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Efficient Compression in Semigroups

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Which finite *semigroups* afford *efficient compression* via *straight-line programs*?

The Membership Problem

The *membership problem* for a class $\mathbf{C}$ of semigroups is the following decision problem.

Input	A semigroup S, a subset $A \subseteq S$, an element $t \in S$.
Promise	The subsemigroup $\langle A \rangle_S$ belongs to the class $\mathbf{C}$.
Question	Is the element t a *member* of the subsemigroup $\langle A \rangle_S$?

* In this talk, we only consider the *Cayley table model* where the semigroup S is given as a complete multiplication table and its elements are given as indices into that table.

Theorem — Jones, Lien, Laaser (1976)

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Theorem — Fleischer (2019)

The membership problem for *groups* is in $\mathrm{NT{\scriptsize IME}}(\mathrm{polylog}\, n) \subseteq \mathrm{qAC}^0$.

Straight-Line Programs

Let S be a semigroup, and let $A \subseteq S$ be a set of constants.

A *straight-line program* $\mathbb{A}$ is a sequence of *instructions* manipulating *registers* $r_0, r_1, \ldots$:

- $r_k \leftarrow a$; (assigns the constant $a \in A$ to register r_k)
- $r_k \leftarrow r_i \cdot r_j$; (assigns the product of registers r_i, r_j to register r_k)

An element $t \in S$ is *computed* by $\mathbb{A}$ if some register r_k holds the value t after executing $\mathbb{A}$.

A straight-line program *computing* *stacs* and $(stacs)^{2026}$
with 25 *instruction* and 2 *registers*:

$$r_0 \leftarrow t;\ r_1 \leftarrow a;\ r_0 \leftarrow r_0 \cdot r_1;\ r_1 \leftarrow c;\ r_0 \leftarrow r_0 \cdot r_1;\ r_1 \leftarrow s;\ r_0 \leftarrow r_1 \cdot r_0;\ r_0 \leftarrow r_0 \cdot r_1;$$
$$r_1 \leftarrow r_0 \cdot r_0;\ r_1 \leftarrow r_1 \cdot r_0;\ r_1 \leftarrow r_1 \cdot r_1;\ r_1 \leftarrow r_1 \cdot r_0;\ r_1 \leftarrow r_1 \cdot r_1;\ r_1 \leftarrow r_1 \cdot r_0;$$
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Let the *length* and *width* of $\mathbb{A}$ be the number of *instructions* and *registers*, respectively.

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- Executing the straight-line program $\mathbb{A}$ on a **register machine** requires:

$$\textit{time} \sim \textit{length},$$

$$\textit{space} \sim \textit{width}.$$

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- Executing the straight-line program $\mathbb{A}$ on a **Turing machine** requires:

$$\textit{time} \sim \textit{length} \times \textit{width} \times \textit{element size},$$

$$\textit{space} \sim \textit{width} \times \textit{element size}.$$

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- Executing the straight-line program $\mathbb{A}$ on a **Turing machine** requires:

$$\textit{time} \sim \textit{length} \times \textit{width} \times \textit{element size},$$

$$\textit{space} \sim \textit{width} \times \textit{element size}.$$

- Assuming *element size* $\sim \log N$ for a semigroup S of size N, this yields

$$\textit{length} \in \mathcal{O}(\operatorname{polylog} N) \implies \textit{time} \in \mathcal{O}(\operatorname{polylog} N),$$

$$\textit{width} \in \mathcal{O}(1) \implies \textit{space} \in \mathcal{O}(\log N).$$

Theorem — Fleischer (2019)

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Theorem — Babai, Szemerédi (1984)

Groups admit straight-line programs of length $\mathcal{O}(\log^2 N)$.

Which *classes* of finite *semigroups* admit *straight-line programs* of *polylogarithmic length*?

pseudovarieties

Which ~~*classes*~~ of finite *semigroups* admit *straight-line programs* of *polylogarithmic length*?

Our Results

Theorem — Fleischer (2019)

Let $\mathbf{V}$ be a pseudovariety of **monoids**. Then the following are equivalent.

- The class $\mathbf{V}$ contains only Clifford monoids or commutative monoids.
- The class $\mathbf{V}$ admits straight-line programs of length $\mathcal{O}(\operatorname{polylog} N)$.
- The membership problem for $\mathbf{V}$ is in $\textsc{NTime}(\operatorname{polylog} n)$.

Lemma

Fleischer (2019)

The following pseudovarieties **don't** admit straight-line programs of length $\mathcal{O}(\operatorname{polylog} N)$.

$$\mathbf{LRB} := [\![x^2 \approx x, xyx \approx xy]\!], \qquad \mathbf{RRB} := [\![x^2 \approx x, xyx \approx yx]\!],$$

$$\mathbf{T} := [\![x^2 \approx xyx \approx 0]\!].$$

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Theorem

T., Weiß (2026)

Let $\mathbf{V}$ be a pseudovariety. Then the following are equivalent.

- The class $\mathbf{V}$ contains neither $\mathbf{LRB}$, $\mathbf{RRB}$, nor $\mathbf{T}$.
- The class $\mathbf{V}$ admits straight-line programs of length $\mathcal{O}(\operatorname{polylog} N)$.
- The membership problem for $\mathbf{V}$ is in $\mathsf{NTime}(\operatorname{polylog} n)$.

Proposition

T., Weiß (2026)

Commutative semigroups admit straight-line programs of length $\mathcal{O}(\log N)$ and width $2.^*$

* Improves Fleischer's (2019) bound of length $\mathcal{O}(\log^2 N)$ and width 3.

Theorem T., Weiß (2026)

Solvable groups admit straight-line programs of

- length $\mathcal{O}(\log N)$,
- length $\mathcal{O}(\operatorname{polylog} N)$ and width $\mathcal{O}(1)$.

Theorem — T., Weiß (2026)

Solvable groups admit straight-line programs of

- length $\mathcal{O}(\log N)$,
- length $\mathcal{O}(\operatorname{polylog} N)$ and width $\mathcal{O}(1)$.

Corollary — T., Weiß (2026)

The membership problem for *solvable groups* is in $\mathsf{NTIMESPACE}(\operatorname{polylog} n, \log n) \subseteq \mathsf{FOLL}$.*

* Conjectured by Barrington, Kadau, Lange and McKenzie (2001).

Questions?