Decidability of Extensions of Presburger Arithmetic by Hardy Field Functions

Or: If it looks like a polynomial, it's problematic

Hera Brown [1]

Jakub Konieczny [1,2]

[1]University of Oxford

[2]Kyiv School of Economics

March 11th, 2026
STACS'26

Decidable and undecidable theories

- In this talk we're interested in extensions of Presburger arithmetic.

Decidable:

$\mathrm{Th}(\mathbb{Z}; +)$
Presburger arithmetic

$\mathrm{Th}(\mathbb{Z}; +, x \mapsto 2^x)$
Semënov arithmetic

$\mathrm{Th}(\mathbb{Z}; +, \lfloor f \rceil)$
(with f a Hardy field function)
???

Undecidable:

$\mathrm{Th}(\mathbb{Z}; +, \cdot)$
Peano arithmetic

$\mathrm{Th}(\mathbb{Z}; +, x \mapsto x^2)$
Presburger + squaring function

- Where is the boundary between decidability and undecidability?

Decidable and undecidable theories

- In this talk we're interested in extensions of Presburger arithmetic.

Decidable:

$\mathsf{Th}(\mathbb{Z};+)$
Presburger arithmetic

$\mathsf{Th}(\mathbb{Z};+,x\mapsto 2^x)$
Semënov arithmetic

$\mathsf{Th}(\mathbb{Z};+,\lfloor f\rceil)$
(with f a Hardy field function)
???

Undecidable:

$\mathsf{Th}(\mathbb{Z};+,\cdot)$
Peano arithmetic

$\mathsf{Th}(\mathbb{Z};+,x\mapsto x^2)$
Presburger + squaring function

- Where is the boundary between decidability and undecidability?

Decidable and undecidable theories

- In this talk we're interested in extensions of Presburger arithmetic.

Decidable:

$\mathrm{Th}(\mathbb{Z};+)$
Presburger arithmetic

$\mathrm{Th}(\mathbb{Z};+,x\mapsto 2^x)$
Semënov arithmetic

$\mathrm{Th}(\mathbb{Z};+,\lfloor f \rceil)$
(with f a Hardy field function)
???

Undecidable:

$\mathrm{Th}(\mathbb{Z};+,\cdot)$
Peano arithmetic

$\mathrm{Th}(\mathbb{Z};+,x\mapsto x^2)$
Presburger + squaring function

- Where is the boundary between decidability and undecidability?

Decidable and undecidable theories

- In this talk we're interested in extensions of Presburger arithmetic.

Decidable:

$\mathrm{Th}(\mathbb{Z}; +)$
Presburger arithmetic

$\mathrm{Th}(\mathbb{Z}; +, x \mapsto 2^x)$
Semënov arithmetic

$\mathrm{Th}(\mathbb{Z}; +, \lfloor f \rceil)$
(with f a Hardy field function)
???

Undecidable:

$\mathrm{Th}(\mathbb{Z}; +, \cdot)$
Peano arithmetic

$\mathrm{Th}(\mathbb{Z}; +, x \mapsto x^2)$
Presburger + squaring function

- Where is the boundary between decidability and undecidability?

Decidable and undecidable theories

- In this talk we're interested in extensions of Presburger arithmetic.

Decidable:

$\mathrm{Th}(\mathbb{Z};+)$
Presburger arithmetic

$\mathrm{Th}(\mathbb{Z};+,x\mapsto 2^x)$
Semënov arithmetic

$\mathrm{Th}(\mathbb{Z};+,\lfloor f\rceil)$
(with f a Hardy field function)
???

Undecidable:

$\mathrm{Th}(\mathbb{Z};+,\cdot)$
Peano arithmetic

$\mathrm{Th}(\mathbb{Z};+,x\mapsto x^2)$
Presburger + squaring function

- Where is the boundary between decidability and undecidability?

Decidable and undecidable theories

- In this talk we're interested in extensions of Presburger arithmetic.

Decidable:

$\mathrm{Th}(\mathbb{Z}; +)$
Presburger arithmetic

$\mathrm{Th}(\mathbb{Z}; +, x \mapsto 2^x)$
Semënov arithmetic

$\mathrm{Th}(\mathbb{Z}; +, \lfloor f \rceil)$
(with f a Hardy field function)
???

Our result →

Undecidable:

$\mathrm{Th}(\mathbb{Z}; +, \cdot)$
Peano arithmetic

$\mathrm{Th}(\mathbb{Z}; +, x \mapsto x^2)$
Presburger + squaring function

- Where is the boundary between decidability and undecidability?

Decidable and undecidable theories

- In this talk we're interested in extensions of Presburger arithmetic.

Decidable:

$\mathrm{Th}(\mathbb{Z};+)$
Presburger arithmetic

$\mathrm{Th}(\mathbb{Z};+,x\mapsto 2^x)$
Semënov arithmetic

Undecidable:

$\mathrm{Th}(\mathbb{Z};+,\cdot)$
Peano arithmetic

$\mathrm{Th}(\mathbb{Z};+,x\mapsto x^2)$
Presburger + squaring function

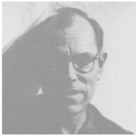
$\mathrm{Th}(\mathbb{Z};+,\lfloor f \rceil)$
(with f a Hardy field function)
???

Our result →

- Where is the boundary between decidability and undecidability?

A history of extensions of Presburger arithmetic






1953
Tarski:
$\mathrm{Th}(\mathbb{Z}; +, x \mapsto x^2)$
is undecidable

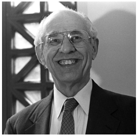
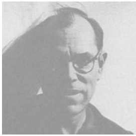
1957
Putnam:
$\mathrm{Th}(\mathbb{Z}; +, P(x))$
(where $P(x)$ is the image of the squaring function)
is undecidable

1960
Büchi:
$\mathrm{Th}(\mathbb{Z}; +, p)$
(where p is any degree-≥ 2 polynomial)
is undecidable

2025
Us:
What about:
- $x^2 \log x$
- $x^{3/2}$
- $\sqrt{x}$

and so on?

