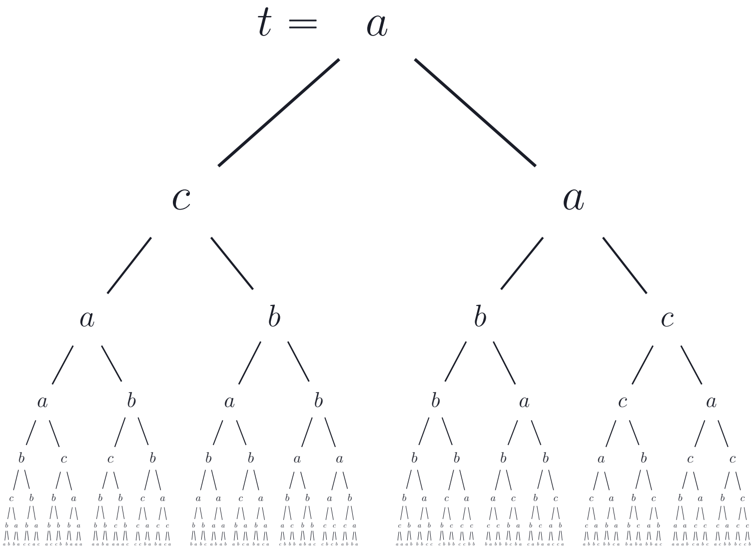
Infinite words w

$$\Sigma^\omega \ni w \colon \{0, 1, \ldots\} \to \Sigma$$

$$w = abaabaaab\cdots$$

Infinite trees t

$$\mathrm{Tr}_\Sigma \ni t \colon \{\mathtt{L}, \mathtt{R}\}^* \to \Sigma$$



Monadic Second-Order logic

$$\psi ::= \neg\psi \mid \psi \vee \psi \mid \exists x.\ \psi \mid \exists X.\ \psi \mid \ldots$$

Theorem (J.R. Büchi [1961])

Satisfiability of **MSO** over **infinite words** is **decidable**.

Theorem (M.O. Rabin [1969])

Satisfiability of **MSO** over **infinite trees** is **decidable**.

Modern proofs:

PARITY AUTOMATA

Parity condition

Parity condition

(E.A. Emerson, C.S. Jutla [1991])

(A.W. Mostowski [1991])

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i, j) = \{i, i{+}1, \ldots, j\}$

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i,j) = \{i, i+1, \ldots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \left\{ (k_0 k_1 \cdots) \in (i,j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2 \right\}$$

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i, j) = \{i, i{+}1, \ldots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \big\{(k_0 k_1 \cdots) \in (i, j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2\big\}$$

- **Büchi** is $(0, 1)$
- **coBüchi** is $(1, 2)$

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i, j) = \{i, i{+}1, \ldots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \left\{(k_0 k_1 \cdots) \in (i, j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2\right\}$$

- **Büchi** is $(0, 1)$
- **coBüchi** is $(1, 2)$

Parity automata

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i, j) = \{i, i+1, \ldots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \left\{(k_0 k_1 \cdots) \in (i, j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2\right\}$$

- **Büchi** is $(0, 1)$
- **coBüchi** is $(1, 2)$

Parity automata

[over infinite **words** / **trees**]

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i, j) = \{i, i{+}1, \ldots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \left\{(k_0 k_1 \cdots) \in (i, j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2\right\}$$

- **Büchi** is $(0, 1)$
- **coBüchi** is $(1, 2)$

Parity automata

[over infinite **words** / **trees**]

Priority mapping $\Omega\colon Q \to (i, j)$

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i, j) = \{i, i{+}1, \dots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \big\{(k_0 k_1 \cdots) \in (i,j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2\big\}$$

- **Büchi** is $(0, 1)$
- **coBüchi** is $(1, 2)$

Parity automata

[over infinite **words** / **trees**]

Priority mapping $\Omega\colon Q \to (i, j)$

A sequence $(q_0 q_1 \cdots) \in Q^\omega$ is **accepting** if

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i, j) = \{i, i+1, \ldots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \left\{(k_0 k_1 \cdots) \in (i, j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2\right\}$$

- **Büchi** is $(0, 1)$
- **coBüchi** is $(1, 2)$

Parity automata

[over infinite **words** / **trees**]

Priority mapping $\Omega \colon Q \to (i, j)$

A sequence $(q_0 q_1 \cdots) \in Q^\omega$ is **accepting** if $\big(\Omega(q_0)\Omega(q_1) \cdots\big) \in P_{(i,j)}$.

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i,j) = \{i, i{+}1, \ldots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \left\{(k_0 k_1 \cdots) \in (i,j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2\right\}$$

- **Büchi** is $(0,1)$
- **coBüchi** is $(1,2)$

Parity automata

[over infinite **words** / **trees**]

Priority mapping $\Omega\colon Q \to (i,j)$

A sequence $(q_0 q_1 \cdots) \in Q^\omega$ is **accepting** if $\big(\Omega(q_0)\Omega(q_1)\cdots\big) \in P_{(i,j)}$.

Parity-index problem

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i, j) = \{i, i{+}1, \ldots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \big\{(k_0 k_1 \cdots) \in (i,j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2\big\}$$

- **Büchi** is $(0, 1)$
- **coBüchi** is $(1, 2)$

Parity automata

[over infinite **words** / **trees**]

Priority mapping $\Omega \colon Q \to (i, j)$

A sequence $(q_0 q_1 \cdots) \in Q^\omega$ is **accepting** if $\big(\Omega(q_0)\Omega(q_1)\cdots\big) \in P_{(i,j)}$.

Parity-index problem

Given a **language** L and an **index** (i, j),

can L be **recognised** by a parity automaton of **index** (i, j)?

Parity condition

(E.A. Emerson, C.S. Jutla [1991])

(A.W. Mostowski [1991])

An **index** is $(i, j) = \{i, i{+}1, \ldots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \Big\{(k_0 k_1 \cdots) \in (i, j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2\Big\}$$

- **Büchi** is $(0, 1)$
- **coBüchi** is $(1, 2)$

Parity automata

[over infinite **words** / **trees**]

Priority mapping $\Omega\colon Q \to (i, j)$

A sequence $(q_0 q_1 \cdots) \in Q^\omega$ is **accepting** if $\big(\Omega(q_0)\Omega(q_1)\cdots\big) \in P_{(i,j)}$.

Parity-index problem

Given a **language** L and an **index** (i, j),

can L be **recognised** by a parity automaton of **index** (i, j)?

⇝ **decision problem** about **semantic property** of languages

Parity condition

(E.A. Emerson, C.S. Jutla [1991])
(A.W. Mostowski [1991])

An **index** is $(i, j) = \{i, i{+}1, \ldots, j\}$

$$P_{(i,j)} \stackrel{\text{def}}{=} \big\{(k_0 k_1 \cdots) \in (i, j)^\omega \mid \liminf_{n\to\infty} k_n \equiv 0 \pmod 2\big\}$$

- **Büchi** is $(0, 1)$
- **coBüchi** is $(1, 2)$

Parity automata

[over infinite **words** / **trees**]

Priority mapping $\Omega\colon Q \to (i, j)$

A sequence $(q_0 q_1 \cdots) \in Q^\omega$ is **accepting** if $\big(\Omega(q_0)\Omega(q_1)\cdots\big) \in P_{(i,j)}$.

Parity-index problem

Given a **language** L and an **index** (i, j),

can L be **recognised** by a parity automaton of **index** (i, j)?

⇝ **decision problem** about **semantic property** of languages

IDEA: make it **harder** by **generalising** it into a **QUANTIFIER**

Generalised quantifiers

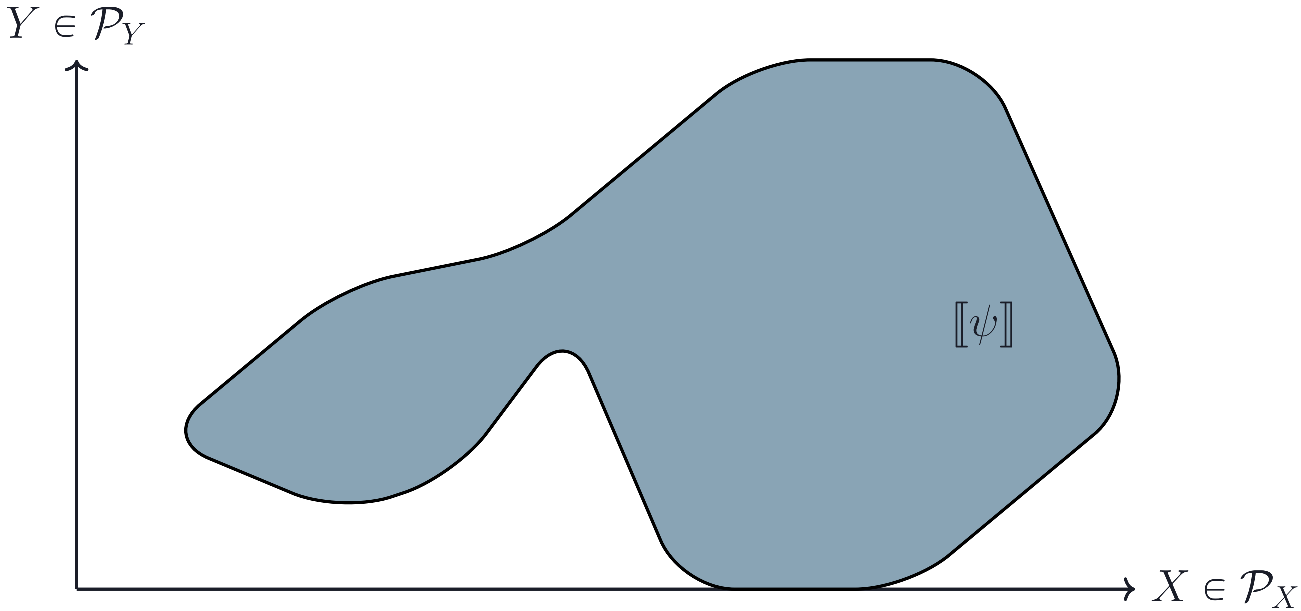
Generalised quantifiers (A. Mostowski [1957])

