Demystifying Codensity Monads via Duality

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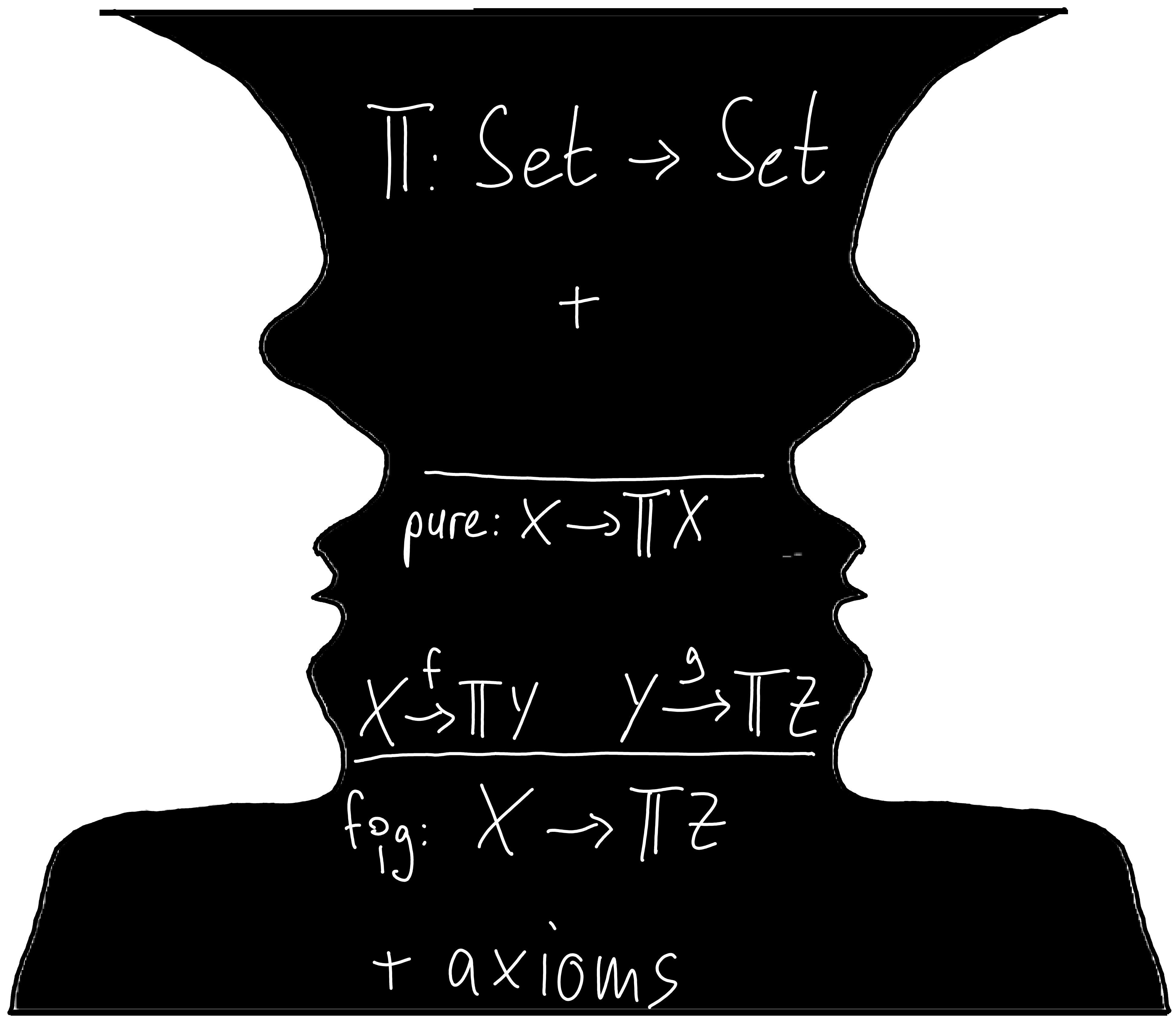
CO_2 ↓↓↓ low emission presentation

