

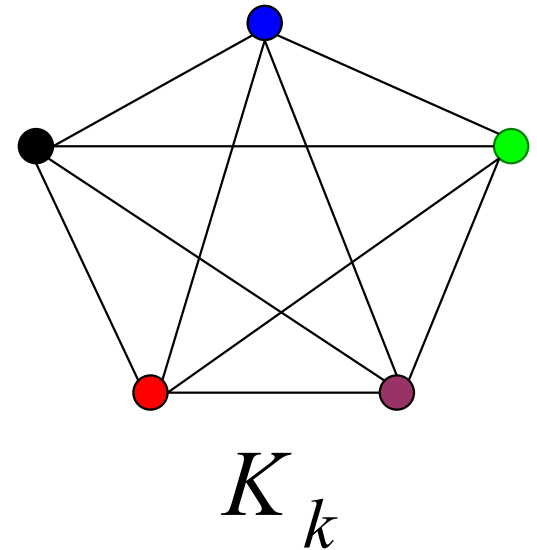
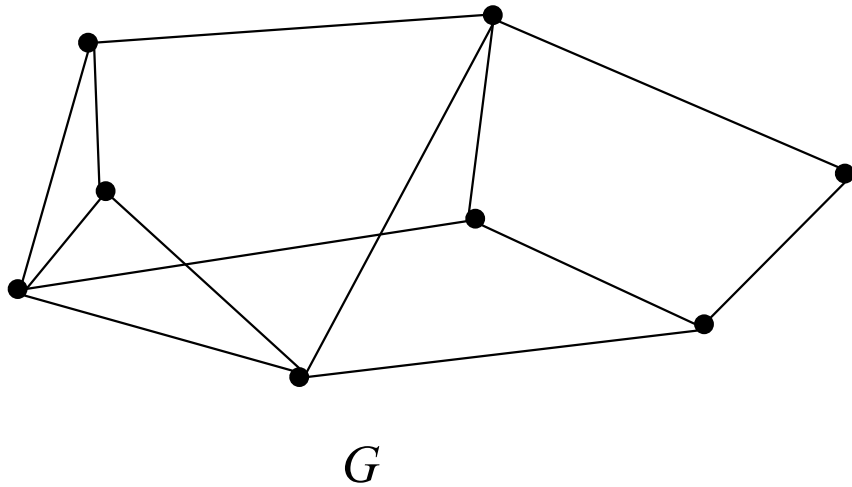
Counting Homomorphisms Modulo Primes

Andrei A. Bulatov
Simon Fraser University

Joint work with Amirhosein Kazemini
University of Oxford

Graph Homomorphisms

A homomorphism from a graph $G = (V(G), E(G))$ to $H = (V(H), E(H))$ is a mapping $\varphi: V(G) \rightarrow V(H)$ such that $\varphi(u)\varphi(v) \in E(H)$ for every $uv \in E(G)$



Counting Homomorphisms

Counting HOM: Given graphs G and H , find the number of homomorphisms from G to H

$\#HOM(H)$ if H is fixed

$\#HOM(H)$ belongs to the class $\#P$

Counting CSP mod p : Given relational structure G and H , find the number of homomorphisms from G to H modulo p , p prime

$\#_pCSP(H)$ if H is fixed

$\#_pCSP(H)$ belongs to the class Mod_pP

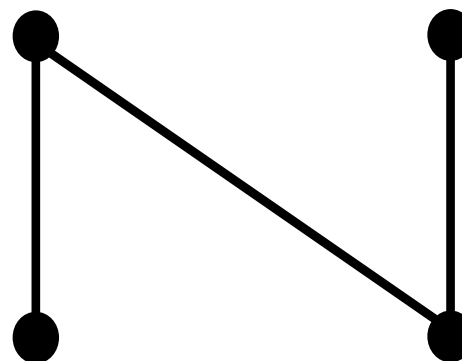
Counting Homomorphisms II

Theorem (Dyer, Greenhill, 2000)

$\#CSP(H)$, H is a graph, is poly time if and only if every connected component of H is a complete reflexive graph or a complete bipartite graph.

