To buy or not to buy: deterministic rent-or-buy problems on node-weighted graphs

STACS 2026

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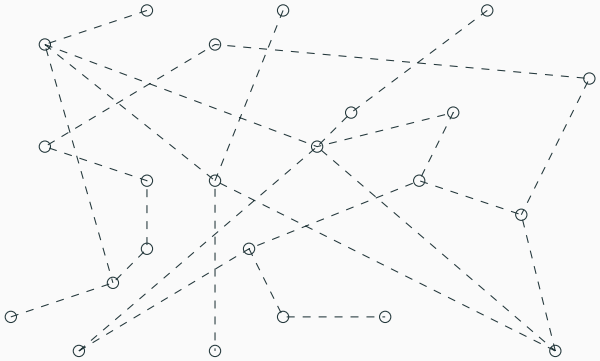
Online Steiner Forest



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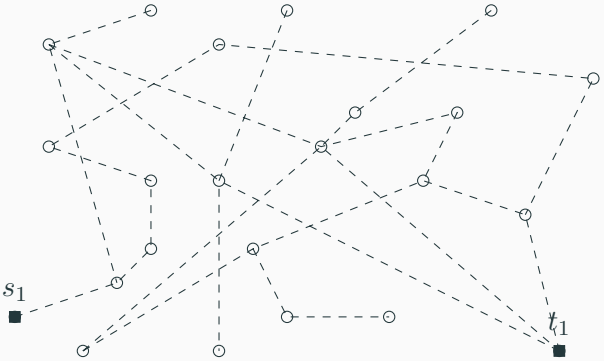

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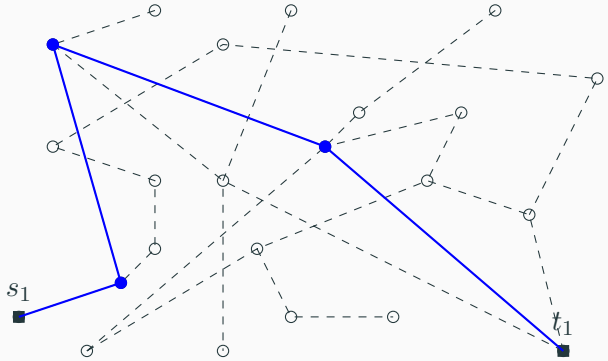

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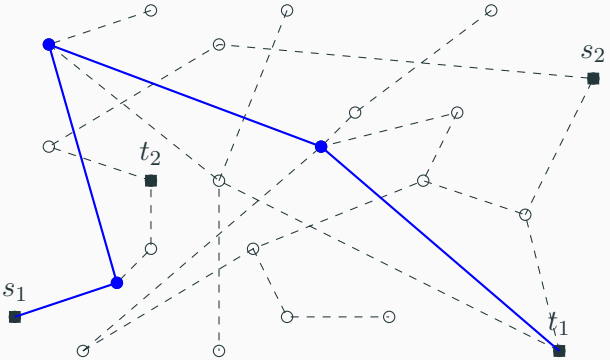

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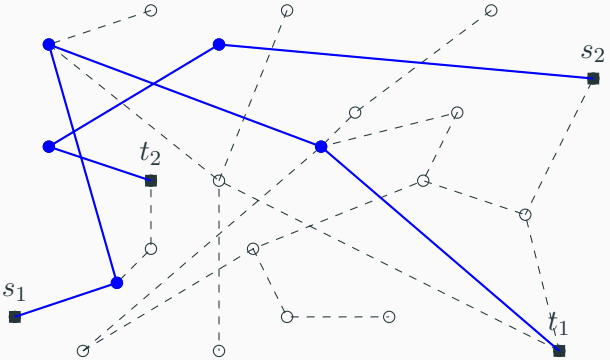