Monads in Computer Science

Monad $\mathbb{T}$:



"Monads $\hat{=}$ Effects"

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Monads in Computer Science

[1] Moggi, 1991, I&C

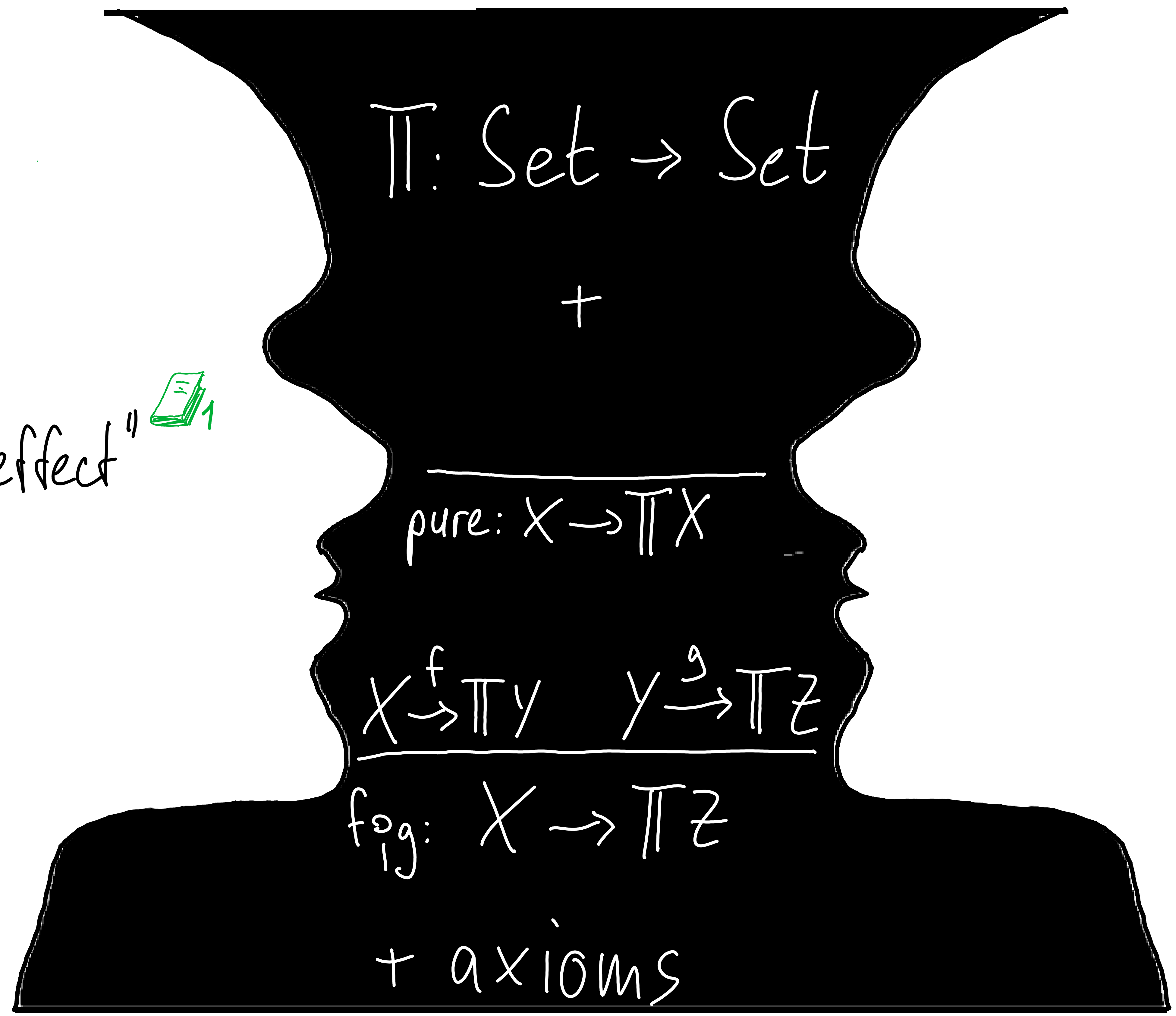
<u>Effectful Computations</u>

Kleisli Maps

$X \to \mathbb{T}Y$

"map $X \to Y$ with $\mathbb{T}$-effect"[1]