A history of extensions of Presburger arithmetic

1953
Tarski:
$\mathrm{Th}(\mathbb{Z}; +, x \mapsto x^2)$
is undecidable

1957
Putnam:
$\mathrm{Th}(\mathbb{Z}; +, P(x))$
(where $P(x)$ is the image of the squaring function)
is undecidable

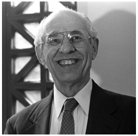
1960
Büchi:
$\mathrm{Th}(\mathbb{Z}; +, p)$
(where p is any degree-≥ 2 polynomial)
is undecidable

2025
Us:
What about:
- $x^2 \log x$
- $x^{3/2}$
- $\sqrt{x}$

and so on?

A history of extensions of Presburger arithmetic

1953
Tarski:
$\mathrm{Th}(\mathbb{Z}; +, x \mapsto x^2)$
is undecidable



1957
Putnam:
$\mathrm{Th}(\mathbb{Z}; +, P(x))$
(where $P(x)$ is the image of the squaring function)
is undecidable

1960
Büchi:
$\mathrm{Th}(\mathbb{Z}; +, p)$
(where p is any degree-≥ 2 polynomial)
is undecidable

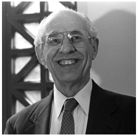

2025
Us:
What about:

- $x^2 \log x$
- $x^{3/2}$
- $\sqrt{x}$

and so on?

A history of extensions of Presburger arithmetic

1953
Tarski:
$\mathrm{Th}(\mathbb{Z}; +, x \mapsto x^2)$
is undecidable



1957
Putnam:
$\mathrm{Th}(\mathbb{Z}; +, P(x))$
(where $P(x)$ is the image of the squaring function)
is undecidable

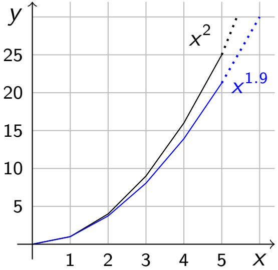
1960
Büchi:
$\mathrm{Th}(\mathbb{Z}; +, p)$
(where p is any degree-≥ 2 polynomial)
is undecidable

2025
Us:
What about:

- $x^2 \log x$
- $x^{3/2}$
- $\sqrt{x}$

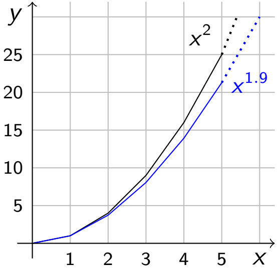
and so on?

Our motivation for the paper



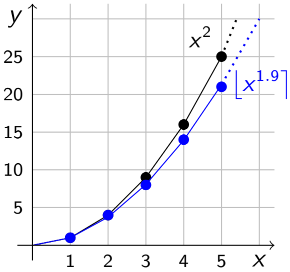
- We want to generalise Büchi's result to real-valued functions. For instance, the function $x^{1.9}$ looks awfully similar to x^2.
- Because Presburger arithmetic can only talk about integers, we have to round our functions to the nearest integer; this doesn't lose us much fidelity.

Our motivation for the paper



- We want to generalise Büchi's result to real-valued functions. For instance, the function $x^{1.9}$ looks awfully similar to x^2.
- Because Presburger arithmetic can only talk about integers, we have to round our functions to the nearest integer; this doesn't lose us much fidelity.

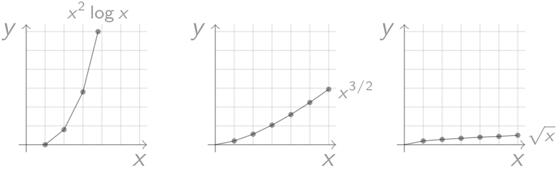
Our motivation for the paper



- We want to generalise Büchi's result to real-valued functions. For instance, the function $x^{1.9}$ looks awfully similar to x^2.
- Because Presburger arithmetic can only talk about integers, we have to round our functions to the nearest integer; this doesn't lose us much fidelity.

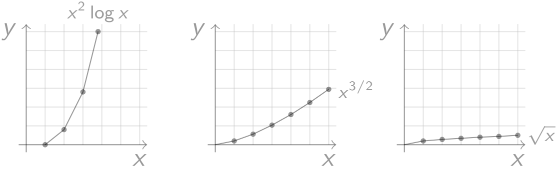
A definition of Hardy field functions

- We want a class of functions that captures everything that looks like a polynomial. The class of Hardy field functions fits the bill.
- There's a technical definition from analysis, but it suffices here that Hardy field functions include all functions built out of:
  - real-valued polynomials,
  - logarithms, and
  - exponentials.
- So all our functions from before are Hardy field functions:



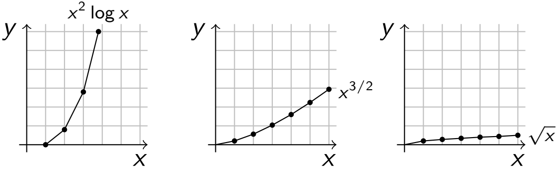
A definition of Hardy field functions

- We want a class of functions that captures everything that looks like a polynomial. The class of Hardy field functions fits the bill.
- There's a technical definition from analysis, but it suffices here that Hardy field functions include all functions built out of:
  - real-valued polynomials,
  - logarithms, and
  - exponentials.
- So all our functions from before are Hardy field functions:



A definition of Hardy field functions

- We want a class of functions that captures everything that looks like a polynomial. The class of Hardy field functions fits the bill.
- There's a technical definition from analysis, but it suffices here that Hardy field functions include all functions built out of:
  - real-valued polynomials,
  - logarithms, and
  - exponentials.
- So all our functions from before are Hardy field functions:

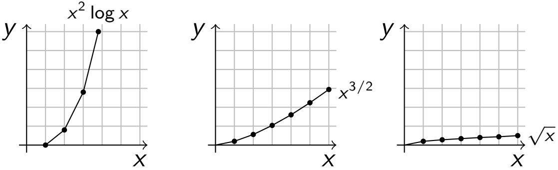


... and restrictions we need for our argument.