Generalised quantifiers (A. Mostowski [1957])

$$\varphi(X) \equiv \mathtt{Q}Y.\, \psi(X, Y)$$

Generalised quantifiers (A. Mostowski [1957])

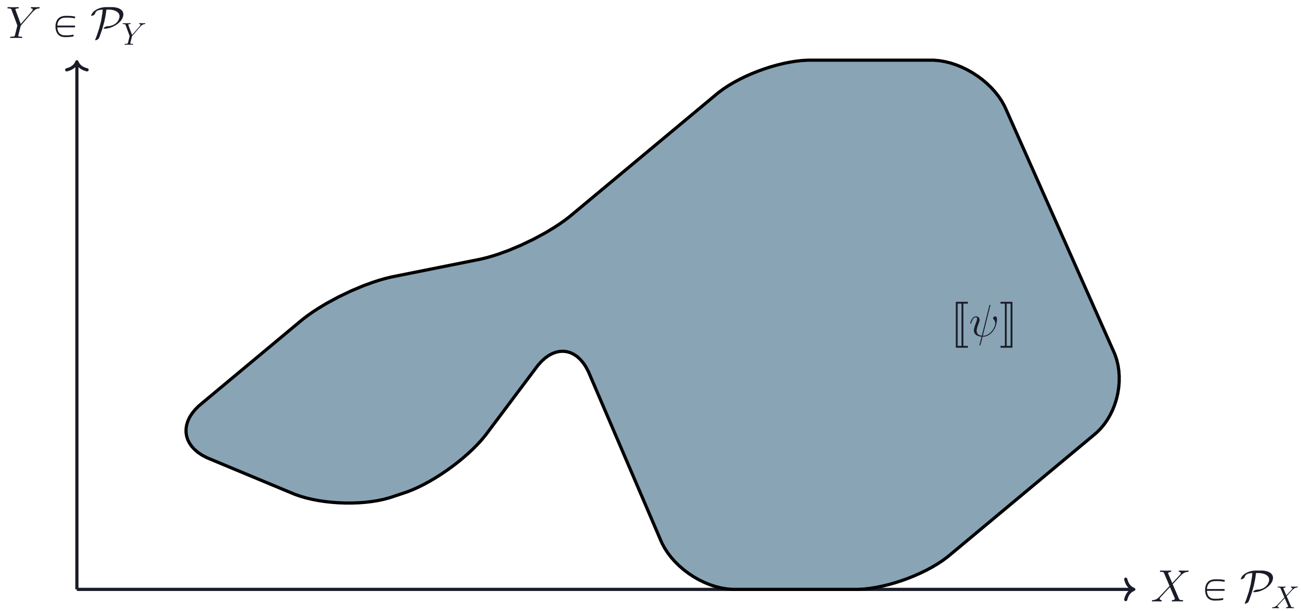
$$\varphi(X) \equiv \mathtt{Q}Y.\ \psi(X, Y)$$



Generalised quantifiers (A. Mostowski [1957])

$$\varphi(X) \equiv \mathtt{Q}Y.\ \psi(X, Y)$$

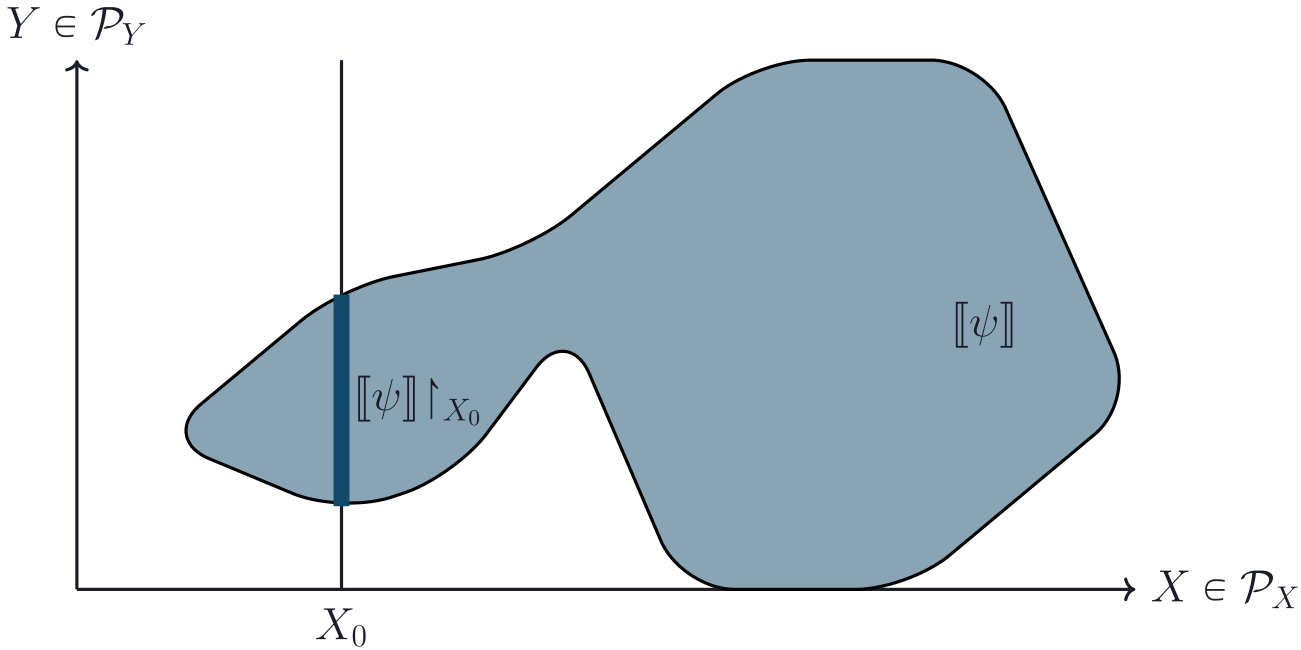
$$R{\upharpoonright}_{X_0} \stackrel{\text{def}}{=} \{Y \mid (X_0, Y) \in R\}$$



Generalised quantifiers (A. Mostowski [1957])

$$\varphi(X) \equiv \mathtt{Q}Y.\, \psi(X, Y)$$

$$R{\restriction}_{X_0} \stackrel{\text{def}}{=} \{Y \mid (X_0, Y) \in R\}$$

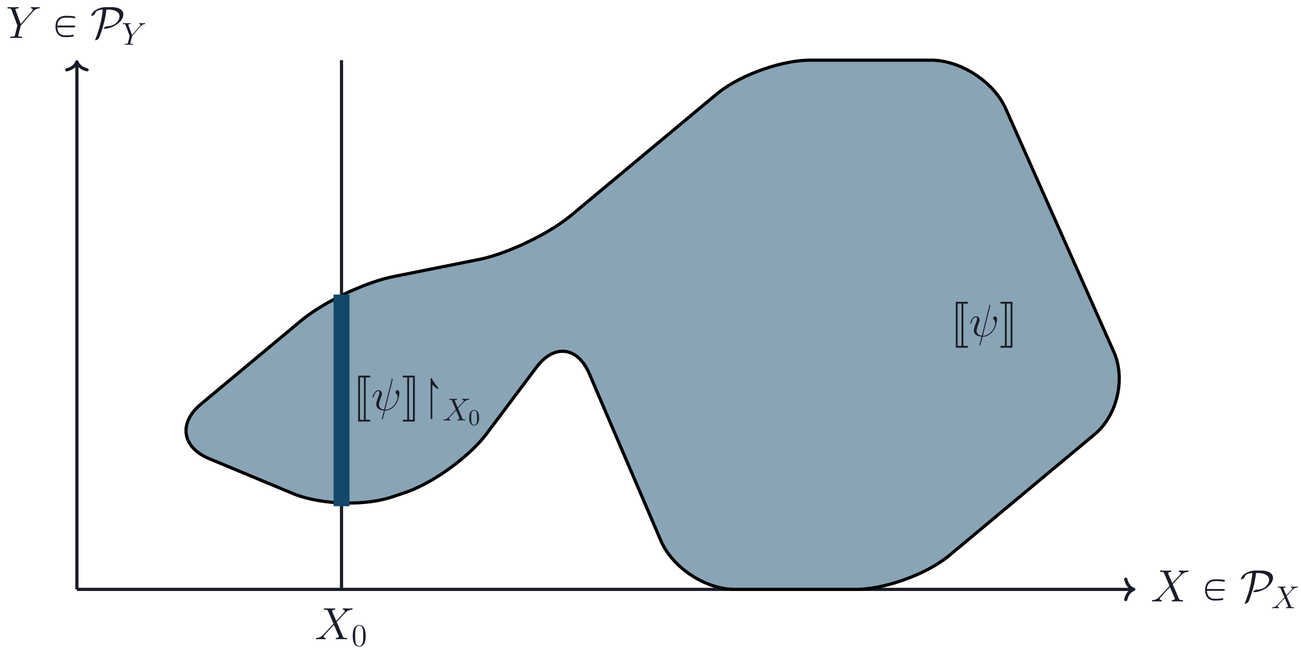


Generalised quantifiers (A. Mostowski [1957])