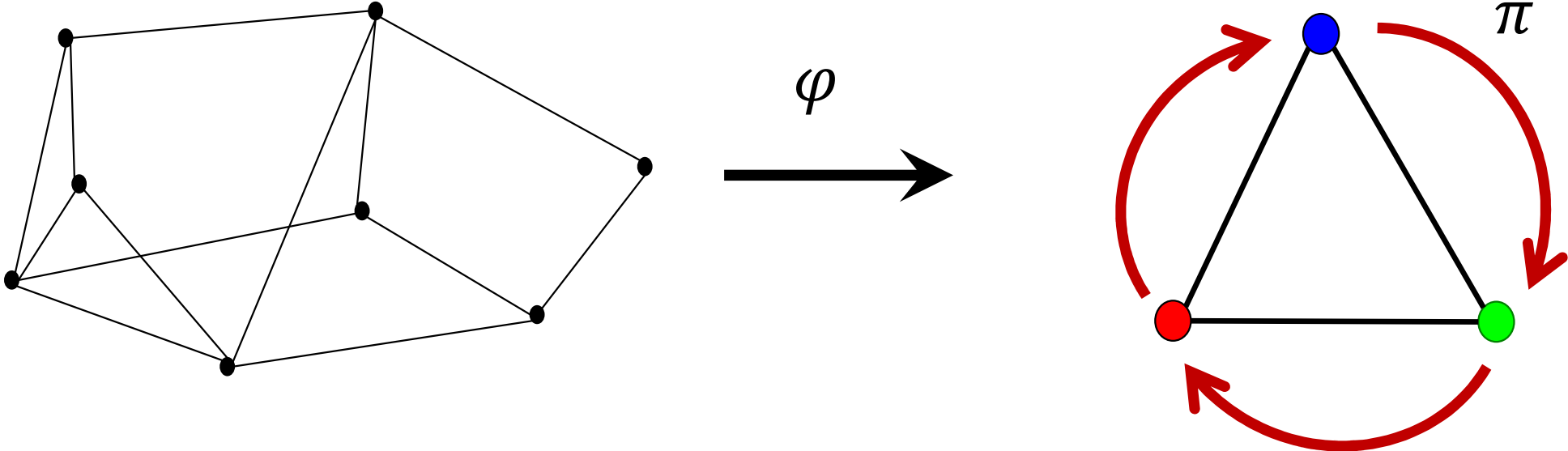
Otherwise $\#CSP(H)$ is $\#P$ -complete

The hardness is from the N-graph



Counting and Automorphisms

Counting 3-colorings mod 3



If φ is a homomorphism, so are $\pi \circ \varphi$ and $\pi^2 \circ \varphi$

So the contribution of φ into the homomorphism count is 0

Conjectures and Results

Conjecture (Faben, Jerrum, 2015): If graph H does not have p -automorphisms (p -rigid), then $\#_p CSP(H)$ is hard whenever $\#CSP(H)$ is hard.

- | | |
|---|--------------------------|
| - trees mod 2 | Faben, Jerrum, 2015 |
| - cactus graphs mod 2 | Göbel et al, 2014 |
| - square-free mod 2 | Göbel et al, 2016 |
| - K_4 -minor free mod 2 | Focke et al, 2021 |
| - trees mod p | Göbel et al, 2018 |
| - square-free mod p | B., K., 2019 |
| - $K_{3,3}-\{e\}$ and domino free mod p | Lagodzinski et al., 2020 |

Conjectures and Results

For a graph H , H^\dagger denotes the graph obtained from H by excluding orbits of size multiple of p

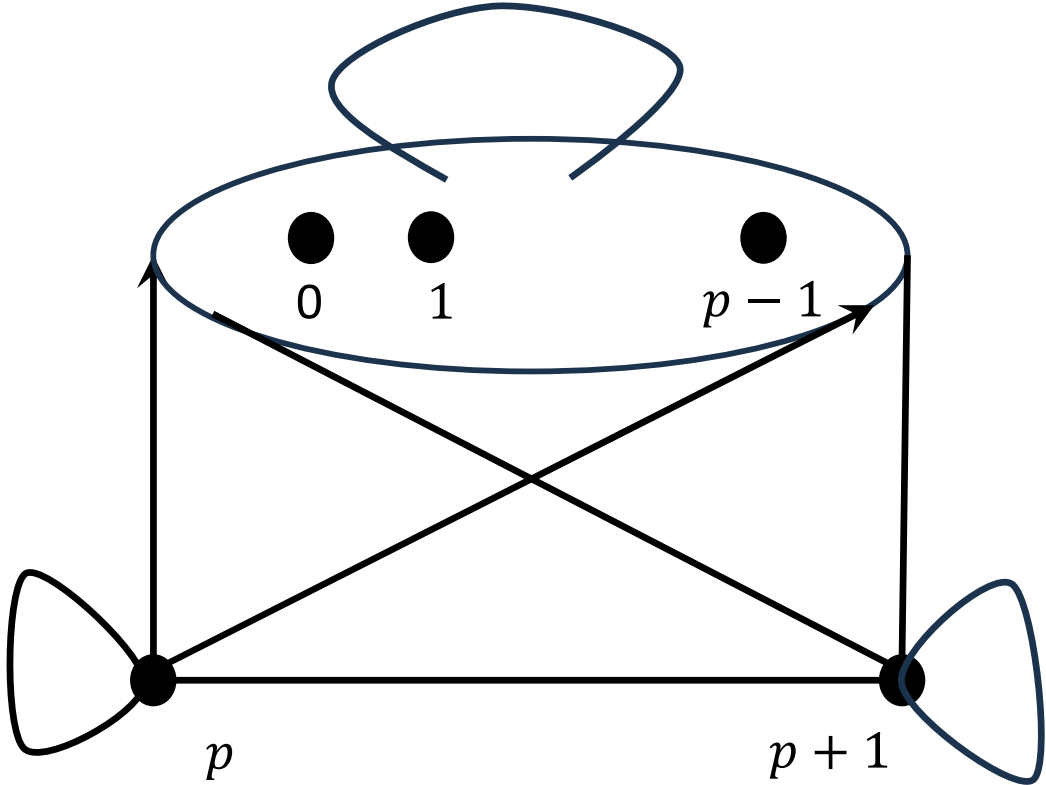
Theorem (B., K., 2022)