- We need a restriction, though. A Hardy field function is subpolynomial if it grows slower than some polynomial and faster than another.

Theorem (Brown & Konieczny, 2025)

The extension of Presburger arithmetic by a rounded sub-polynomial non-constant non-linear Hardy field function is undecidable.

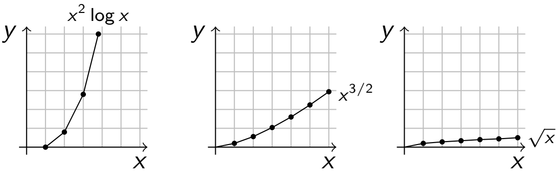


… and restrictions we need for our argument.

- We need a restriction, though. A Hardy field function is subpolynomial if it grows slower than some polynomial and faster than another.

Theorem (Brown & Konieczny, 2025)

The extension of Presburger arithmetic by a rounded sub-polynomial non-constant non-linear Hardy field function is undecidable.



... and restrictions we need for our argument.

- We need a restriction, though. A Hardy field function is subpolynomial if it grows slower than some polynomial and faster than another.

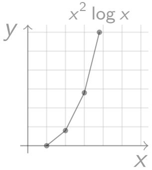
Theorem (Brown & Konieczny, 2025)

The extension of Presburger arithmetic by a rounded sub-polynomial non-constant non-linear Hardy field function is undecidable.

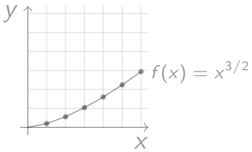
An overview of cases in our proof

We have three cases to consider:

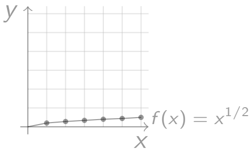
1. When $f(x)$ grows faster than x^2:



2. When $f(x)$ grows between x and x^2:



3. When $f(x)$ grows slower than x:

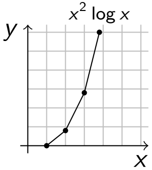


We'll now go through each of these cases.

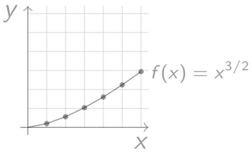
An overview of cases in our proof

We have three cases to consider:

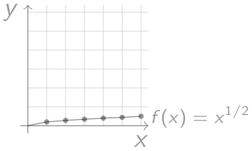
1. When $f(x)$ grows faster than x^2:



2. When $f(x)$ grows between x and x^2:



3. When $f(x)$ grows slower than x:

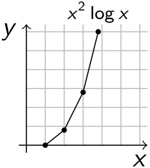


We'll now go through each of these cases.

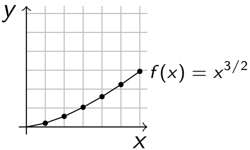
An overview of cases in our proof

We have three cases to consider:

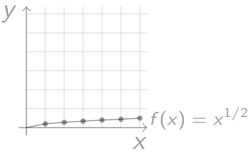
1. When $f(x)$ grows faster than x^2:



2. When $f(x)$ grows between x and x^2:



3. When $f(x)$ grows slower than x:

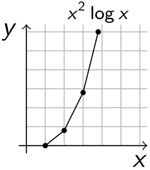


We'll now go through each of these cases.

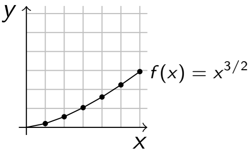
An overview of cases in our proof

We have three cases to consider:

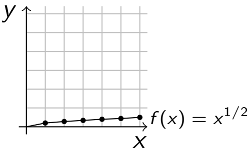
1. When $f(x)$ grows faster than x^2:



2. When $f(x)$ grows between x and x^2:



3. When $f(x)$ grows slower than x:



We'll now go through each of these cases.

1 When f grows quickly

- When f grows faster than x^2, but slower than x^d for some $d \geq 2$, we approximate it by a degree-d Taylor polynomial.
- Surprisingly, we can approximate our Hardy field function arbitrarily well by a fixed-degree Taylor polynomial.

1 When f grows quickly

- When f grows faster than x^2, but slower than x^d for some $d \geq 2$, we approximate it by a degree-d Taylor polynomial.
- Surprisingly, we can approximate our Hardy field function arbitrarily well by a fixed-degree Taylor polynomial.

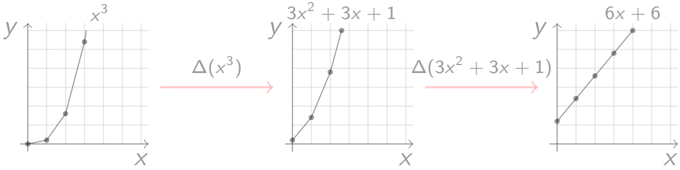
1 The idea behind our proof

- The squaring function is undecidable because we can define multiplication with it:

$$a \cdot b = c \iff (a+b)^2 - a^2 - b^2 = 2c.$$

- One way to apply Büchi's idea to higher-degree polynomials is to *discretely differentiate* them, defined as follows:

$$\Delta(f)(x) \stackrel{\text{def}}{=} f(x+1) - f(x).$$



This will let us extract the squaring function from our Taylor polynomials.