$$\varphi(X) \equiv \mathtt{Q}Y.\ \psi(X, Y)$$

$$R\restriction_{X_0} \stackrel{\text{def}}{=} \{Y \mid (X_0, Y) \in R\}$$

$$[\![\varphi]\!] \stackrel{\text{def}}{=} \{X \mid [\![\psi]\!]\restriction_X \in [\![\mathtt{Q}]\!]\}$$



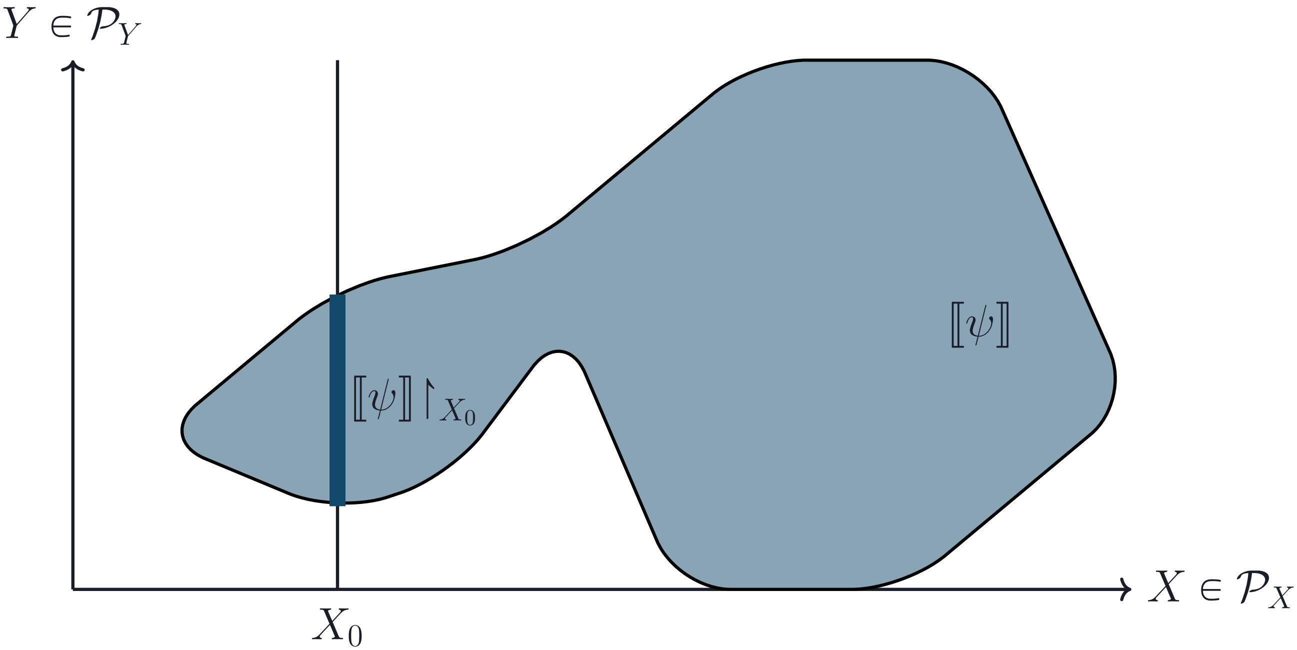
Generalised quantifiers (A. Mostowski [1957])

$$\varphi(X) \equiv \mathtt{Q}Y.\ \psi(X,Y)$$

$$R{\restriction}_{X_0} \stackrel{\text{def}}{=} \{Y \mid (X_0, Y) \in R\}$$

$$[\![\varphi]\!] \stackrel{\text{def}}{=} \left\{X \mid [\![\psi]\!]{\restriction}_X \in [\![\mathtt{Q}]\!]\right\}$$

$[\![\exists]\!] = \{\mathcal{Y} \subseteq \mathcal{P}_Y \mid \mathcal{Y} \neq \varnothing\}$,



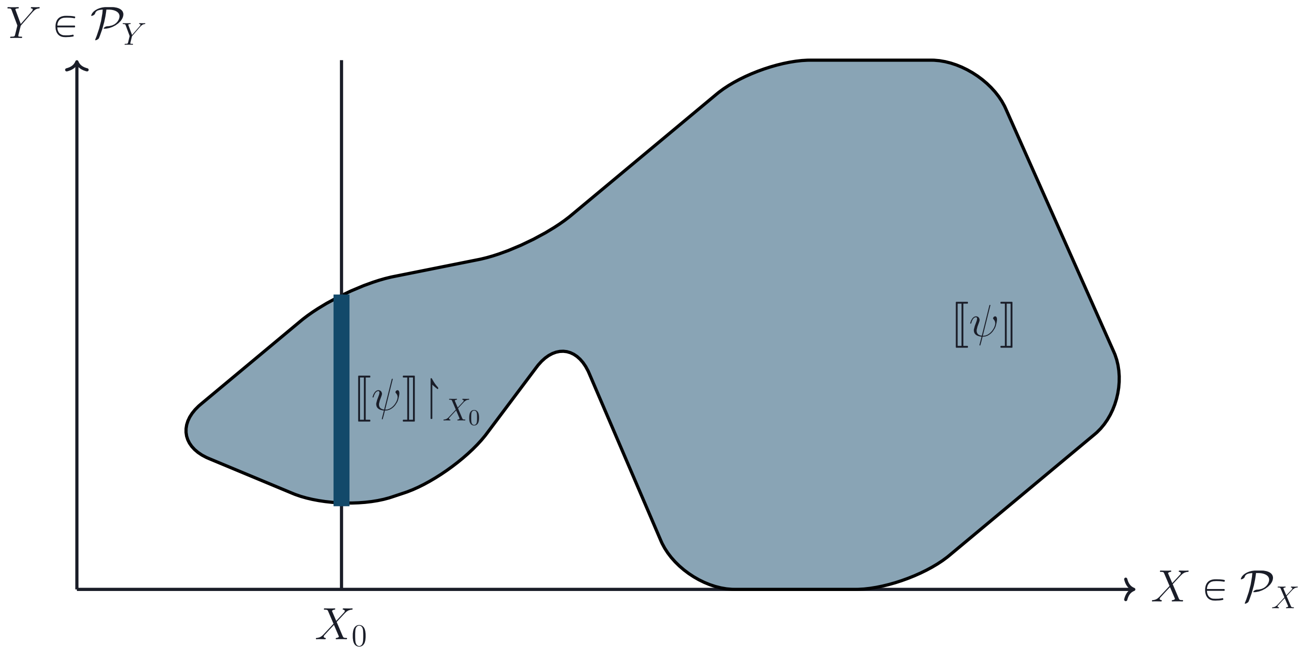
Generalised quantifiers (A. Mostowski [1957])

$$\varphi(X) \equiv \mathtt{Q}Y.\ \psi(X,Y)$$

$$R{\upharpoonright}_{X_0} \stackrel{\text{def}}{=} \{Y \mid (X_0, Y) \in R\}$$

$$[\![\varphi]\!] \stackrel{\text{def}}{=} \left\{X \mid [\![\psi]\!]{\upharpoonright}_X \in [\![\mathtt{Q}]\!]\right\}$$

$$[\![\exists]\!] = \left\{\mathcal{Y} \subseteq \mathcal{P}_Y \mid \mathcal{Y} \neq \varnothing\right\}, \quad [\![\forall]\!] = \left\{\mathcal{P}_Y\right\},$$



