On Effective Banach-Mazur Games and an application to the Poincaré Recurrence Theorem for Category

Prajval Koul & Satyadev Nandakumar

(Department of Computer Science and Engineering
Indian Institute of Technology Kanpur)

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Contents

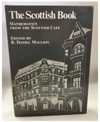
Introduction

Topics covered:

- Effective Meager sets.
- Strongly computable T_0-spaces.
- Effective Banach-Mazur games.
- Effective Poincaré recurrence.

The Banach-Mazur Game

$BM \langle M, C \rangle$



The Banach-Mazur Game

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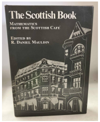
- Let I_0 be a closed interval.
- Player 1 is given an arbitrary subset $M \subseteq I_0$.
- Player 2 gets $B = I_0 \setminus M$.

- Now, a game is played as follows:
  - Player 1 chooses a closed interval $I_1 \subseteq I_0$.
  - Player 2 chooses a closed interval $I_2 \subseteq I_1$.
  - Player 1 chooses a closed interval $I_3 \subseteq I_2$...
- Player 1 and 2 determine a nested sequence $I_1 \supseteq I_2 \supseteq \cdots$ of closed intervals, where odd indices are picked by player 1.



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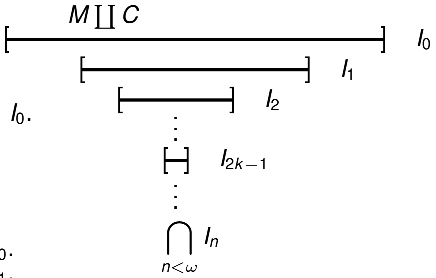

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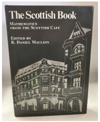

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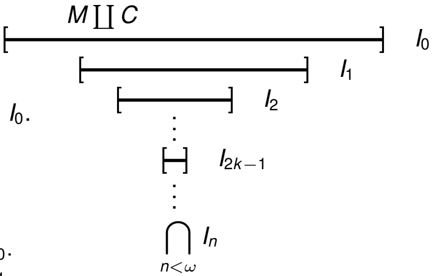

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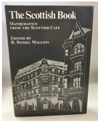

- If $M \cap \bigcap_n I_n \neq \varnothing$, player 1 wins. Else, player 2 wins.

The Banach-Mazur Game

- Can this game be effectivized?
- What does an effective strategy entail?
- What can we say about the sets "classified" by this effectivization?

- What constitutes an "effective" set?

Effectivizing the Game

Representation[1]

For a countable class $\mathcal{U}$, the partial computable surjection $\nu : \Sigma^* \dashrightarrow \mathcal{U}$, where $(\forall w \in dom(\nu))\,(\nu(w) \neq \varnothing)$, is said to be the *representation* of $\mathcal{U}$.

For $U \in \mathcal{U}$, $\nu^{-1}(U)$ refers to a *name* of U.

[1] Tanja Grubba, Matthias Schröder, and Klaus Weihrauch. Computable metrization. Math. Log. Q., 53(4-5):381–395, 2007. doi:10.1002/MALQ.200710009.

[2] K, Nandakumar, 2026.

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Computable T_0 Space[1]

A *computable T_0 space* is a tuple (X, τ, β, ν) such that:

- (X, τ) is a second countable T_0 space.
- β is a c.e. basis for the topology.
- $U \neq \varnothing$ for $U \in \beta$.
- For some c.e. set B, $(\forall u, v \in dom(\nu))\,(\nu(u) \cap \nu(v) = \bigcup\{\nu(w) : (u, v, w) \in B\})$.

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(X, τ, β, ν) is said to be a *strongly computable T_0 space*[2] if $(\forall u, v \in dom(\nu))(\nu(u) \cap \nu(v)$ is decidable$)$.

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Effectivizing the Game

Effective Nowhere Dense Set[1]

Let (X, τ, β, ν) be a computable T_0 space. A set $A \subseteq X$ is said to be *effectively nowhere dense* in X if there is a computable function $f : \Sigma^* \to \Sigma^*$ such that

$$(\forall w \in dom(\nu)) \left(\nu\left(f(w)\right) \subseteq \left(\nu(w) \setminus A\right)^o\right).$$

Recall that a set $A \subseteq X$ is nowhere dense if $\left(\bar{A}\right)^o = \varnothing$.

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Effective First Category Set[1]

A set is said to be of *effective first category*, if it can be represented as a c.e. union of effective nowhere dense sets.

Effective first category sets are also called effective meager sets.

[1] K, Nandakumar, 2026.

Effectivizing the Game

- Base assumptions:
  - The parent space (X, τ, β, ν) is a strongly computable T_0 space.
  - Players pick sets from $\mathcal{G} = \{G : G^0 \neq \varnothing\}$ such that $(\forall X \in \tau)\,(\exists G \in \mathcal{G})\,(G \subseteq X)$.
  - Player 2 has a c.e. strategy.

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c.e. Strategy for Player 2

A *c.e. strategy* for P_2 involves a sequence of c.e. choice functions

$$\left\{ f_n : f_n : (\Sigma^*)^{2n-1} \to \Sigma^* \right\}_{n \geq 1}$$

uniformly computable in n, such that for every response set of P_1, P_2's response is a c.e. set.