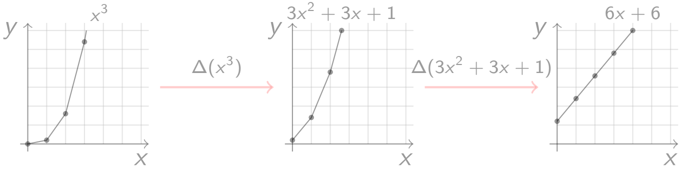
1 The idea behind our proof

- The squaring function is undecidable because we can define multiplication with it:

$$a \cdot b = c \iff (a+b)^2 - a^2 - b^2 = 2c.$$

- One way to apply Büchi's idea to higher-degree polynomials is to *discretely differentiate* them, defined as follows:

$$\Delta(f)(x) \stackrel{\text{def}}{=} f(x+1) - f(x).$$



This will let us extract the squaring function from our Taylor polynomials.

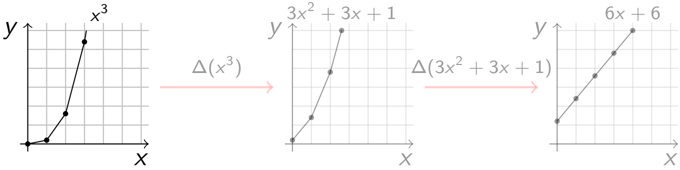
1 The idea behind our proof

- The squaring function is undecidable because we can define multiplication with it:

$$a \cdot b = c \iff (a+b)^2 - a^2 - b^2 = 2c.$$

- One way to apply Büchi's idea to higher-degree polynomials is to *discretely differentiate* them, defined as follows:

$$\Delta(f)(x) \stackrel{\text{def}}{=} f(x+1) - f(x).$$



This will let us extract the squaring function from our Taylor polynomials.

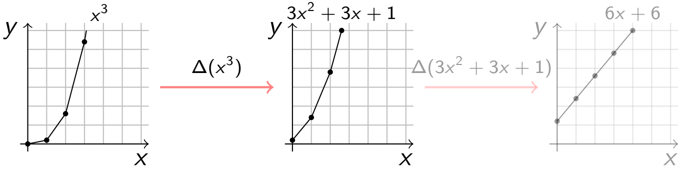
1 The idea behind our proof

- The squaring function is undecidable because we can define multiplication with it:

$$a \cdot b = c \iff (a+b)^2 - a^2 - b^2 = 2c.$$

- One way to apply Büchi's idea to higher-degree polynomials is to *discretely differentiate* them, defined as follows:

$$\Delta(f)(x) \stackrel{\text{def}}{=} f(x+1) - f(x).$$



This will let us extract the squaring function from our Taylor polynomials.

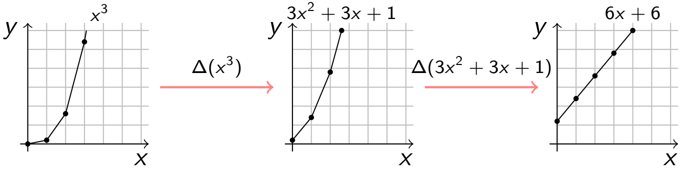
1 The idea behind our proof

- The squaring function is undecidable because we can define multiplication with it:

$$a \cdot b = c \iff (a+b)^2 - a^2 - b^2 = 2c.$$

- One way to apply Büchi's idea to higher-degree polynomials is to *discretely differentiate* them, defined as follows:

$$\Delta(f)(x) \stackrel{\text{def}}{=} f(x+1) - f(x).$$



This will let us extract the squaring function from our Taylor polynomials.

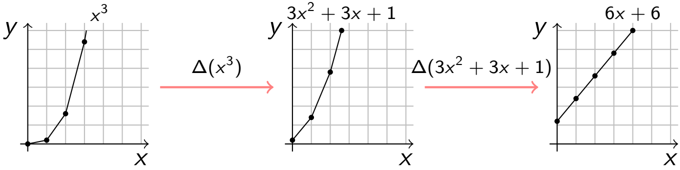
1 The idea behind our proof

- The squaring function is undecidable because we can define multiplication with it:

$$a \cdot b = c \iff (a+b)^2 - a^2 - b^2 = 2c.$$

- One way to apply Büchi's idea to higher-degree polynomials is to *discretely differentiate* them, defined as follows:

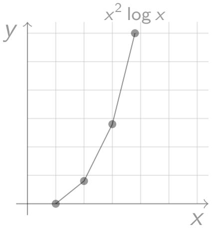
$$\Delta(f)(x) \stackrel{\text{def}}{=} f(x+1) - f(x).$$



This will let us extract the squaring function from our Taylor polynomials.

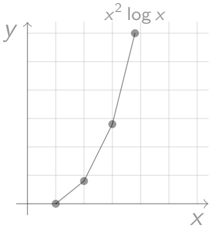
1 Defining differentiation in Presburger arithmetic

- With that we have all we need to define multiplication in Presburger arithmetic. We have $ab = c$ *iff*:
  - There exists an interval of length $\max(a, b, c)$ where f is approximated by an integer polynomial p,
  - we can differentiate p to get out the squaring function, and
  - $(a + b)^2 - a^2 - b^2 = 2c$.
- So super-linear functions get us an undecidable theory.



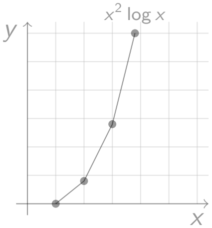
1 Defining differentiation in Presburger arithmetic

- With that we have all we need to define multiplication in Presburger arithmetic. We have $ab = c$ *iff*:
  - There exists an interval of length $\max(a, b, c)$ where f is approximated by an integer polynomial p,
  - we can differentiate p to get out the squaring function, and
  - $(a+b)^2 - a^2 - b^2 = 2c$.
- So super-linear functions get us an undecidable theory.