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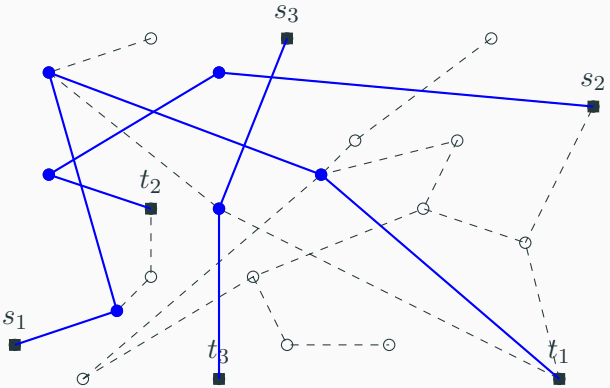

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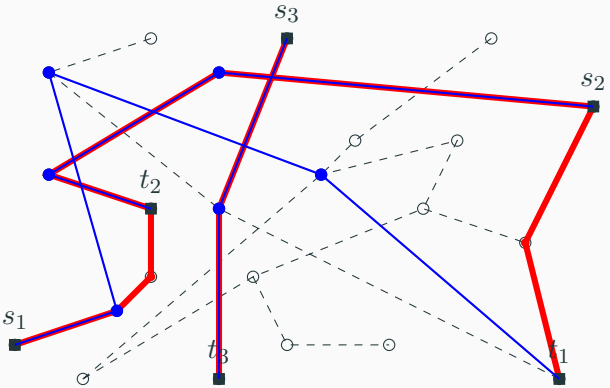

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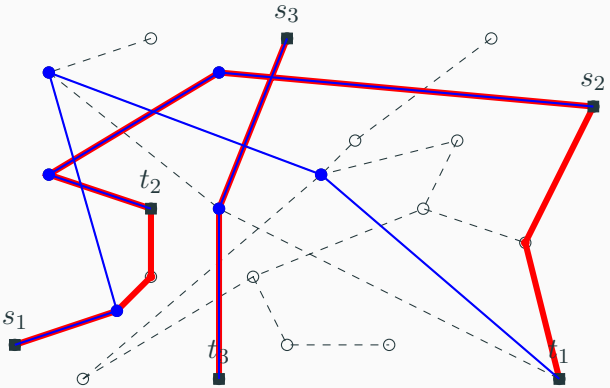

- Objective: buy edges/nodes to connect s_i and t_i for all i.
- Algorithm is α-competitive if cost(ALG) $\leq \alpha \cdot$ OPT for all inputs.
- Special case: online Steiner tree

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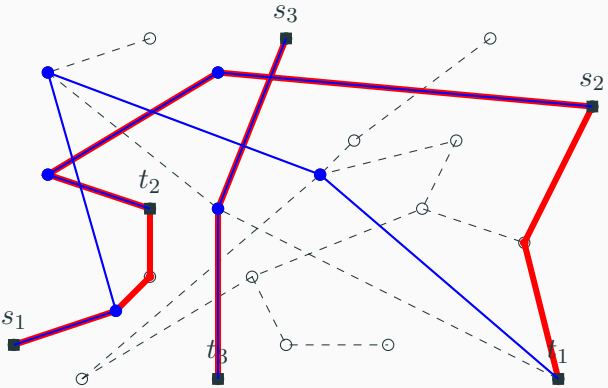

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  k: # node pairs

  (Imase, Waxman 1991)

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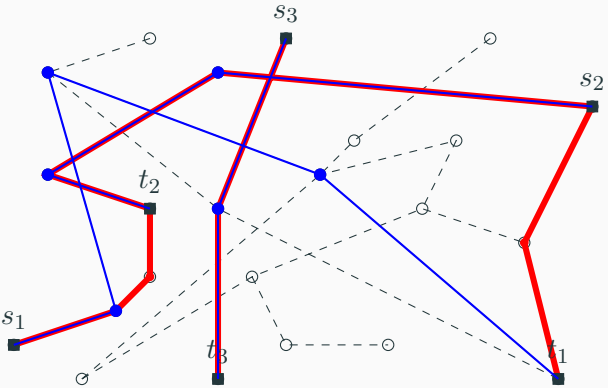

- Edge weights:
  $O(\log(k))$
  k: # node pairs
  (Imase, Waxman 1991)
- Node weights:
  $O(\log(n)^2)$
  n: # nodes
  (B., Eliáš, Venzin, 2025)

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Rent or Buy

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- Reusing bought edges/nodes for later requests is free.