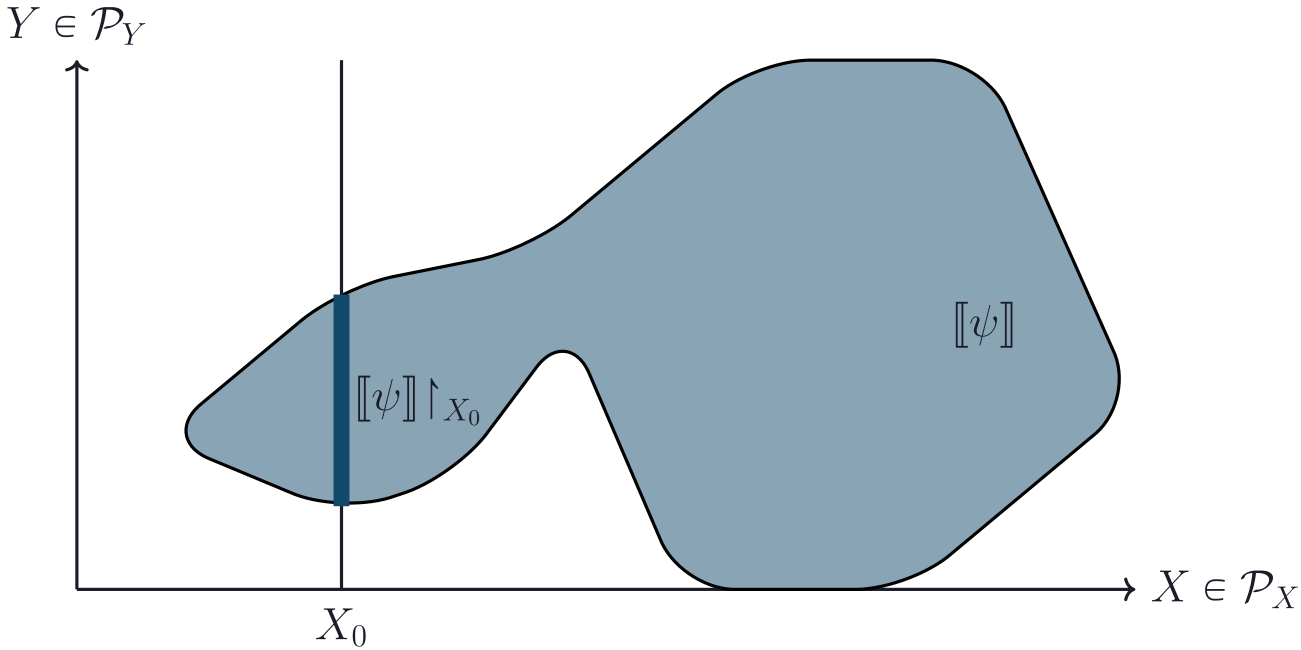
Generalised quantifiers (A. Mostowski [1957])

$$\varphi(X) \equiv \mathtt{Q}Y.\ \psi(X, Y)$$

$$R{\restriction}_{X_0} \stackrel{\text{def}}{=} \{Y \mid (X_0, Y) \in R\}$$

$$[\![\varphi]\!] \stackrel{\text{def}}{=} \left\{X \mid [\![\psi]\!]{\restriction}_X \in [\![\mathtt{Q}]\!]\right\}$$

$$[\![\exists]\!] = \left\{\mathcal{Y} \subseteq \mathcal{P}_Y \mid \mathcal{Y} \neq \varnothing\right\}, \quad [\![\forall]\!] = \left\{\mathcal{P}_Y\right\}, \quad [\![\exists^\infty]\!] = \left\{\mathcal{Y} \subseteq \mathcal{P}_Y \mid |\mathcal{Y}| = \infty\right\},$$



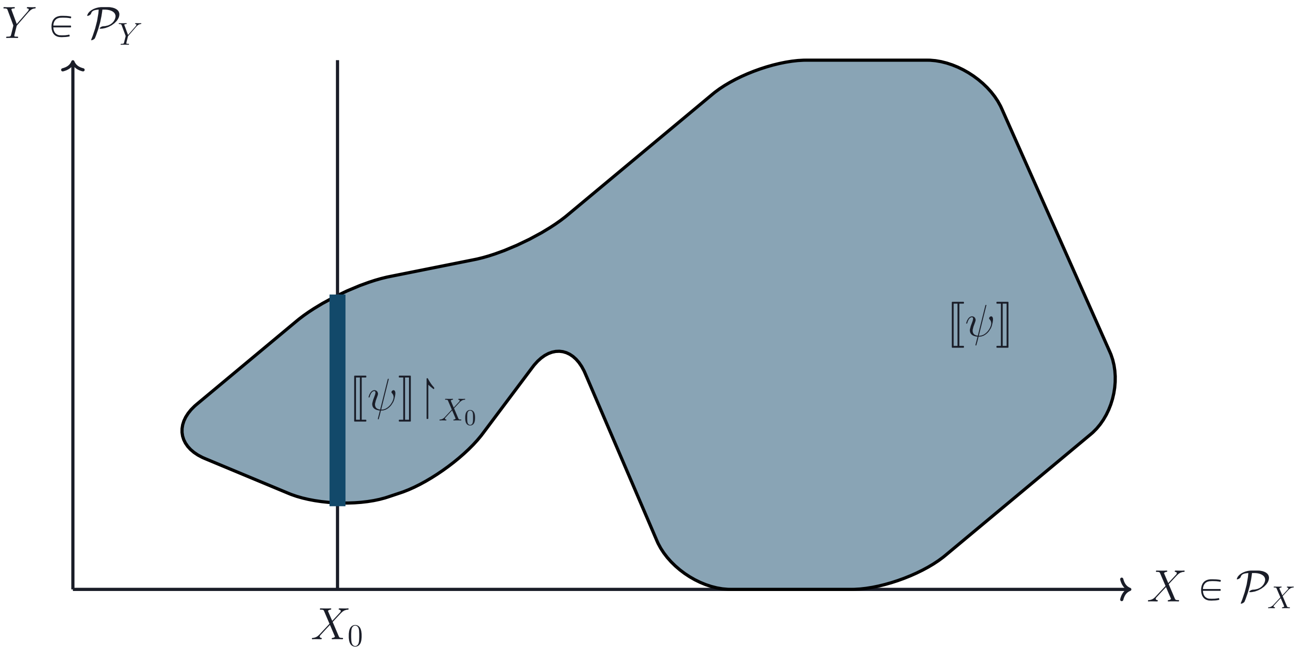
Generalised quantifiers (A. Mostowski [1957])

$$\varphi(X) \equiv \mathtt{Q}Y.\ \psi(X, Y)$$

$$R{\restriction}_{X_0} \stackrel{\text{def}}{=} \{Y \mid (X_0, Y) \in R\}$$

$$[\![\varphi]\!] \stackrel{\text{def}}{=} \big\{X \mid [\![\psi]\!]{\restriction}_X \in [\![\mathtt{Q}]\!]\big\}$$

$$[\![\exists]\!] = \big\{\mathcal{Y} \subseteq \mathcal{P}_Y \mid \mathcal{Y} \neq \varnothing\big\}, \quad [\![\forall]\!] = \big\{\mathcal{P}_Y\big\}, \quad [\![\exists^\infty]\!] = \big\{\mathcal{Y} \subseteq \mathcal{P}_Y \mid |\mathcal{Y}| = \infty\big\}, \quad \ldots$$



Index quantifiers

Index quantifiers (This Work [2026])

Index quantifiers

(This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

$$\sim \{X_0 \mid \text{“index } (i, j) \text{ suffices to check if } \psi(X_0, Y) \text{ holds”}\}$$

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

$$\sim \{X_0 \mid \text{"index } (i, j) \text{ suffices to check if } \psi(X_0, Y) \text{ holds"}\}$$

E.g. $\mathtt{I}_{(0,1)} Y.\ X = Y$

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

$$\sim \{X_0 \mid \text{“index } (i, j) \text{ suffices to check if } \psi(X_0, Y) \text{ holds”}\}$$

E.g. $\mathtt{I}_{(0,1)} Y.\ X{=}Y$ // $X{=}Y$ is a safety property

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

$$\sim \{X_0 \mid \text{“index } (i, j) \text{ suffices to check if } \psi(X_0, Y) \text{ holds”}\}$$

E.g. $\mathtt{I}_{(0,1)} Y.\ X{=}Y$ // $X{=}Y$ is a safety property

However: $[\![X{=}Y]\!]\restriction_{X_0} = \{X_0\}$

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)} Y.\ \psi(X,Y)$$

$$\sim \{X_0 \mid \text{“index } (i,j) \text{ suffices to check if } \psi(X_0,Y) \text{ holds”}\}$$

E.g. $\mathtt{I}_{(0,1)} Y.\ X{=}Y$ // $X{=}Y$ is a safety property

However: $[\![X{=}Y]\!]\restriction_{X_0} = \{X_0\}$ // typically **non-regular** **!!!**

Index quantifiers

(This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

$$\sim \{X_0 \mid \text{“index } (i,j) \text{ suffices to check if } \underbrace{\psi(X_0, Y) \text{ holds”}}_{\text{given } X_0 \text{ and } Y}\}$$

E.g. $\mathtt{I}_{(0,1)} Y.\ X{=}Y$ // $X{=}Y$ is a safety property

However: $[\![X{=}Y]\!]{\restriction}_{X_0} = \{X_0\}$ // typically **non-regular** **!!!**

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)}Y.\ \psi(X,Y)$$

$$\sim \{X_0 \mid \text{“index } (i,j) \text{ suffices to check if } \underbrace{\psi(X_0,Y)}_{\text{given } X_0 \text{ and } Y} \text{ holds”}\}$$