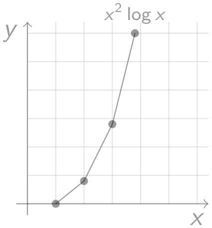
1 Defining differentiation in Presburger arithmetic

- With that we have all we need to define multiplication in Presburger arithmetic. We have $ab = c$ *iff*:
  - There exists an interval of length $\max(a, b, c)$ where f is approximated by an integer polynomial p,
  - we can differentiate p to get out the squaring function, and
  - $(a+b)^2 - a^2 - b^2 = 2c$.
- So super-linear functions get us an undecidable theory.



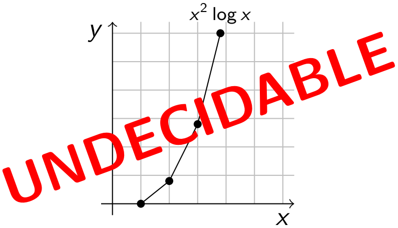
1 Defining differentiation in Presburger arithmetic

- With that we have all we need to define multiplication in Presburger arithmetic. We have $ab = c$ *iff*:
  - There exists an interval of length $\max(a, b, c)$ where f is approximated by an integer polynomial p,
  - we can differentiate p to get out the squaring function, and
  - $(a + b)^2 - a^2 - b^2 = 2c$.
- So super-linear functions get us an undecidable theory.



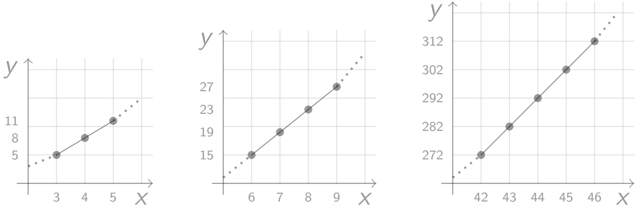
1 Defining differentiation in Presburger arithmetic

- With that we have all we need to define multiplication in Presburger arithmetic. We have $ab = c$ *iff*:
  - There exists an interval of length $\max(a, b, c)$ where f is approximated by an integer polynomial p,
  - we can differentiate p to get out the squaring function, and
  - $(a + b)^2 - a^2 - b^2 = 2c$.
- So super-linear functions get us an undecidable theory.



2 When f grows near-linearly

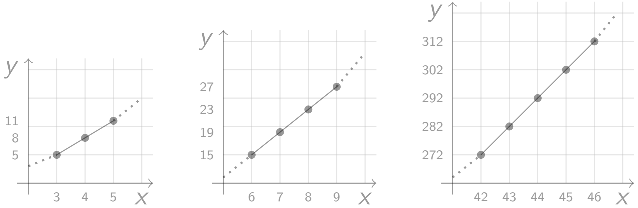
- If f grows slower than x^2 we can't use a polynomial approximation like above, because it'd be linear.
- But if we look closely we see there are intervals defining arithmetic progressions. For instance, when $f(x) = x^{3/2}$ we have:



- We can exploit this.

2 When f grows near-linearly

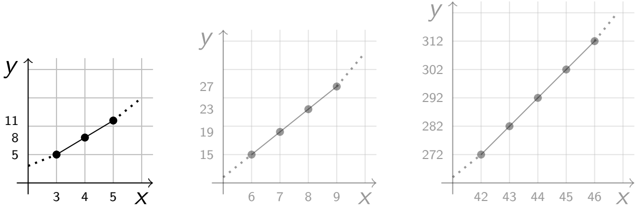
- If f grows slower than x^2 we can't use a polynomial approximation like above, because it'd be linear.
- But if we look closely we see there are intervals defining arithmetic progressions. For instance, when $f(x) = x^{3/2}$ we have:



- We can exploit this.

2 When f grows near-linearly

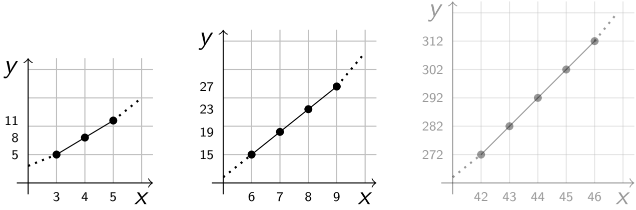
- If f grows slower than x^2 we can't use a polynomial approximation like above, because it'd be linear.
- But if we look closely we see there are intervals defining arithmetic progressions. For instance, when $f(x) = x^{3/2}$ we have:



- We can exploit this.

2 When f grows near-linearly

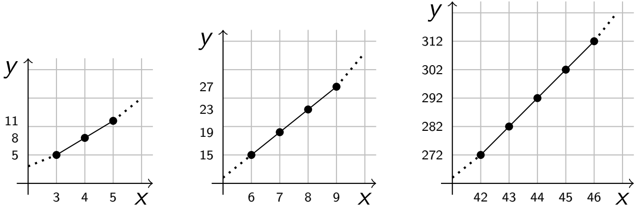
- If f grows slower than x^2 we can't use a polynomial approximation like above, because it'd be linear.
- But if we look closely we see there are intervals defining arithmetic progressions. For instance, when $f(x) = x^{3/2}$ we have:



- We can exploit this.

2 When f grows near-linearly

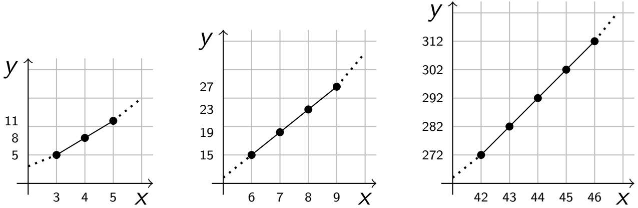
- If f grows slower than x^2 we can't use a polynomial approximation like above, because it'd be linear.
- But if we look closely we see there are intervals defining arithmetic progressions. For instance, when $f(x) = x^{3/2}$ we have:



- We can exploit this.