Monad $\mathbb{T}$:



"Monads $\hat{=}$ Effects"

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Monads in Computer Science

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Effectful Computations

Monad $\mathbb{T}$:

Effectful Structures

Kleisli Maps

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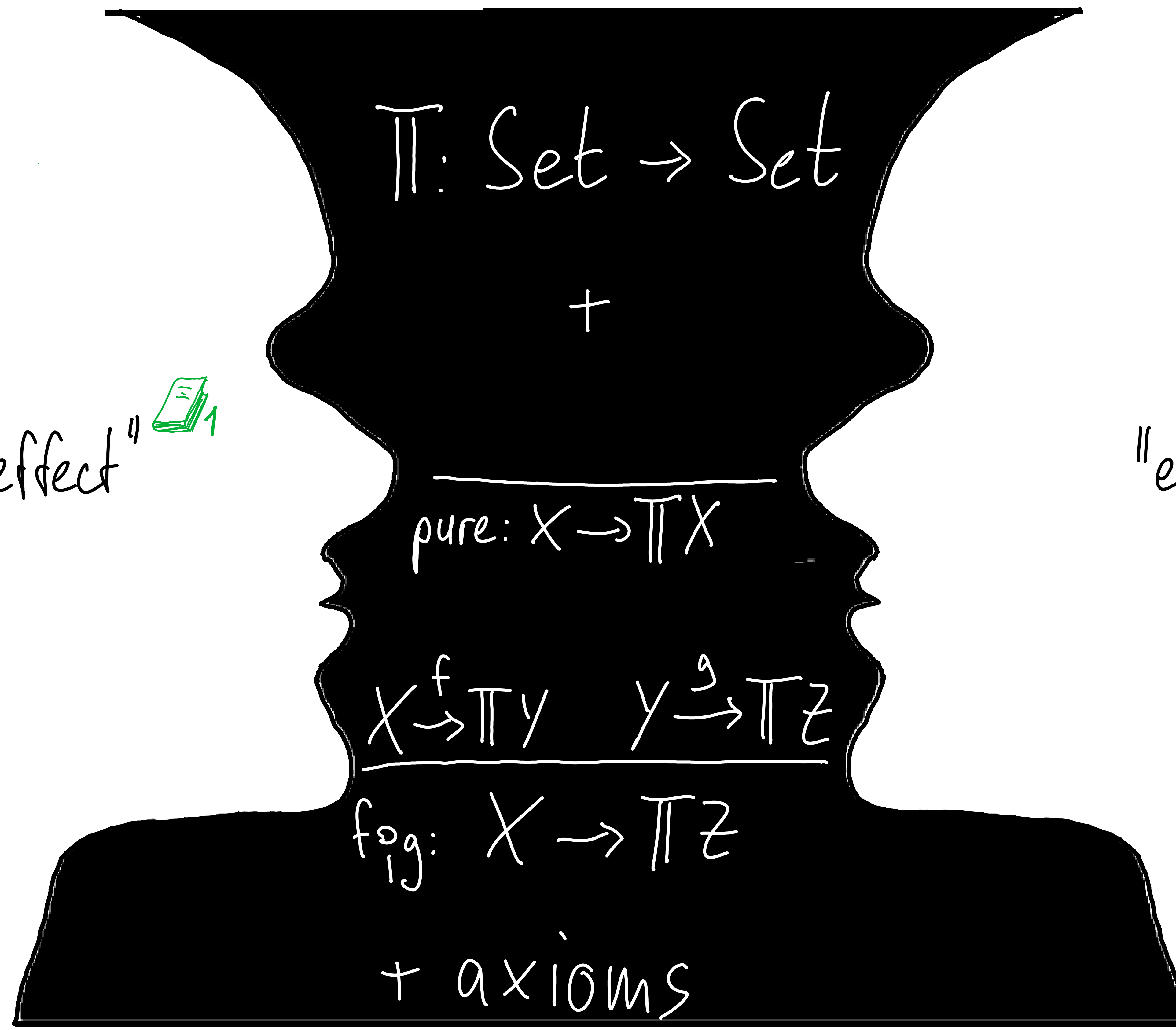