E.g. $\mathtt{I}_{(0,1)}Y.\ X{=}Y$ // $X{=}Y$ is a safety property

However: $[\![X{=}Y]\!]\restriction_{X_0} = \{X_0\}$ // typically **non-regular !!!**

Definition

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)} Y.\ \psi(X,Y)$$

$$\sim \{X_0 \mid \text{“index } (i,j) \text{ suffices to check if } \underbrace{\psi(X_0,Y)}_{\text{given } X_0 \text{ and } Y} \text{ holds”}\}$$

E.g. $\mathtt{I}_{(0,1)} Y.\ X{=}Y$ // $X{=}Y$ is a safety property

However: $[\![X{=}Y]\!]\restriction_{X_0} = \{X_0\}$ // typically **non-regular** **!!!**

Definition

$$[\![\mathtt{I}_{(i,j)} Y.\ \psi(X,Y)]\!]$$

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)}Y.\ \psi(X,Y)$$

$$\sim \{X_0 \mid \text{“index } (i,j) \text{ suffices to check if } \underbrace{\psi(X_0,Y)}_{\text{given } X_0 \text{ and } Y} \text{ holds”}\}$$

E.g. $\mathtt{I}_{(0,1)}Y.\ X{=}Y$ // $X{=}Y$ is a safety property

However: $[\![X{=}Y]\!]\restriction_{X_0} = \{X_0\}$ // typically **non-regular** **!!!**

Definition

$$[\![\mathtt{I}_{(i,j)}Y.\ \psi(X,Y)]\!] \stackrel{\text{def}}{=} \{X_0 \mid$$

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)} Y.\ \psi(X,Y)$$

$$\sim \{X_0 \mid \text{“index } (i,j) \text{ suffices to check if } \underbrace{\psi(X_0,Y)}_{\text{given } X_0 \text{ and } Y} \text{ holds”}\}$$

E.g. $\mathtt{I}_{(0,1)} Y.\ X{=}Y$ // $X{=}Y$ is a safety property

However: $[\![X{=}Y]\!]\restriction_{X_0} = \{X_0\}$ // typically **non-regular** **!!!**

Definition

$$[\![\mathtt{I}_{(i,j)} Y.\ \psi(X,Y)]\!] \stackrel{\text{def}}{=} \{X_0 \mid \text{there exists an automaton } \mathcal{A} \text{ of index } (i,j)$$

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)}Y.\ \psi(X,Y)$$

$$\sim \{X_0 \mid \text{“index } (i,j) \text{ suffices to check if } \underbrace{\psi(X_0,Y)}_{\text{given } X_0 \text{ and } Y} \text{ holds”}\}$$

E.g. $\mathtt{I}_{(0,1)}Y.\ X{=}Y$ // $X{=}Y$ is a safety property

However: $[\![X{=}Y]\!]\restriction_{X_0} = \{X_0\}$ // typically **non-regular !!!**

Definition

$$[\![\mathtt{I}_{(i,j)}Y.\ \psi(X,Y)]\!] \stackrel{\text{def}}{=} \{X_0 \mid \text{there exists an automaton } \mathcal{A} \text{ of index } (i,j) \text{ which reads pairs } (X,Y) \text{ such that}$$

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

$$\sim \{X_0 \mid \text{“index } (i,j) \text{ suffices to check if } \underbrace{\psi(X_0, Y)}_{\text{given } X_0 \text{ and } Y} \text{ holds”}\}$$

E.g. $\mathtt{I}_{(0,1)} Y.\ X{=}Y$ // $X{=}Y$ is a safety property

However: $[\![X{=}Y]\!]\!\restriction_{X_0} = \{X_0\}$ // typically **non-regular** **!!!**

Definition

$[\![\mathtt{I}_{(i,j)} Y.\ \psi(X, Y)]\!] \stackrel{\text{def}}{=} \{X_0 \mid$ there exists an automaton $\mathcal{A}$ of index (i,j) which reads pairs (X, Y) such that for every Y we have:

Index quantifiers (This Work [2026])

$$\varphi(X) \equiv \mathtt{I}_{(i,j)}Y.\ \psi(X,Y)$$

$$\sim \{X_0 \mid \text{“index } (i,j) \text{ suffices to check if } \underbrace{\psi(X_0,Y)}_{\text{given } X_0 \text{ and } Y} \text{ holds”}\}$$

E.g. $\mathtt{I}_{(0,1)}Y.\ X{=}Y$ // $X{=}Y$ is a safety property

However: $[\![X{=}Y]\!]\restriction_{X_0} = \{X_0\}$ // typically **non-regular** **!!!**

Definition

$$[\![\mathtt{I}_{(i,j)}Y.\ \psi(X,Y)]\!] \stackrel{\text{def}}{=} \{X_0 \mid \text{there exists an automaton } \mathcal{A} \text{ of index } (i,j)$$

which reads pairs (X,Y) such that

for every Y we have:

$$(\mathcal{A}\ \textbf{accepts}\ (X_0,Y)) \iff \psi(X_0,Y)\}$$

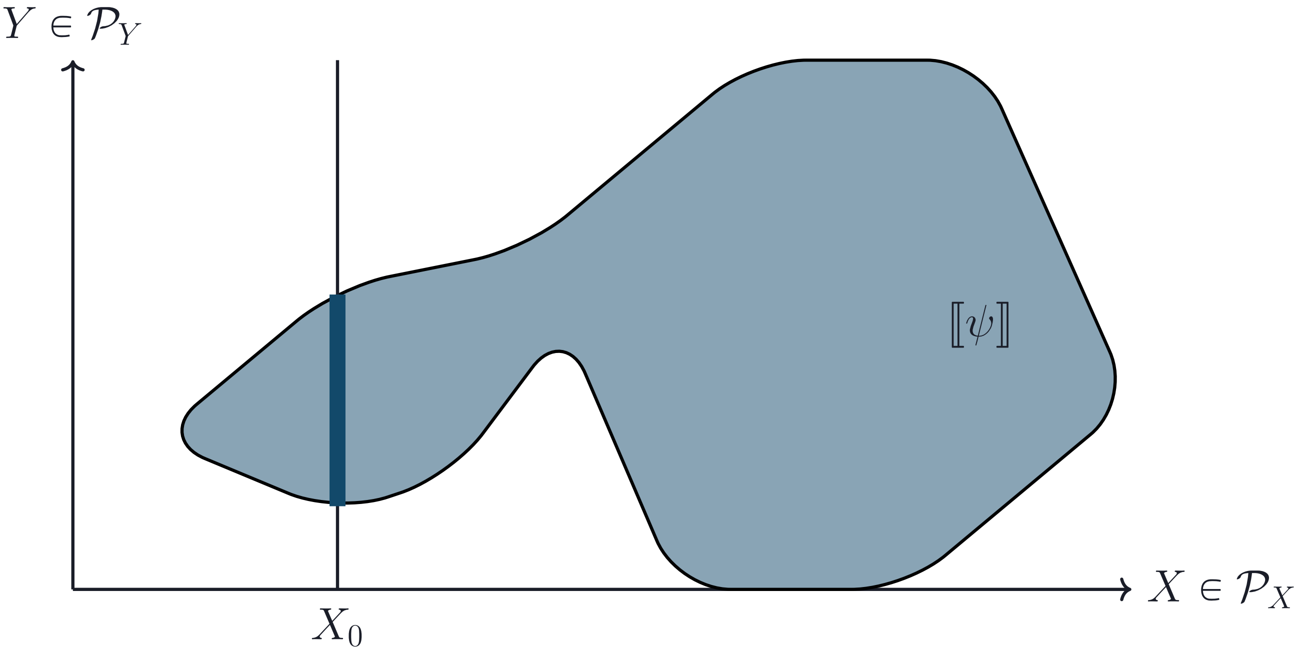
$\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a **fixed** X_0