Theorem

Theorem 1

In a strongly computable T_0 space (X, τ, β, ν) with $M \coprod C = X$, the game $BM\langle M, C\rangle$ has an effective winning strategy for P_2 if and only if M is an effective first category set in X.

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Proof:

- Let $M = \bigcup_{n \geq 1} M_n$ be an effective first category set.
  Each M_n here is an effective nowhere dense set.
- Denote by $\mathcal{G}$, the class of playable sets.
- Stage k:
  - P_1 plays $G_{2k-1} \in \mathcal{G}$.
  - P_2 responds with G_{2k} (the first set in the enumeration of $G_{2k-1} \setminus \overline{M_k}$).
- Key Observation: $M \bigcap \left(\bigcap_{n \geq 1} G_n \right) = \varnothing$.
- P_2 wins.

Theorem

Proof (cont...):

- Now, let P_2 have an effective winning strategy $G_2, G_4, G_6, \ldots$, where $G_{2k} = f_k(G_1, G_2, \cdots, G_{2k-1})$.
- Let $\mathcal{H}_n$ be a *maximal* family of c.e. open dense sets in the parent space.
- With $G = \bigcap_{n\in\mathbb{N}} \bigcup \mathcal{H}_n$,
  since P_2 has a winning strategy, $G = \bigcap_{n\in\mathbb{N}} \bigcup \mathcal{H}_n \subseteq C$.
- Therefore, $G^c \supseteq M$, and hence M, is an effective first category set in X. $\square$

Theorem

- There is another version of this effectivization that talks about the winning criterion for P_1.
- P_1 wins the game iff C is of effective first category at some point in X.

Poincaré Recurrence

Poincaré Recurrence Theorem[1]

Let $X \subseteq \mathbb{R}^n$ be a bounded open region equipped with a measure preserving homeomorphism T onto itself, such that it admits no non-empty wandering open set. Then, all the points of X, except a set of measure zero and first category, are recurrent under T.

Recall...

- Recurrence
  - For a space $X \subseteq \mathbb{R}^n$ equipped with a homeomorphism T onto itself, and and open set $G \subseteq X$, a point $x \in G$ is said to be *recurrent* with respect to G, if $(\exists^\infty i \in \mathbb{N}) \left(T^i x \in G\right)$.
  - $x \in X$ is said to be recurrent under T, if for every open $U \ni x$, x is recurrent with respect to U.

- Wandering Set
  - For a space X equipped with a surjective map $T : X \to X$, an open set $E \subseteq X$ is said to be *wandering* if the sets in the sequence $\{T^{-i}E\}_{i \geq 0}$ are disjoint.

[1] H. Poincaré. Les méthodes nouvelles de la méchanique céleste, volume 3. Gauthier-Villars, 1899.

Poincaré Recurrence

Effective Poincaré Recurrence Theorem for Category[1]

Let $X \subseteq \mathbb{R}^n$ be a bounded c.e. open region equipped with a computable homeomorphism T onto itself, such that it admits no non-empty wandering open set. Then all the points of X, except a set of effective first category, are recurrent under T.

[1] The analogous theorem for effective measure also holds true and was shown by "Nitesh Vijayvargiya. Poincare non-recurrent points form an effective measure zero set. Master's thesis, Indian Institute of Technology Kanpur, 2014".

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Proof:

- Let $E \subsetneq X$ be a c.e. open set.
- Denote by $N(E)$ the set of non-recurrent points in E.
- Let $F_n(E) = \{x \in E : (\forall j > n)(T^{-j}x \notin E)\}$.
  Then $N(E) = \bigcup_{n \in \mathbb{N}} F_n(E)$.
- Key observation: $F_k(E)$ is not effective dense in E.

[1] The analogous theorem for effective measure also holds true and was shown by "Nitesh Vijayvargiya. Poincare non-recurrent points form an effective measure zero set. Master's thesis, Indian Institute of Technology Kanpur, 2014".

Poincaré Recurrence

Proof (cont...):

- Consider the game $BM\,\langle N(E), X \setminus N(E)\rangle$.
- At stage k,
  - P_1 plays the set G_{2k-1}.
  - P_2 plays the set G_{2k}, the first set in the enumeration of $G_{2k-1} \setminus \overline{F_k(E)}$.
    (Since $F_k(E)$ is not dense in E, $(\exists G_{2k} \in \mathcal{G})\,(G_{2k} \neq \varnothing)$.)
- Recall that $N(E) = \bigcup_{n\in\mathbb{N}} F_n(E)$.
- Since $N(E) \cap \left(\bigcap_{k\in\mathbb{N}} G_k\right) = \varnothing$, P_2 wins the game.
- Hence $N(E)$ is of effective first category. $\square$

Key References

1. Tanja Grubba, Matthias Schröder, and Klaus Weihrauch. Computable metrization. Math. Log. Q., 53(4-5):381–395, 2007. doi:10.1002/MALQ.200710009.

2. Ryszard Engelking. General topology, volume 6 of Sigma Series in Pure Mathematics. Helder-mann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

3. S. Banach. Théoreme sur les ensembles des première catégorie. Fund. Math., 16:395–398, 1930.

4. Peter Maličký. Category version of the Poincaré recurrence theorem. Topology Appl., 154(14):2709–2713, 2007. doi:10.1016/j.topol.2007.05.004.

5. H. Poincaré. Les méthodes nouvelles de la méchanique céleste, volume 3. Gauthier-Villars, 1899.

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