Algebras

$\mathbb{T}A \xrightarrow{a} A$ (+axioms)

"evaluate $\mathbb{T}$-effects"

Syntax "Monads $\hat{=}$ Effects" Semantics

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Monads: Examples

$\mathbb{T}X$	Computations	Algebras
$\mathcal{P}_f X = \{A \subseteq X \mid A \text{ finite}\}$	nondeterminism	semilattices
$X^* = \{x_1 \cdots x_n \mid x_i \in X\}$	list of results	monoids
$\mathcal{D}X = \{\sum_{i=1}^{n} r_i x_i \mid r_i \in [0,1], x_i \in X, \sum_{i=1}^{n} r_i = 1\}$	probabilities	convex sets

Monads: Examples

[1] Jacobs-Mandemaker I&C'16

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$C^{C^X} = \{\text{maps } C^X \to C\}$	"CPS-style"	complete atomic Boolean algebras
$\mathcal{U}X = \{U \subseteq 2^X \mid U \text{ ultrafilter}\}$		compact Hausdorff spaces
$\mathbb{E}X = \{\text{fin. add. prob. measures } 2^X \to [0,1]\}$	"probabilistic CPS" [1]	observable convex CH spaces
$\widehat{X^*}$ = free profinite monoid on X		

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Leinster: Where do Monads come from?

generators equations

Monoid $\mathbb{N}\times\mathbb{N} \cong \langle a, b \mid ab = ba \rangle$

Vector space $\mathbb{R}^2 \cong \langle x, y, z \mid 10x = -3z \rangle$

e.g. e.g.

Idea from algebra:
<u>Presentations</u>

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presentations describe properties

Idea from algebra: Presentations

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has a presentation

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Idea from algebra:
<u>Presentations</u>

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$\Rightarrow$

Presenting
Monads!

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Monads: Examples

[1] Altenkirch et al. FOSSACS'10
[2] Fiore et al. LICS'99

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well-understood:

(finitary!) algebraic theories

For $(-)^*$: $\langle \cdot/2, e/0 \mid x \cdot e = e \cdot x = e, \; x \cdot (y \cdot z) = (x \cdot y) \cdot z \rangle$

generators/operations

equations

Categorically stable model:

Lawvere Theories

$(\mathbf{Set}_{fin} \hookrightarrow \mathbf{Set})$-relative monads [1]

monoids [2] in $\mathbf{Set}^{\mathbf{Set}_{fin}}$

Monads: Examples

Very complicated presentations as highly infinitary algebraic theories

class of operations
+ unbounded arities

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$\boxed{\text{Find different type of presentation?}}$

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Codensity Presentations

[1] Kock '66

F functor: preserves id + composition

Kock[1]: Every compatible assignment $\mathrm{Set}_{fin} \xrightarrow{F} \mathrm{Set}$ induces codensity monad $\hat{F}$ of F

Def:

$$\boxed{\hat{F}X = \{(a_f)_f \mid f: X \to FA,\ A \in \mathrm{Set}_{fin},\ a_f \in FA.\ \forall g: A \to B: F(g)(a_f) = a_{g \circ f}\} \quad (= \mathrm{Ran}_F F)}$$