$\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a **fixed** X_0

IFF

$\mathtt{I}_{(i,j)} Y.\, \psi(X_0, Y)$ **holds** for a **fixed** X_0

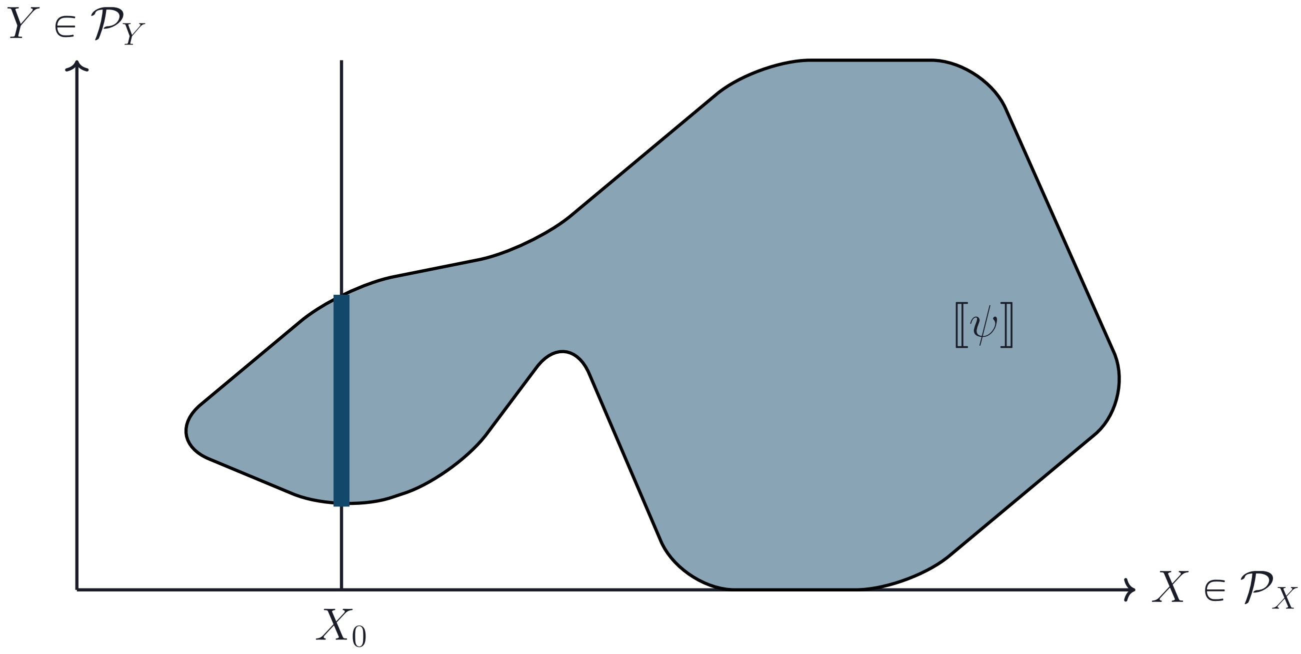
IFF



$\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a **fixed** X_0

IFF

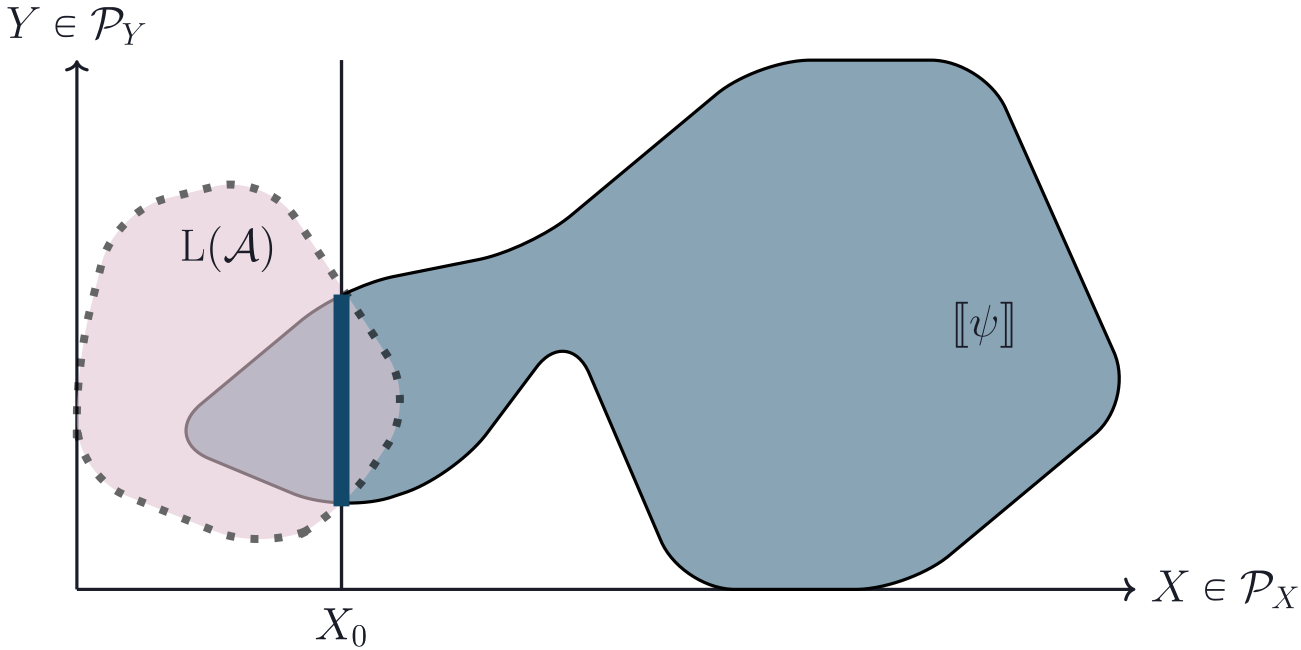
there exists $\mathcal{A}$ of **index** (i, j) such that



$\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a **fixed** X_0

IFF

there exists $\mathcal{A}$ of **index** (i, j) such that

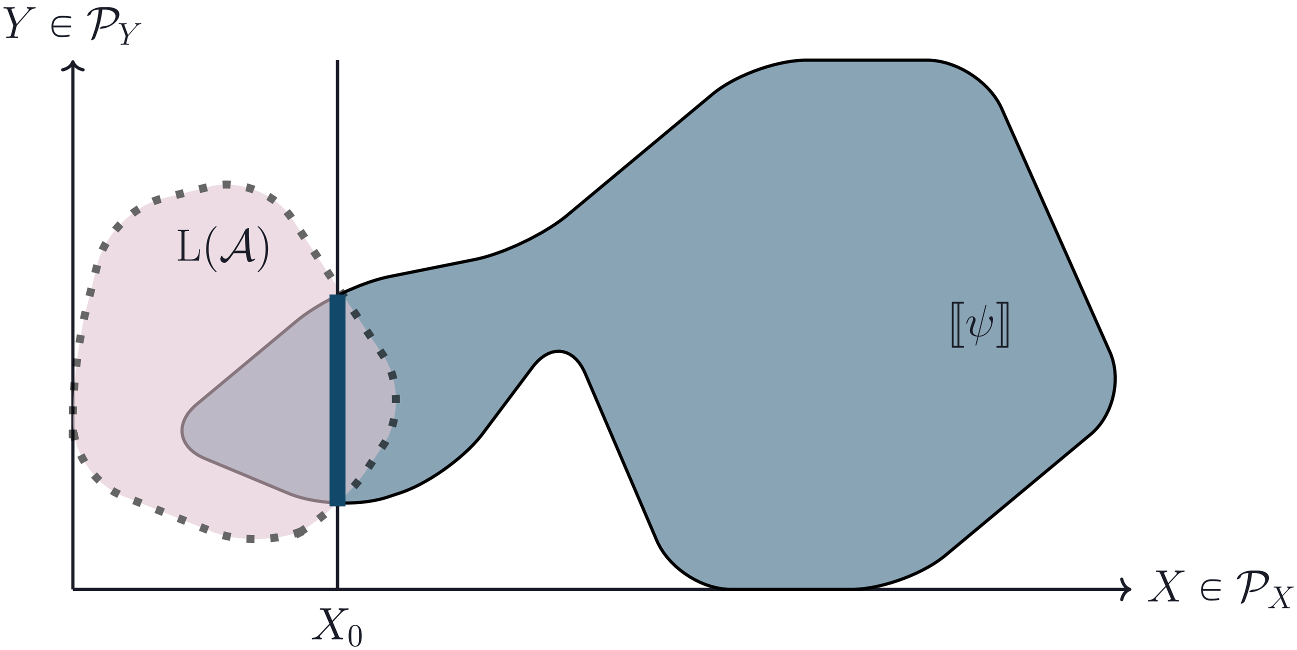


$\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a **fixed** X_0

IFF

there exists $\mathcal{A}$ of **index** (i, j) such that

$$\{Y \mid (X_0, Y) \in \mathrm{L}(\mathcal{A})\} = [\![\psi]\!]\restriction_{X_0}$$



