Effective Versions of Strong Measure Zero

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Smallness Notions and Their Effectivizations

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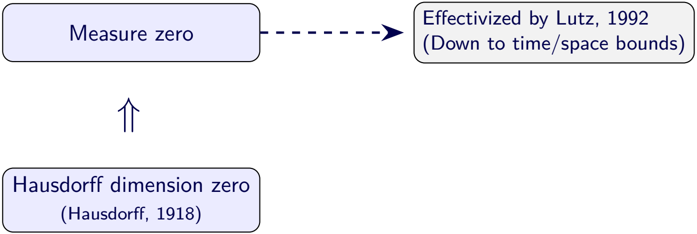

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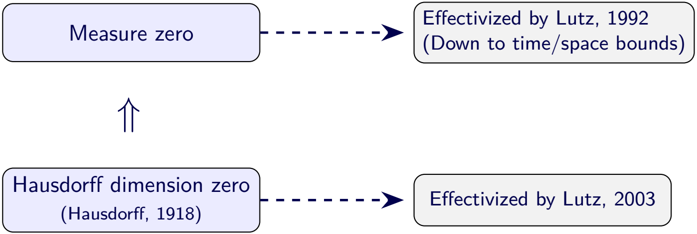

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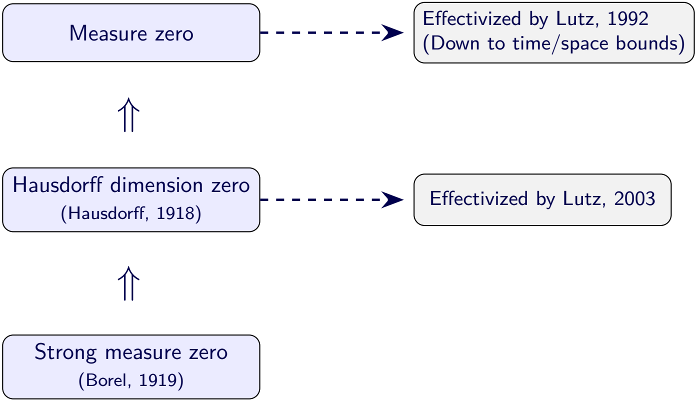

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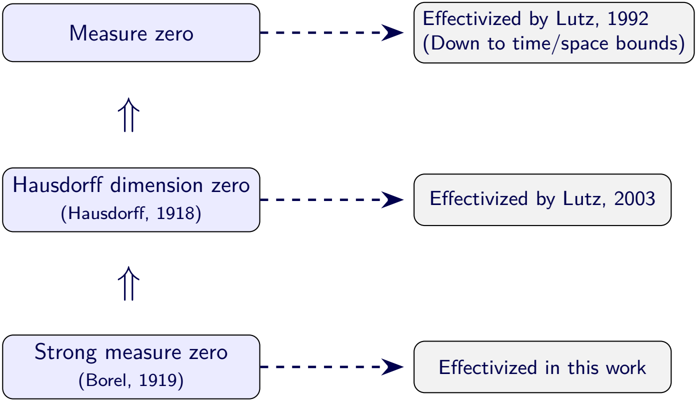

Measure zero → Effectivized by Lutz, 1992 (Down to time/space bounds)

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Strong measure zero (Borel, 1919) → Effectivized in this work

Strong Measure zero

Let $X \subseteq \mathbf{C}$ and $C_w = \{S \in \mathbf{C} | w \sqsubseteq S\}$.

- X has *measure zero* if for every $\epsilon > 0$ there is a set $A \subseteq \{0,1\}^*$ with
  1. $X \subseteq \bigcup_{w \in A} C_w$
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- (Borel 1919) X has *strong measure zero* (SMZ) if for every sequence $(\epsilon_n)_{n \in \mathbb{N}}$ there is $(w_n)_{n \in \mathbb{N}}$ with
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Observation: If X is countable then X has strong measure zero.

Borel's Conjecture

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- Borel 1919: Conjectured that a set has strong measure zero if and only if it is countable.
- Sierpiński 1928: There are uncountable strong measure zero sets assuming CH.
- Laver 1976: There is a model of ZFC where the conjecture holds.
- In this work effective versions of SMZ are defined with

Theorem

$$\begin{tikzcd}
\Delta_R\text{-}\mathit{SMZ} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \mathit{comp\text{-}SMZ} \arrow[r, Rightarrow] \arrow[d, Rightarrow, shift left, "/"{marking}] & \mathit{algorithmic\text{-}SMZ} \arrow[r, Rightarrow] \arrow[d, Rightarrow, shift left] & \mathit{SMZ} \arrow[d, Rightarrow, dashed, shift left] \\
\Delta_R\text{-}\mathit{countable} \arrow[r, Rightarrow] & \mathit{comp\text{-}countable} \arrow[r, Rightarrow] \arrow[u, Rightarrow, shift left] & \mathit{countable} \arrow[u, Rightarrow, shift left, "/"{marking}] & \mathit{countable} \arrow[u, Rightarrow, shift left]
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$$algorithmic - SMZ \iff NCR$$

Odds Supermartingales (Higuchi and Kihara)

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$$d(w) \geq \frac{d(w0)}{O(w0)} + \frac{d(w1)}{O(w1)}$$
for all w.