2 When f grows near-linearly

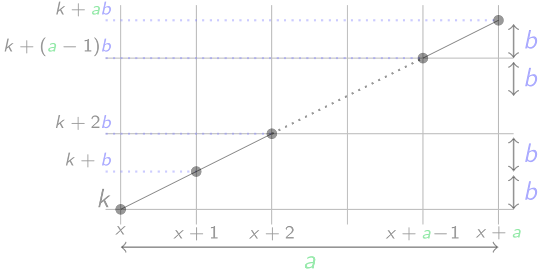
- If f grows slower than x^2 we can't use a polynomial approximation like above, because it'd be linear.
- But if we look closely we see there are intervals defining arithmetic progressions. For instance, when $f(x) = x^{3/2}$ we have:



- We can exploit this.

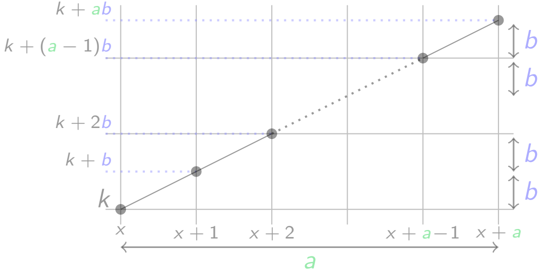
2 Guaranteeing that linear intervals exist

- We define multiplication using these intervals.
  Namely, $a \cdot b = c$ just when $f(x+a) - f(x) = c$, where $[x, x+a]$ is a multiplicative interval with step b. The following diagram illustrates this:



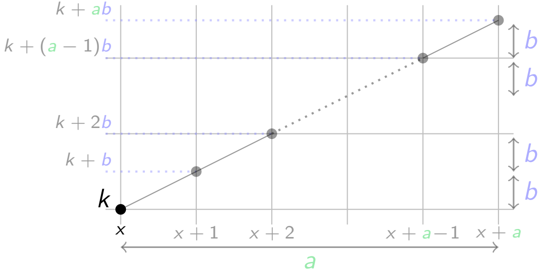
2 Guaranteeing that linear intervals exist

- We define multiplication using these intervals.
  Namely, $a \cdot b = c$ just when $f(x+a) - f(x) = c$, where $[x, x+a]$ is a multiplicative interval with step b. The following diagram illustrates this:



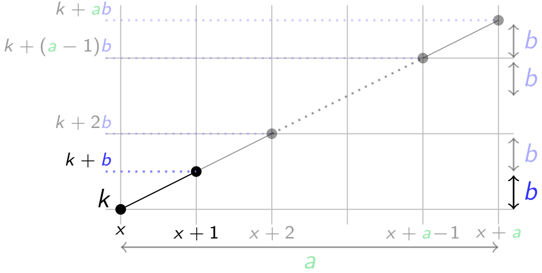
2 Guaranteeing that linear intervals exist

- We define multiplication using these intervals. Namely, $a \cdot b = c$ just when $f(x+a) - f(x) = c$, where $[x, x+a]$ is a multiplicative interval with step b. The following diagram illustrates this:



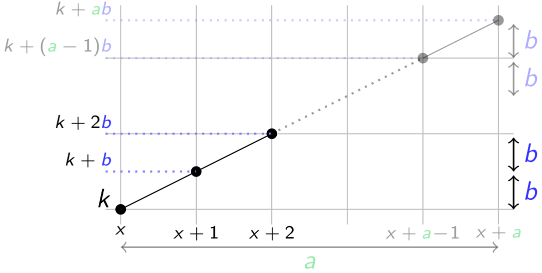
2 Guaranteeing that linear intervals exist

- We define multiplication using these intervals. Namely, $a \cdot b = c$ just when $f(x+a) - f(x) = c$, where $[x, x+a]$ is a multiplicative interval with step b. The following diagram illustrates this:



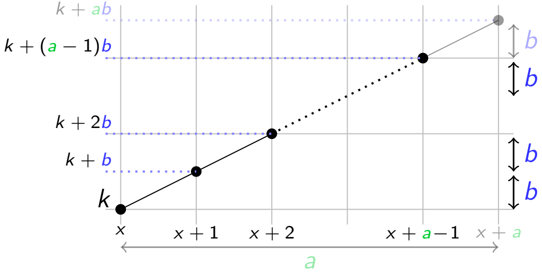
2 Guaranteeing that linear intervals exist

- We define multiplication using these intervals.
  Namely, $a \cdot b = c$ just when $f(x+a) - f(x) = c$, where $[x, x+a]$ is a multiplicative interval with step b. The following diagram illustrates this:



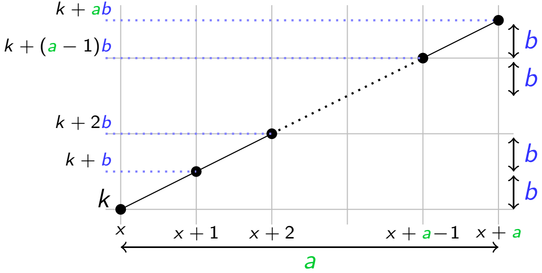
2 Guaranteeing that linear intervals exist

- We define multiplication using these intervals.
  Namely, $a \cdot b = c$ just when $f(x+a) - f(x) = c$, where $[x, x+a]$ is a multiplicative interval with step b. The following diagram illustrates this:

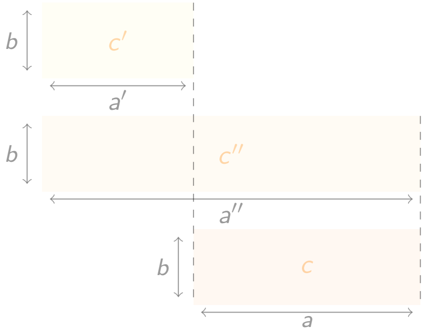


2 Guaranteeing that linear intervals exist

- We define multiplication using these intervals. Namely, $a \cdot b = c$ just when $f(x+a) - f(x) = c$, where $[x, x+a]$ is a multiplicative interval with step b. The following diagram illustrates this:



2 ... and what to do when intervals don't exist.