$\#_p CSP(H)$, H is a graph, is poly time if and only if every connected component of H^\dagger is a complete reflexive graph or a complete bipartite graph.

Otherwise $\#_p CSP(H)$ is $\text{Mod}_p P$ -complete

Digraph Counterexample

Unfortunately, a similar result cannot be true for digraphs



$0, \dots, p - 1$ need colours

Counting CSPs

Consider relational structures instead of graphs: allow any number of relations of any arity

(Modular) Counting CSP: Given relational structure G and H , find the number of homomorphisms from G to H (modulo p)

$\#CSP(H)$ ($\#_pCSP(H)$) if H is fixed

$\#CSP(H)$ belongs to the class $\#P$,

$\#_pCSP(H)$ belongs to the class Mod_pP

- Complexity of $\#_pCSP(H)$ for H

Faben 2015

Polymorphisms

For a relational structure H , a homomorphism $f: H^n \rightarrow H$ is said to be a (n -ary) *polymorphism* of H .

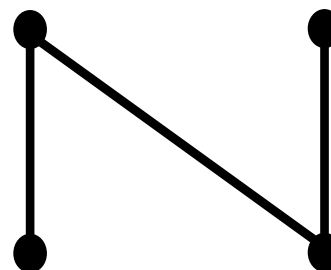
For counting the most important type of polymorphisms is Mal'tsev polymorphisms:

$$m(x, y, y) = m(y, y, x) = x$$

No Mal'tsev poly \implies exact counting is hard

Dyer/Greenhill graphs are exactly those having a Mal'tsev poly

The N-graph does not have a Mal'tsev poly



Conservative Structures

We call a structure H conservative if every unary relation is in H

Theorem

Let H be a conservative structure. Then, if H does not have a Mal'tsev polymorphism then $\#_p CSP(H)$ is hard. Otherwise if $p = 2$, $\#_2 CSP(H)$ is poly time solvable

Remark: The gap is due to partition functions (to be discussed later)

3-Element Structures

Theorem

Let H be a 3-element structure. Then one of the following holds

- (a) H does not have a p -automorphic polynomial and H does not have a Mal'tsev polymorphism, then $\#_p CSP(H)$ is hard, or
- (b) H has a special kind of polymorphism and then $\#_p CSP(H)$ is equivalent to then $\#_p CSP(H')$ for a 2-element structure H' , or
- (c) H does not have that polymorphism, H has a Mal'tsev polymorphism, and $p = 2$, then $\#_p CSP(H)$ is solvable in polynomial time.

Remark: The gap is again due to partition functions

Technical Contributions

- New kind of polymorphisms that can be used for reductions similar to those through p -automorphisms
- Deeper study of modular primitive positive definitions in relational structures

Obstacles: Partition Functions over Finite Fields

Let $M = (M_{ij})$ be a $k \times k$ -matrix.

The problem $Part(M)$ given a graph $G = (V, E)$ asks to compute the partition function

$$Z_M(G) = \sum_{\varphi: V \rightarrow [k]} \prod_{vw \in E} M_{\varphi(v)\varphi(w)}$$

The complexity of (exact) computing partition functions depends on the rank of M

For modular counting matrices M are over finite fields that complicates everything a lot

Thank You!