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Intuition: generalizes closure operators on topological spaces

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Codensity Presentations

[1] Koch '66
[2] Avery JPAA'16 [3] van Belle TAC'22 [4] Shirazi '24

$\mathcal{F}$ functor: preserves id + composition

Kock[1]: Every compatible assignment $\mathrm{Set}_{fin} \xrightarrow{\mathcal{F}} \mathrm{Set}$ induces codensity monad $\hat{\mathcal{F}}$ of $\mathcal{F}$

Def:

$$\boxed{\hat{\mathcal{F}}X = \{(a_f)_f \mid f: X \to \mathcal{F}A,\ A \in \mathrm{Set}_{fin},\ a_f \in \mathcal{F}A.\ \forall g: A \to B: \mathcal{F}(g)(a_f) = a_{g \circ f}\} \quad (= \mathrm{Ran}_{\mathcal{F}}\mathcal{F})}$$

✓ "$\mathcal{F}$ presentation for $\hat{\mathcal{F}}$"

↳ every monad $\mathbb{T} \cong \hat{\mathcal{F}}$ for some $\mathcal{F}$

✓ Non-trivial properties of $\hat{\mathcal{F}}$ deriveable from props. of $\mathcal{F}$[4]

Codensity Presentations

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✓ "F presentation for $\hat{F}$"

↳ every monad $\mathbb{T} \cong \hat{F}$ for some F

✓ Non-trivial properties of $\hat{F}$ deriveable from props. of F[4]

✗ Proofs $\hat{F} = \mathbb{T}$ complicated

↳ e.g.:

$F: \mathrm{Set}_{fin} \to \mathrm{Set}$		$\hat{F}$
A	$\mapsto A$	$\mathcal{U}$
A	$\mapsto C$	2^{2^C}

↳ worse in algebraic/continuous settings: requires domain-specific knowledge[2,3,4]

Our Contribution

Freyd: "Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial"

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Idea: Reduce proofs $\mathbb{T} = \hat{\mathbb{F}}$ to two ingredients

$$\boxed{\text{Codensity Monads} = \text{Density} + \text{Duality}}$$

Our Contribution

[1]Avery'16, [2]Adamek-Sousa'21, [3]Gehrke et al.'20, [4]Kock'66, [5]Leinster '16
[6]Reggio'20, [7]Shirazi'24, [8]Sipoş'18, [9]van Belle'21

Freyd: "Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial"

<u>Idea</u>: Reduce proofs $\mathbb{T} = \hat{\mathbb{F}}$ to two ingredients

$$\boxed{\text{Codensity Monads} = \text{Density} + \text{Duality}}$$

- Capture + simplify <u>all</u> instances in the literature [1,2,3,4,5,6,7,8,9]
- Several <u>new</u> presentations for important monads

Warning:

The next slide contains category theory

Main Theorem

[1] Mateo TAC '25 proved different formulation

$$
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F"] & \mathcal{C} \arrow[loop right, "\widehat{F}"] \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{D}_0^{op} \arrow[u, "E"] \arrow[r, squiggly, "G^{op}"] & \mathcal{D}^{op} \arrow[u, bend right, "R"']
\end{tikzcd}
$$

Theorem:

$$\boxed{G \text{ dense} + RG^{op} \cong FE \implies RL \cong \widehat{F}\ [1]}$$

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Theorem:

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<u>known</u>

$$\begin{tikzcd} \mathbf{Set}_f \arrow[r, hook, "\iota"] & \mathbf{Set} \arrow[loop right, "\widehat{\iota}"] \arrow[d, shift right, "2^-"'] \arrow[d, phantom, "\dashv"] \\ \mathbf{BA}_f^{op} \arrow[u, phantom, "\wr"] \arrow[r, hook] & \mathbf{BA}^{op} \arrow[u, shift right, "\mathbf{BA}(-,2)"'] \end{tikzcd}$$

density: trivial

$\implies \widehat{\iota} \cong \mathbf{BA}(2^-, 2) \cong \mathcal{U}$

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Main Theorem

[1] Mateo TAC '25 proved different formulation
[2] Staton-Uijlen I&C'18 [3] Jacobs TCS'10