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Black-box reduction to buy-only (Awerbuch, Azar, Bartal 2004)

With probability $1/M$ buy using algorithm for buy-only, otherwise rent.

- If buy-only algorithm is c-competitive then the above is $O(c)$-competitive.
- Resulting algorithm is randomized.

How to derandomize?

- Deterministic c-competitive buy-only algorithm $\implies$ deterministic $O(c^2)$-competitive algorithm for rent-or-buy (Bartal, Charikar, Indyk 2001)
  - $O(\log(k)^2)$ for edge-weighted graphs.
  - $O(\log(n)^4)$ for node-weighted graphs.

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- For edge-weighted graphs: deterministic $O(\log(k))$-competitive algorithm (Umboh 2014)

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- For edge-weighted graphs: deterministic $O(\log(k))$-competitive algorithm

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- Our main result:
  For node-weighted graphs: deterministic $O(\log(n)^2)$-competitive algorithm

This presentation

- Restrict to online Steiner tree for simplicity.

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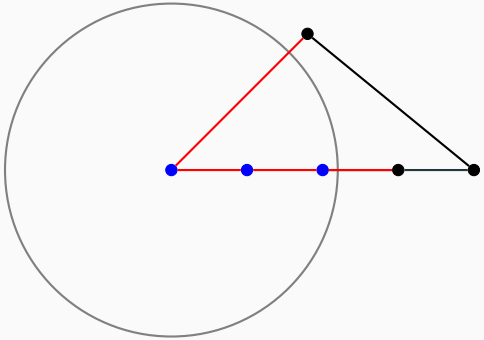
1. Algorithm for edge-weighted case (Umboh 2014).
2. What makes node-weighted case more challenging?
3. Our algorithm for deterministic rent-or-buy on node-weighted graphs.
4. Open questions

Deterministic algorithm for edge-weighted graphs

- Let $B(v, r)$ be the set of all nodes within distance r of v ($B(v, r) = \{u \in V : d(u, v) \leq r\}$).
- Let $B_{\text{edge}}(v, r)$ be the set of all edges within distance r of v ($B(v, r) = \{(u, w) \in E : \min(d(u, v), d(w, v)) \leq r\}$).

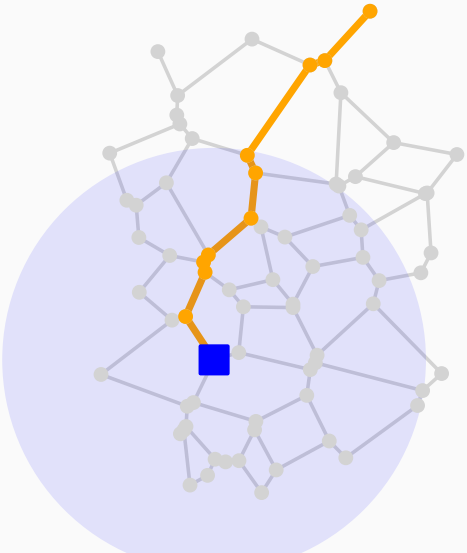


Deterministic algorithm for edge-weighted graphs

Algorithm for edge-weighted case

1: **when** terminal i arrives **do**
2: Set j s.t. distance to bought tree $\approx 2^j$.
3: **if** $R_j \cap B(i, 2^{j-1}) \geq M$ **then**
4: Buy path to bought tree.
5: **else**
6: Rent path to bought tree.
7: Add i to R_j.