- In some cases intervals of the right size doesn't exist.
- In this case we use *difference sets*. Specifically we have $a \cdot b = c$ when there are a' and a'' with:

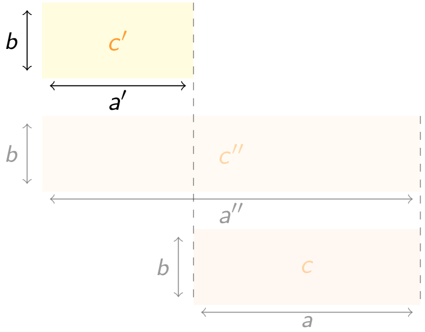
$$a' \cdot b = c'$$
$$a'' \cdot b = c''$$

such that

$$a'' - a' = a$$
$$c'' - c' = c.$$

- Multiplicative intervals appear often enough: there's always a difference set of the right size.

2 ... and what to do when intervals don't exist.



- In some cases intervals of the right size doesn't exist.
- In this case we use *difference sets*. Specifically we have $a \cdot b = c$ when there are a' and a'' with:

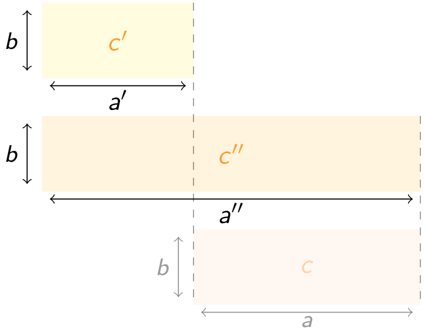
$$a' \cdot b = c'$$
$$a'' \cdot b = c''$$

such that

$$a'' - a' = a$$
$$c'' - c' = c.$$

- Multiplicative intervals appear often enough: there's always a difference set of the right size.

2 ... and what to do when intervals don't exist.



- In some cases intervals of the right size doesn't exist.
- In this case we use *difference sets*. Specifically we have $a \cdot b = c$ when there are a' and a'' with:

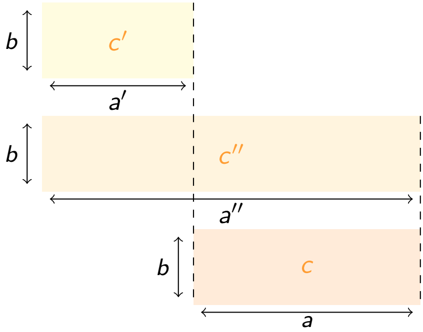
$$a' \cdot b = c'$$
$$a'' \cdot b = c''$$

such that

$$a'' - a' = a$$
$$c'' - c' = c.$$

- Multiplicative intervals appear often enough: there's always a difference set of the right size.

2 … and what to do when intervals don’t exist.



- In some cases intervals of the right size doesn’t exist.
- In this case we use *difference sets*. Specifically we have $a \cdot b = c$ when there are a' and a'' with:

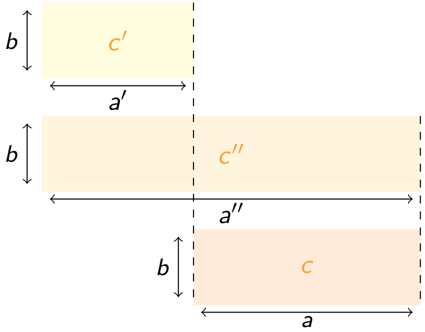
$$a' \cdot b = c'$$
$$a'' \cdot b = c''$$

such that

$$a'' - a' = a$$
$$c'' - c' = c.$$

- Multiplicative intervals appear often enough: there’s always a difference set of the right size.

2 … and what to do when intervals don't exist.



- In some cases intervals of the right size doesn't exist.
- In this case we use *difference sets*. Specifically we have $a \cdot b = c$ when there are a' and a'' with:

$$a' \cdot b = c'$$
$$a'' \cdot b = c''$$

  such that

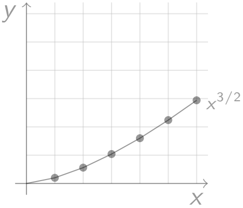
$$a'' - a' = a$$
$$c'' - c' = c.$$

- Multiplicative intervals appear often enough: there's always a difference set of the right size.

2 So near-linear Hardy field functions are undecidable.

- With this we can define multiplication using a near-linear Hardy field function:
  - Find two multiplicative intervals,
  - Take the difference of them, and
  - Use the difference to define multiplication.

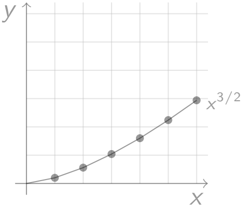
  So near-linear functions get us an undecidable theory.



2 So near-linear Hardy field functions are undecidable.

- With this we can define multiplication using a near-linear Hardy field function:
  - Find two multiplicative intervals,
  - Take the difference of them, and
  - Use the difference to define multiplication.

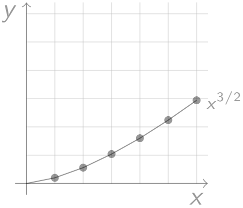
So near-linear functions get us an undecidable theory.



2 So near-linear Hardy field functions are undecidable.

- With this we can define multiplication using a near-linear Hardy field function:
  - Find two multiplicative intervals,
  - Take the difference of them, and
  - Use the difference to define multiplication.

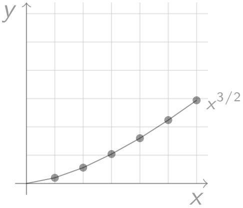
So near-linear functions get us an undecidable theory.