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density: trivial

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finite stochastic matrices

new

$$
\begin{tikzcd}
\mathrm{Stoch}_f \arrow[r, "D"] & \mathrm{Set} \arrow[loop right, "\hat{D}"] \arrow[d, bend right, "{[0,1]^-}"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{EMod}_{fd}^{op} \arrow[u, "\wr"] \arrow[r, hook] & \mathrm{EMod}^{op} \arrow[u, bend right, "\mathrm{State}\ [3]"']
\end{tikzcd}
$$

"probabilistic vector spaces"

density: [2]

$$\implies \hat{D} \cong \mathrm{State}([0,1]^-) \cong \mathbb{E}$$

Two Ingredients:

① Dual adjunctions restricting to simple dualities

$$
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F"] \arrow[d, phantom, "\cong"] & \mathcal{C} \arrow[loop right, "\hat{F}"] \arrow[d, shift right, "L"'] \arrow[d, phantom, "\dashv"] \\
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algebra	vs. topology
Boolean algebras	sets
frames	topological spaces
ω-Boolean algebras	measurable spaces

Two Ingredients:

① Dual adjunctions restricting to simple dualities

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "F"] \arrow[d, phantom, "\cong"] & \mathcal{C} \arrow[loop above, "\hat{F}"] \arrow[d, shift left, "R"] \arrow[d, phantom, "\dashv"] \\ \mathcal{D}_0^{op} \arrow[r, "G^{op}"] & \mathcal{D}^{op} \arrow[u, shift left, "L"] \end{tikzcd}$$

algebra	vs. topology
Boolean algebras	sets
frames	topological spaces
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② Compute density instead of codensity

measured by $\hat{F}$

$X \in \mathcal{D}$ "determined" by G-subobjects

- $\mathbb{Q} \hookrightarrow \mathbb{R}$ dense
- $\{1\} \hookrightarrow Set$ dense
- $Alg_{fp} \hookrightarrow Alg$ dense

formally dual, but "less natural"

e.g. $\{2\} \subseteq Stone$

$\{S^1 \times S^1\} \subseteq AbGrp(CHaus)$

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1. Dual adjunctions restricting to simple dualities

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2. Compute density

$X \in \mathcal{D}$ "determined" by G-subobjects

$$\begin{tikzcd} GA \arrow[dr] \arrow[drr, bend left] & & \\ & GC \arrow[r] & X \\ GB \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

- $\mathbb{Q} \hookrightarrow \mathbb{R}$ dense
- $\{1\} \hookrightarrow \mathrm{Set}$ dense
- $\{a,b\}^* \hookrightarrow \mathrm{Mon}$ dense

well-studied!

The Power of Category Theory!

Freyd: "Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial"

Clarity: separate trivial overhead from domain-specific requirements:

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Clarity: separate trivial overhead from domain-specific requirements:

- $\overbrace{\mathrm{Set}_{\mathrm{Fin}} \hookrightarrow \mathrm{Set} \cong \mathcal{U}}$ is "completely categorical"

The Power of Category Theory!

> Freyd: "Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial"

Clarity: separate trivial overhead from domain-specific requirements:

$\widehat{\mathrm{Set}_{fin} \hookrightarrow \mathrm{Set}} \cong \mathcal{U}$ is "completely categorical"

vs.

$\widehat{\mathrm{Stoch}_{fin} \xrightarrow{D} \mathrm{Set}} \cong \mathbb{E}$ requires integral representation theorem

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Takeaway

$$
\begin{tikzcd}
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Codensity monads $\triangleq$ duality + density

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