Deterministic algorithm for edge-weighted graphs



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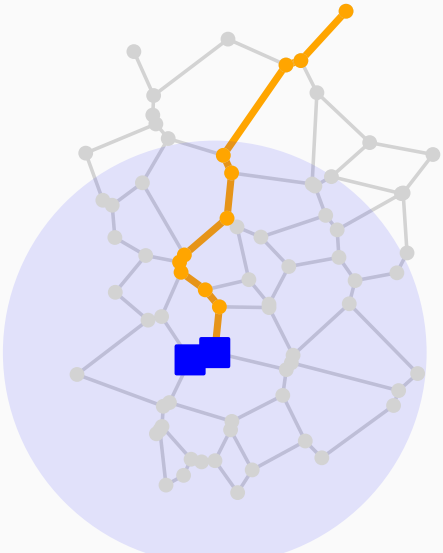

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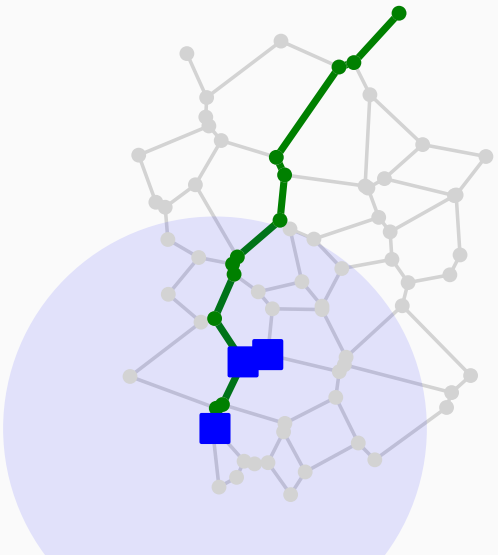

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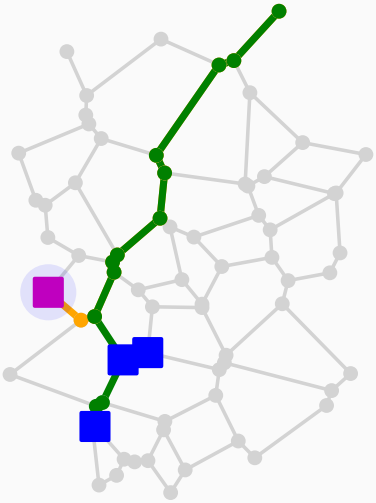

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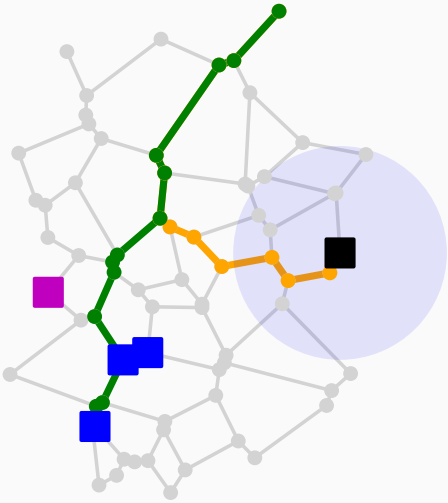

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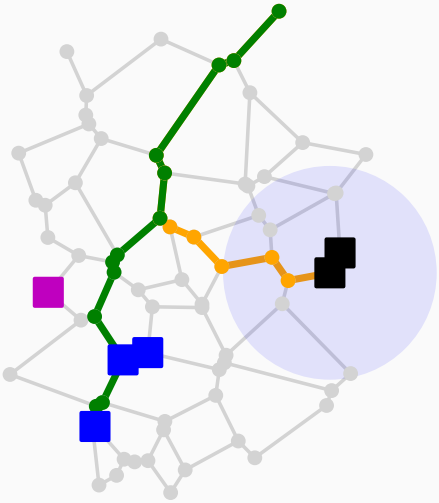

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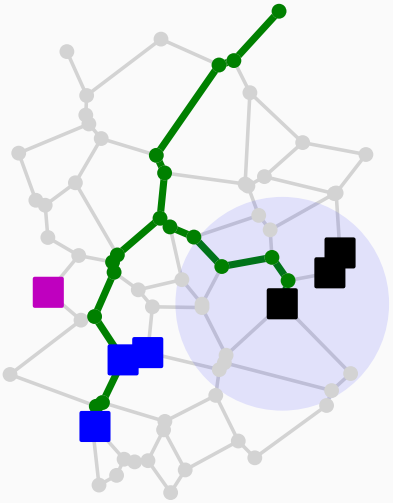