2 So near-linear Hardy field functions are undecidable.

- With this we can define multiplication using a near-linear Hardy field function:
  - Find two multiplicative intervals,
  - Take the difference of them, and
  - Use the difference to define multiplication.

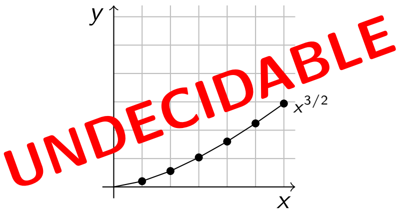
So near-linear functions get us an undecidable theory.



2 So near-linear Hardy field functions are undecidable.

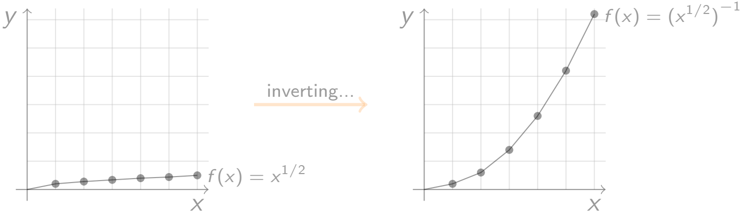
- With this we can define multiplication using a near-linear Hardy field function:
  - Find two multiplicative intervals,
  - Take the difference of them, and
  - Use the difference to define multiplication.

So near-linear functions get us an undecidable theory.



3 When f grows slowly

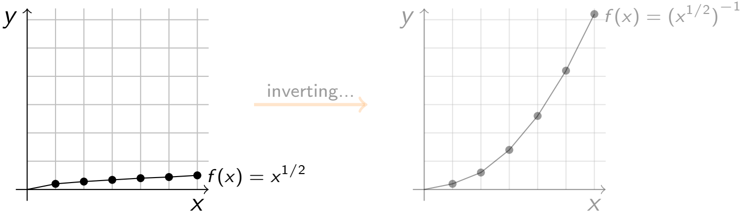
- When f grows slower than x, note that the inverse f^{-1} grows faster than x.



- The inverse f^{-1} is a Hardy field function and the inverse $\lfloor f \rceil$ is a rounded Hardy field function. Our earlier proofs work here.

3 When f grows slowly

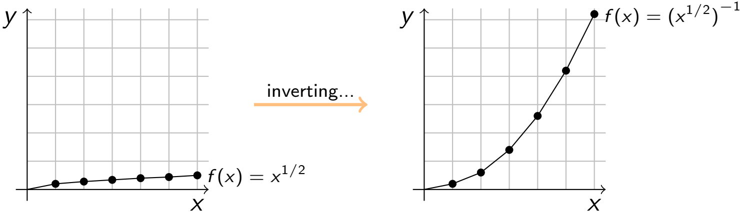
- When f grows slower than x, note that the inverse f^{-1} grows faster than x.



- The inverse f^{-1} is a Hardy field function and the inverse $\lfloor f \rceil$ is a rounded Hardy field function. Our earlier proofs work here.

3 When f grows slowly

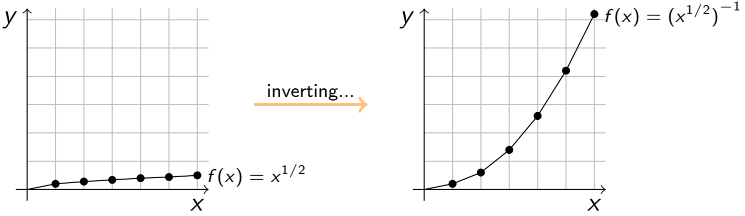
- When f grows slower than x, note that the inverse f^{-1} grows faster than x.



- The inverse f^{-1} is a Hardy field function and the inverse $\lfloor f \rceil$ is a rounded Hardy field function. Our earlier proofs work here.

3 When f grows slowly

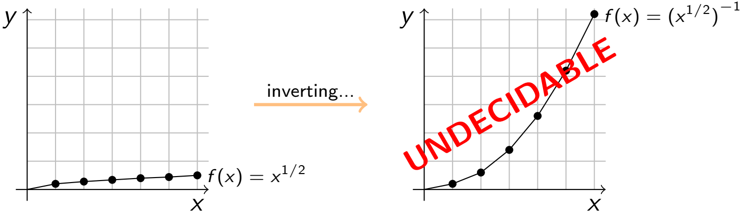
- When f grows slower than x, note that the inverse f^{-1} grows faster than x.



- The inverse f^{-1} is a Hardy field function and the inverse $\lfloor f \rceil$ is a rounded Hardy field function. Our earlier proofs work here.

3 When f grows slowly

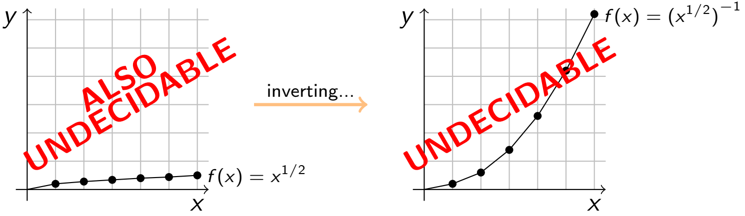
- When f grows slower than x, note that the inverse f^{-1} grows faster than x.



- The inverse f^{-1} is a Hardy field function and the inverse $\lfloor f \rceil$ is a rounded Hardy field function. Our earlier proofs work here.

3 When f grows slowly

- When f grows slower than x, note that the inverse f^{-1} grows faster than x.



- The inverse f^{-1} is a Hardy field function and the inverse $\lfloor f \rceil$ is a rounded Hardy field function. Our earlier proofs work here.

What we've proved

- Presburger arithmetic extended by a sub-polynomial Hardy field function is undecidable.
- This confirms our intuition: if it looks like a polynomial, then it's problematic.

Thank you!