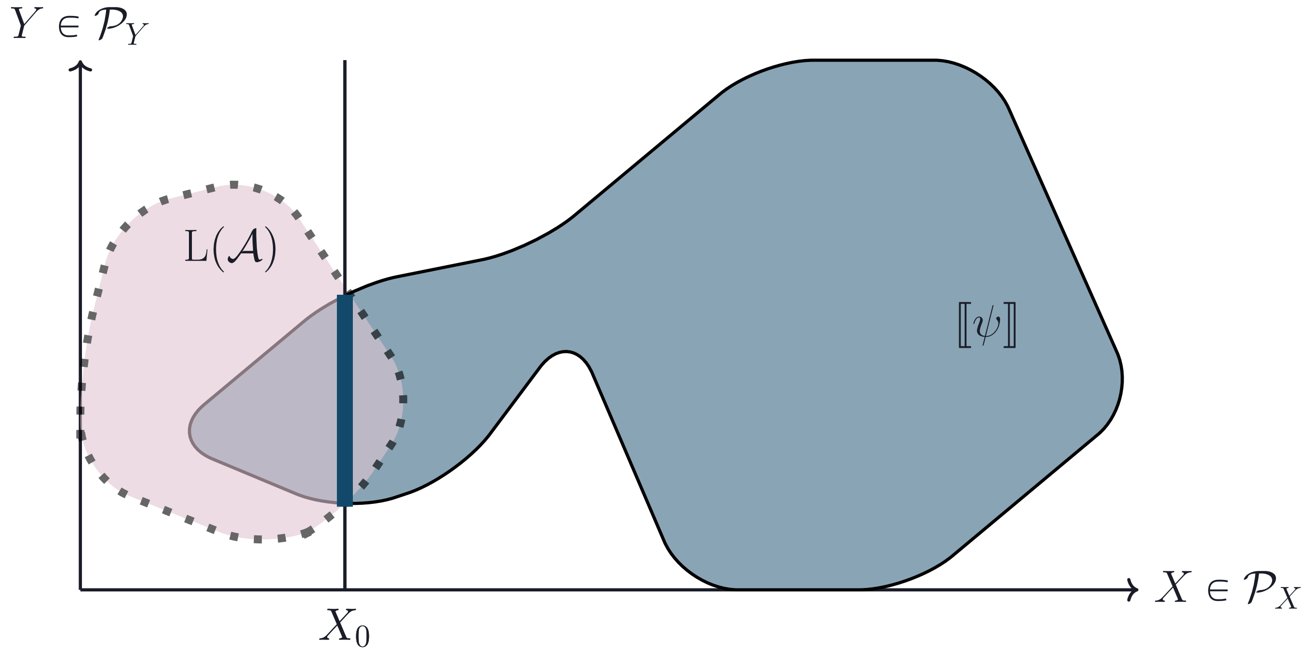
$\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a **fixed** X_0

IFF

there exists $\mathcal{A}$ of **index** (i,j) such that

$$\{Y \mid (X_0, Y) \in \mathrm{L}(\mathcal{A})\} = [\![\psi]\!]\restriction_{X_0}$$

⇝ if ψ is (i,j)**-recognisable**, then $\mathtt{I}_{(i,j)}Y.\ \psi(X,Y)$ is a **tautology**



Index quantifiers

Index quantifiers

Extention of **index problem**:

$\mathtt{I}_{(i,j)} Y.\ \psi(Y)$ **holds** **IFF** ψ can be recognised by an (i, j) **automaton**

Index quantifiers

Extention of **index problem**:

$\mathtt{I}_{(i,j)} Y.\, \psi(Y)$ **holds** **IFF** ψ can be recognised by an (i, j) **automaton**

Theorem

MSO+$\mathtt{I}$ is **equivalent** to **MSO** over **infinite words**.

Index quantifiers

Extention of **index problem**:

$\mathtt{I}_{(i,j)} Y.\ \psi(Y)$ **holds** **IFF** ψ can be recognised by an (i, j) **automaton**

Theorem

MSO+$\mathtt{I}$ is **equivalent** to **MSO** over **infinite words**.

[**effective quantifier-elimination procedure**]

Index quantifiers

Extention of **index problem**:

$\mathtt{I}_{(i,j)} Y.\, \psi(Y)$ **holds** **IFF** ψ can be recognised by an (i, j) **automaton**

Theorem

MSO+I is **equivalent** to **MSO** over **infinite words**.

[**effective quantifier-elimination procedure**]

Theorem

MSO+I is **undecidable** over **infinite trees**.

Index quantifiers

Extention of **index problem**:

$\mathtt{I}_{(i,j)} Y.\ \psi(Y)$ **holds** **IFF** ψ can be recognised by an (i,j) **automaton**

Theorem

MSO+$\mathtt{I}$ is **equivalent** to **MSO** over **infinite words**.

[**effective quantifier-elimination procedure**]

Theorem

MSO+$\mathtt{I}$ is **undecidable** over **infinite trees**.

[one of **the first** **undecidability results**]

Index quantifiers

Extention of **index problem**:

$\mathtt{I}_{(i,j)} Y.\ \psi(Y)$ **holds** **IFF** ψ can be recognised by an (i, j) **automaton**

Theorem

MSO+I is **equivalent** to **MSO** over **infinite words**.

[**effective quantifier-elimination procedure**]

Theorem

MSO+I is **undecidable** over **infinite trees**.

[one of **the first undecidability results**]

Proof idea:

Index quantifiers

Extention of **index problem**:

$\mathtt{I}_{(i,j)}Y.\ \psi(Y)$ **holds** **IFF** ψ can be recognised by an (i,j) **automaton**

Theorem

MSO+I is **equivalent** to **MSO** over **infinite words**.

[**effective quantifier-elimination procedure**]

Theorem

MSO+I is **undecidable** over **infinite trees**.

[one of **the first undecidability results**]

Proof idea:

Reduce from **MSO+U**. (M. Bojańczyk, P. Parys, Sz. Toruńczyk [2016])

Index quantifiers

Extention of **index problem**:

$\mathtt{I}_{(i,j)} Y.\, \psi(Y)$ **holds** **IFF** ψ can be recognised by an (i, j) **automaton**

Theorem

MSO+I is **equivalent** to **MSO** over **infinite words**.

[**effective quantifier-elimination procedure**]

Theorem

MSO+I is **undecidable** over **infinite trees**.

[one of **the first undecidability results**]

Proof idea:

Reduce from **MSO+U**. (M. Bojańczyk, P. Parys, Sz. Toruńczyk [2016])

A **safety** automaton can recognise a **reachability** condition

Index quantifiers

Extention of **index problem**:

$\mathtt{I}_{(i,j)} Y.\, \psi(Y)$ **holds** **IFF** ψ can be recognised by an (i, j) **automaton**

Theorem

MSO+I is **equivalent** to **MSO** over **infinite words**.

[**effective quantifier-elimination procedure**]

Theorem

MSO+I is **undecidable** over **infinite trees**.

[one of **the first undecidability results**]

Proof idea:

Reduce from **MSO+U**. (M. Bojańczyk, P. Parys, Sz. Toruńczyk [2016])

A **safety** automaton can recognise a **reachability** condition

IFF the **wait times** are **uniformly bounded**!

Index quantifiers

Extention of **index problem**:

$\mathtt{I}_{(i,j)}Y.\ \psi(Y)$ **holds** **IFF** ψ can be recognised by an (i, j) **automaton**

Theorem

MSO+I is **equivalent** to **MSO** over **infinite words**.

[**effective quantifier-elimination procedure**]

Theorem

MSO+I is **undecidable** over **infinite trees**.

[one of **the first undecidability results**]

Proof idea:

Reduce from **MSO+U**. (M. Bojańczyk, P. Parys, Sz. Toruńczyk [2016])

A **safety** automaton can recognise a **reachability** condition

IFF the **wait times** are **uniformly bounded**!

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

I: $\in A^\omega$

II: $\in (i, j)^\omega$

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

I: a_0 $\in A^\omega$

II: $\in (i, j)^\omega$

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

I: a_0 $\in A^\omega$

II: k_0 $\in (i, j)^\omega$

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

I:	$a_0 \quad a_1$	$\in A^\omega$
II:	k_0	$\in (i, j)^\omega$

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

I:	a_0	a_1	$\in A^\omega$
II:	k_0	k_1	$\in (i, j)^\omega$

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

$$\begin{array}{lcccccccccccccl} \mathbf{I}: & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & \cdots \in A^\omega \\ \mathbf{II}: & k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_{10} & k_{11} & k_{12} & \cdots \in (i, j)^\omega \end{array}$$

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

$$\begin{array}{lllllllllllllll} \textbf{I}: & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & \cdots & \in A^\omega \\ \textbf{II}: & k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_{10} & k_{11} & k_{12} & \cdots & \in (i, j)^\omega \end{array}$$

Player II wins if:

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

$$\begin{array}{lcccccccccccccccc} \textbf{I}: & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & \cdots & \in A^\omega \\ \textbf{II}: & k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_{10} & k_{11} & k_{12} & \cdots & \in (i,j)^\omega \end{array}$$

Player II wins if:

$$(a_0 a_1 a_2 \cdots) \in L \iff (k_0 k_1 \cdots) \in P_{(i,j)}$$

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

$$\begin{array}{lllllllllllllll} \textbf{I}: & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & \cdots \in A^\omega \\ \textbf{II}: & k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_{10} & k_{11} & k_{12} & \cdots \in (i, j)^\omega \end{array}$$

Player II wins if:

$$\underbrace{(a_0 a_1 a_2 \cdots) \in L \iff (k_0 k_1 \cdots) \in P_{(i,j)}}_{\omega\textbf{-regular}}$$

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

$$\begin{array}{lllllllllllllll} \textbf{I}: & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & \cdots \in A^\omega \\ \textbf{II}: & k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_{10} & k_{11} & k_{12} & \cdots \in (i,j)^\omega \end{array}$$

Player II wins if:

$$\underbrace{(a_0 a_1 a_2 \cdots) \in L \iff (k_0 k_1 \cdots) \in P_{(i,j)}}_{\omega\text{-regular}}$$

Theorem (Büchi-Landweber [1969])

Games with **ω-regular** conditions are **finite-memory determined**.