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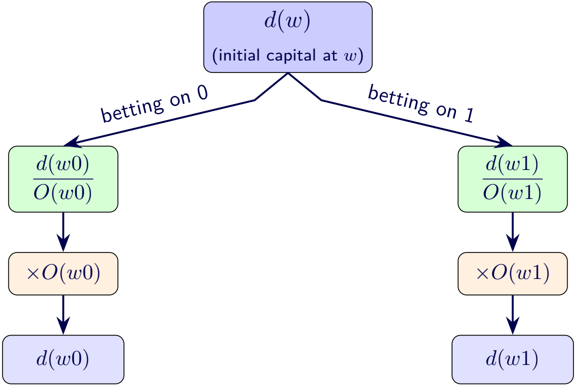

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- An acceptable odds function is a function $O : \{0,1\}^* \to [1,\infty)$ with $\prod_{w \sqsubset S} O(w) = \infty$ for all $S \in \mathbf{C}$.

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Theorem (Higuchi and Kihara 2014)

$X \subseteq \mathbf{C}$ has strong measure zero if and only if for every acceptable odds function O there is a O-supermartingale d with

$$\limsup_{n\to\infty} d(S[0\ldots n]) = \infty$$

for all $S \in X$.

Our Effective Strong Measure Zero

- Let Δ be a level of effectivity (complexity/computability).

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Definition

$X \subseteq \mathbf{C}$ has Δ-*strong measure zero* if there is a functional $F : (\{0,1\}^* \to [1,\infty)) \to (\{0,1\}^* \to [0,\infty))$ in Δ such that for every acceptable odds function O:

1. $F(O)$ is an O-supermartingale.
2. $\limsup_{n\to\infty} F(O)(S[0\ldots n]) = \infty$ for all $S \in X$.

Effective Borel Conjecture

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- The *result* of a constructor δ is the sequence $R(\delta) \in \mathbf{C}$ such that $\delta^i(\lambda) \sqsubset R(\delta)$ for all $i \in \mathbb{N}$.

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  3. $X \subseteq \{R(\delta_k) \mid k \in \mathbb{N}\}$.

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Lemma

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- Let $\delta : \mathbb{N} \times \{0,1\}^* \to \{0,1\}^*$ witness X being Δ-countable.
- Let O be an odds function.

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- for each $k \in \mathbb{N}$ let d_k be defined as:
  - $d_k(\lambda) = 2^{-k}$.
  - $d_k(w) = 2^{-k} \prod_{x \sqsubset w} O(x)$ if $w \sqsubset R(\delta_k)$ and 0 otherwise.

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Time/Space Bounded Borel Conjecture

Lemma

If X has Δ-strong measure zero for time/space bounded Δ then it is Δ-countable.

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Corollary

NP *has strong measure zero in* E *if and only if*

$$\mathrm{NP} \cap \mathrm{E} \subseteq \mathrm{DTIME}(2^{kn})$$

for some fixed $k \in \mathbb{N}$*.*

Computable Borel Conjecture

Does NOT hold.

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Definition

$S \in \mathbf{C}$ is *almost constructible* if there exists a computable

$$\delta : \mathbb{N} \times \{0,1\}^* \to \{0,1\} \times \{0,1\}^*$$

and for every $n \in \mathbb{N}, w \sqsubseteq S$ we have

1. $\delta(n, w) = (b, x)$ where $|x| = n$ and
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- Example: Let

  $$f(n) = \max\{t \in \mathbb{N} \mid \exists k \leq n \text{ and } M_k(k) \text{ halts in exactly } t \text{ steps}\}$$

  and

  $$S = \mathrm{Graph}(f) = \{\langle n, f(n)\rangle \mid n \in \mathbb{N}\}.$$

  Then S is almost constructible but not decidable.

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Every almost constructible sequence has computable strong measure zero

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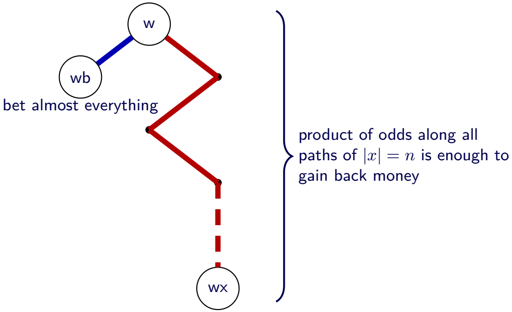

Outer Premeasures

An *outer premeasure* is function $\mu : \{0,1\}^* \to [0,\infty)$ satisfying the following for all $w \in \{0,1\}^*, a \in \{0,1\}$ and $S \in \mathbf{C}$.