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Analysis for edge-weighted graphs

Lemma

For a set of terminals S with radius $y_s \geq 0$ for $s \in S$, if every edge $e \in E$ is contained in at most M balls $B_{\text{edge}}(s, y_s)$, then:

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- Our construction ensures: $|B(s, 2^{j-1}) \cap R_j| \leq M$ for each $s \in R_j$.
- For each j, setting $y_s := 2^{j-2}\mathbf{1}_{s \in R_j}$ satisfies the above condition.
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For node-weighted graphs

Lemma

For a set of terminals S with radius $y_s \geq 0$ for $s \in S$, if every node $e \in E$ is contained in at most M balls $B(s, y_s)$, then:

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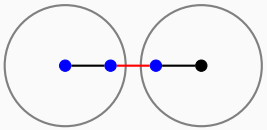
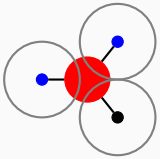
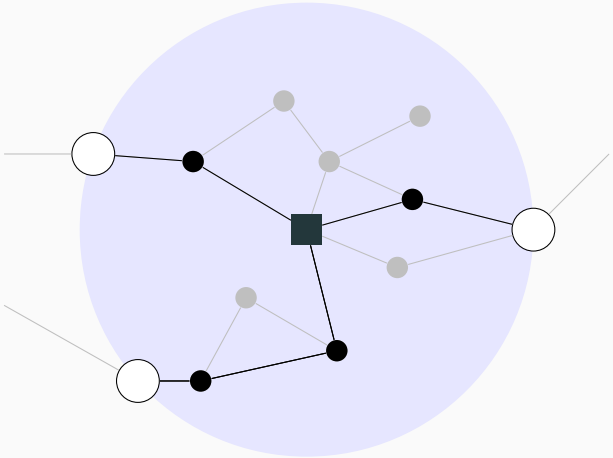
- For each j, setting $y_s := 2^{j-2}\mathbf{1}_{s \in R_j}$ no longer satisfies the above condition.





Choosing nodes on the boundary using prize-collecting set cover



- ■ Arriving terminal
- ○ Node on the bounary

- On arrival of terminal ■:
  Either:
  - buy boundary node ○
  - or pay a penalty for renting.

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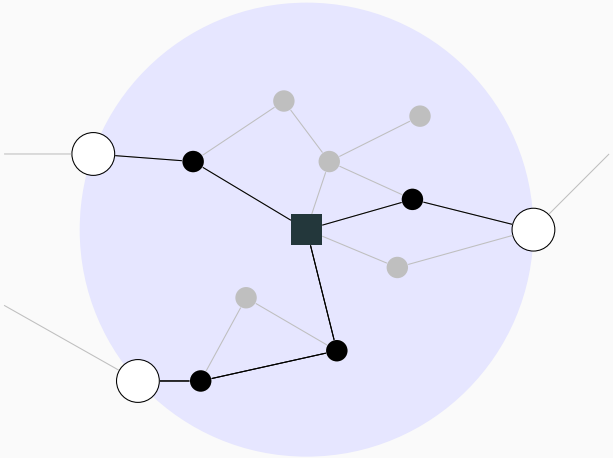

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- Prize-collecting set cover
- $\log(n)\log(m)$-competitive online algorithm

(n: # sets, m: # elements)

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There is a $\log n \log \bar{n}$ competitive deterministic algorithm for node-weighted + edge-weighted rent-or-buy. (n : #nodes, $\bar{n}$: #nodes with nonzero weight)

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Open questions

- Better deterministic black-box reduction from rent-or-buy to buy-only (for monotonic task systems)
- Buy-at-bulk (generalizing to concave node costs)
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Thanks for your attention!
Questions?