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

$$\begin{array}{lllllllllllllll} \textbf{I}: & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & \cdots \in A^\omega \\ \textbf{II}: & k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_{10} & k_{11} & k_{12} & \cdots \in (i, j)^\omega \end{array}$$

Player II **wins** if:

$$\underbrace{(a_0 a_1 a_2 \cdots) \in L \iff (k_0 k_1 \cdots) \in P_{(i,j)}}_{\omega\text{-regular}}$$

Theorem (Büchi-Landweber [1969])

Games with **ω-regular** conditions are $\underbrace{\textbf{finite-memory determined}}_{\textbf{sequential transducers}}$.

When $L \subseteq A^\omega$ is **recognisable** by a **deterministic** parity (i, j) automaton?

Game: (W. Wadge [1983])

$$\begin{array}{lllllllllllllll} \textbf{I}: & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & \cdots \in A^\omega \\ \textbf{II}: & k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_{10} & k_{11} & k_{12} & \cdots \in (i,j)^\omega \end{array}$$

Player II wins if:

$$\underbrace{(a_0 a_1 a_2 \cdots) \in L \iff (k_0 k_1 \cdots) \in P_{(i,j)}}_{\omega\textbf{-regular}}$$

Theorem (Büchi-Landweber [1969])

Games with ω**-regular** conditions are $\underbrace{\textbf{finite-memory determined}}_{\textbf{sequential transducers}}$.

Corollary

$L \subseteq A^\omega$ is (i, j)-recognisable **IFF** **Player II wins** the above game.

When $\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

When $\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

When $\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$$\mathbf{X_0}: \quad x_0 \ x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8 \ x_9 \ x_{10} \ x_{11} \ x_{12} \ \cdots \ = X_0$$

I:

II:

When $\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots = X_0$
I:	y_0													
II:														

When $\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0														
II:	k_0														

When $\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$$\begin{array}{rllllllllllllll} \mathbf{X_0}: & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & \cdots = X_0 \\ \mathbf{I}: & y_0 & y_1 \\ \mathbf{II}: & k_0 \end{array}$$

When $\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0	y_1													
II:	k_0	k_1													

When $\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots$	$= Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots$	$= K$

When $\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots$	$= Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots$	$= K$

Player II wins if:

When $\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots$	$= Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots$	$= K$

Player II wins if: $\psi(X_0, Y) \iff K \in P_{(i,j)}$

When $\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots = X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots = Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots = K$

Player II wins if: $\psi(X_0, Y) \iff K \in P_{(i,j)}$

Remark

In general **NO finite-memory** strategies!

When $\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots$	$= Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots$	$= K$

Player II wins if: $\psi(X_0, Y) \iff K \in P_{(i,j)}$

Remark

In general **NO finite-memory** strategies!

E.g. when the condition is: $\forall n \in \mathbb{N}.\ k_n = x_{n+1}$

When $\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots$	$= Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots$	$= K$

Player II wins if:

$$\underbrace{\psi(X_0, Y) \iff K \in P_{(i,j)}}_{\textbf{special shape}}$$

Remark

In general **NO finite-memory** strategies!

E.g. when the condition is: $\forall n \in \mathbb{N}.\ k_n = x_{n+1}$

When $\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots = X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots = Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots = K$

Player II wins if:

$$\underbrace{\psi(X_0, Y) \iff K \in P_{(i,j)}}_{\textbf{special shape}}$$

When $\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots$	$= Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots$	$= K$

Player II wins if:

$$\underbrace{\psi(X_0, Y) \iff K \in P_{(i,j)}}_{\textbf{special shape}}$$

Theorem

If the **winning condition** is

a **Boolean combination** of formulae $\psi_i(X_0, Y)$ and $\varphi_i(K)$

When $\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots$	$= Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots$	$= K$

Player II wins if:

$$\underbrace{\psi(X_0, Y) \iff K \in P_{(i,j)}}_{\textbf{special shape}}$$

Theorem

If the **winning condition** is

$$\underbrace{\text{a } \textbf{Boolean combination} \text{ of formulae } \psi_i(X_0, Y) \text{ and } \varphi_i(K)}_{\text{it ``depends } \textbf{seperately on } K\text{''}}$$

When $\mathtt{I}_{(i,j)}Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots$	$= Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots$	$= K$

Player II wins if: $\underbrace{\psi(X_0, Y) \iff K \in P_{(i,j)}}_{\textbf{special shape}}$

Theorem

If the **winning condition** is

$\underbrace{\text{a \textbf{Boolean combination} of formulae } \psi_i(X_0, Y) \text{ and } \varphi_i(K)}_{\text{it ``depends \textbf{seperately on} } K\text{''}}$

and **Player II wins**,

When $\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots$	$= X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots$	$= Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots$	$= K$

Player II wins if:
$$\underbrace{\psi(X_0, Y) \iff K \in P_{(i,j)}}_{\textbf{special shape}}$$

Theorem

If the **winning condition** is

$$\underbrace{\text{a } \textbf{Boolean combination} \text{ of formulae } \psi_i(X_0, Y) \text{ and } \varphi_i(K)}_{\text{it “depends } \textbf{seperately on } K\text{”}}$$

and **Player II wins**,

then **Player II** has a **finite-memory** winning strategy.

When $\mathtt{I}_{(i,j)} Y.\ \psi(X_0, Y)$ **holds** for a given X_0?

Game (**parametrised** by X_0):

$\mathbf{X_0}$:	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\cdots = X_0$
I:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	$\cdots = Y$
II:	k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	k_{12}	$\cdots = K$

Player II wins if: $\underbrace{\psi(X_0, Y) \iff K \in P_{(i,j)}}_{\textbf{special shape}}$

Theorem

If the **winning condition** is

$$\underbrace{\text{a } \textbf{Boolean combination} \text{ of formulae } \psi_i(X_0, Y) \text{ and } \varphi_i(K)}_{\text{it “depends } \textbf{seperately on } K\text{”}}$$

and **Player II wins**,

then **Player II** has a $\underbrace{\textbf{finite-memory} \text{ winning strategy}}_{\textbf{sequential transducer } (X_0, Y) \longrightarrow K}$.

Paper idea: **extend index problem into** a **generalised quantifier**

$$\mathtt{I}_{(i,j)} Y.\, \psi(X, Y)$$

Paper idea: **extend** **index problem** **into** a **generalised quantifier**

$$\mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

Theorem

MSO+$\mathtt{I}$ is **undecidable** over **infinite trees**.

Paper idea: **extend index problem into** a **generalised quantifier**

$$\mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

Theorem

MSO+$\mathtt{I}$ is **undecidable** over **infinite trees**.

$\rightsquigarrow$ **the first undecidability result** relating **index problem**

Paper idea: **extend index problem into** a **generalised quantifier**

$$\mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

Theorem

MSO+$\mathtt{I}$ is **undecidable** over **infinite trees**.

$\rightsquigarrow$ **the first undecidability result** relating **index problem**

Theorem

MSO+$\mathtt{I}$ effectively **reduces** to **MSO** over **infinite words**.

Paper idea: **extend** **index problem** **into** a **generalised quantifier**

$$\mathtt{I}_{(i,j)}Y.\ \psi(X,Y)$$

Theorem

MSO+$\mathtt{I}$ is **undecidable** over **infinite trees**.

$\rightsquigarrow$ **the first** **undecidability result** relating **index problem**

Theorem

MSO+$\mathtt{I}$ effectively **reduces** to **MSO** over **infinite words**.

$\rightsquigarrow$ potential **applications** in **model-checking**

Paper idea: **extend** **index problem** **into** a **generalised quantifier**

$$\mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

Theorem

MSO+$\mathtt{I}$ is **undecidable** over **infinite trees**.

⇝ **the first** **undecidability result** relating **index problem**

Theorem

MSO+$\mathtt{I}$ effectively **reduces** to **MSO** over **infinite words**.

⇝ potential **applications** in **model-checking**

Theorem

Parametrised games with conditions "**depending separately** on **II**" are **finite-memory determined** for **II**.

Paper idea: **extend** **index problem** **into** a **generalised quantifier**

$$\mathtt{I}_{(i,j)}Y.\ \psi(X,Y)$$

Theorem

MSO+$\mathtt{I}$ is **undecidable** over **infinite trees**.

⇝ **the first** **undecidability result** relating **index problem**

Theorem

MSO+$\mathtt{I}$ effectively **reduces** to **MSO** over **infinite words**.

⇝ potential **applications** in **model-checking**

Theorem

Parametrised games with conditions "**depending separately** on **II**" are **finite-memory determined** for **II**.

⇝ **new** **parametrised** variant of **Büchi-Landweber**

Paper idea: **extend** **index problem** **into** a **generalised quantifier**

$$\mathtt{I}_{(i,j)} Y.\ \psi(X, Y)$$

Theorem

MSO+I is **undecidable** over **infinite trees**.

⇝ **the first** **undecidability result** relating **index problem**

Theorem

MSO+I effectively **reduces** to **MSO** over **infinite words**.

⇝ potential **applications** in **model-checking**

Theorem

Parametrised games with conditions "**depending separately** on **II**" are **finite-memory determined** for **II**.

⇝ **new** **parametrised** variant of **Büchi-Landweber**

⇝ **lookahead** in games, **transducers**, **synthesis**, ...