1. (monotone) $\mu(w) \geq \mu(wa)$.
2. (subadditive) $\mu(w) \leq \mu(w0) + \mu(w1)$.
3. (atomless) $\liminf_{n\to\infty} \mu(S[0\ldots n]) = 0$.

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Lemma (Higuchi and Khiara 2014)

$X \subseteq \mathbf{C}$ has strong measure zero if and only if it has measure zero with respect to all outer premeasures.

Never Continuously Random (NCR)

NCR (Reimann, Slaman 2008):

- A Martin-Löf-μ test relative to r_μ is a sequence of uniformly Σ_1^0 sets $(W_n)_{n\in\mathbb{N}}$ relative to r_μ with $\mu(W_n) \leq 2^{-n}$ for all n.
- $S \in \mathbf{C}$ passes a Martin-Löf-μ test relative to r_μ if $S \notin \bigcap_{n\in\mathbb{N}} W_n$.
- $S \in \mathbf{C}$ is μ-random if it passes every Martin-Löf-μ test relative to r_μ for some representation r_μ of μ.
- $S \in \mathbf{C}$ is in NCR if it its not μ-random for any continuous **probability measure** μ.

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- $S \in \mathbf{C}$ is in NCR if it its not μ-random for any continuous **probability measure** μ.

Theorem

S has algorithmic (lower semicomputable) strong measure zero if and only if it is NCR.

However, classically NCR corresponds to universal measure zero.

Effective Coverings

Δ-strong measure zero can be characterized in terms of effective covers similar to Borel's original definition.

Theorem

A set X has Δ-strong measure zero if and only if there is an oracle Turing machine M in Δ such that for every infinite $A \subseteq \mathbb{N}$:

1. *If $n \in A$ then $M^A(n) = w$ for some w with $|w| = n$ (or does not halt for lower semicomputable).*
2. *For every $S \in X$ there exits an $n \in A$ such that $M^A(n) \sqsubseteq S$.*

Thank you!

Strong Packing Dimension Zero

Require success in the limit inferior instead:

Theorem

$$\begin{tikzcd}
\Delta_R\text{-\textit{SPDZ}} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{\textit{computable-SPDZ}} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{\textit{algorithmic-SPDZ}} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{\textit{SPDZ}} \arrow[d, Rightarrow, dashed] \\
\Delta_R\text{-\textit{countable}} \arrow[r, Rightarrow] & \text{\textit{weak computably-countable}} \arrow[r, Rightarrow] & \subseteq\text{\textit{DEC}} \arrow[r, Rightarrow] & \text{\textit{countable}} \arrow[u, Rightarrow]
\end{tikzcd}$$

- A *weak constructor* is a function $\delta : \mathbb{N} \to (\{0,1\}^*)^b$ for some $b \in \mathbb{N}$ satisfying $w \in \delta(n) \Rightarrow \exists u \in \delta(n-1)$ with $u \sqsubset w$ for all $n \in \mathbb{N}^+$.
- The *result* of a weak constructor is the set $\{S \in \mathbf{C} \mid \forall n \in \mathbb{N} \;\; \exists w \in \delta(n) \;\; w \sqsubseteq S\}$.
- A set $X \subseteq \mathbf{C}$ is weakly Δ-countable if there exists a function $\delta : \mathbb{N} \times \mathbb{N} \to \mathcal{P}\{0,1\}^*$ meeting the following properties.
    1. $\delta \in \Delta$.
    2. For each $k \in \mathbb{N}$, if we write $\delta_k(n) = \delta(k,n)$, then the function δ_k is a weak constructor.
    3. $X \subseteq \bigcup_{k \in \mathbb{N}} R(\delta_k)$.

Complexity of Computable SMZ Sequences

Lemma

Let X *be a* Σ^0_2 *set. Then the following are equivalent.*

1. X *is strongly coverable*
2. X *is algorithmically strongly coverable*
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Moreover 1 and 2 are equivalent for every union of Π^0_1 *sets.*

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Theorem (Cenzer et al. [**?**])

Let α be a computable ordinal. Then there is a countable Π^0_1 class containing x with $x \equiv_{\mathrm{T}} \emptyset^{(\alpha)}$.

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Corollary

for every computable ordinal α there is a x with $x \equiv_{\mathrm{T}} \emptyset^{(\alpha)}$ and $\{x\}$ having computable strong measure zero.

This matches a result of NCR (Kjos-Hanssen, Montalbán 2005).