Conditional Complexity Hardness: Monotone Circuit Size, Matrix Rigidity, and Tensor Rank

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Show that there exists an *interesting* function that cannot be computed by any *fast* Turing machine.

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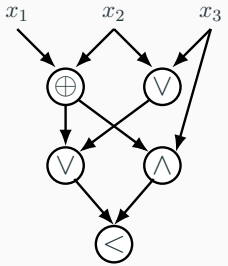
Show that there exists an *interesting* function that cannot be computed by any *fast* Turing machine.

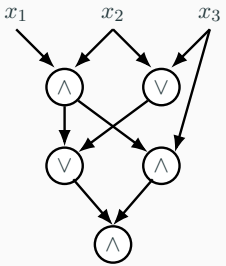
Dream (II)
Show that there exists an *explicit* function that cannot be computed by circuits of *linear size*.



Monotone Functions

Definition
A Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is *monotone* if $x \leq y$ coordinate-wise implies $f(x) \leq f(y)$.

A Boolean circuit is *monotone* if it uses only $\wedge$ and $\vee$ gates.



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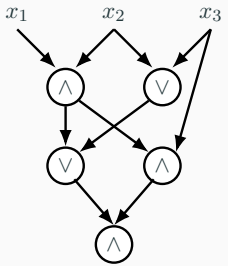

Theorem (Razborov, 1985 [2])
There is an explicit monotone function with monotone circuit size $n^{\Omega(\log n)}$.

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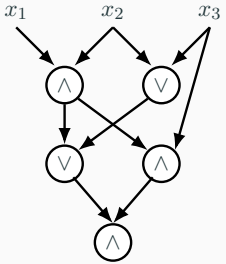

Theorem (Andreev, 1985 [3])
There is an explicit monotone function with monotone circuit size $2^{n^{1/8-o(1)}}$.

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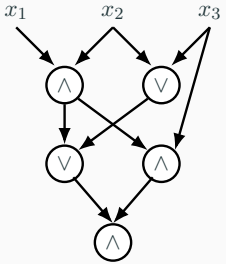

Theorem (Alon, Boppana, [4], Andreev [5], Harnik and Raz, 2000 [6])
There is an explicit monotone function with monotone circuit size $2^{n^{1/3-o(1)}}$.

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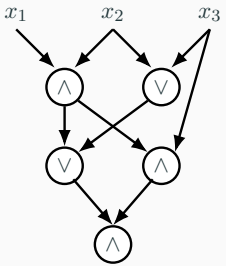

Theorem (Cavalar, Kumar and Rossman, 2022 [1])
There is an explicit monotone function with monotone circuit size $2^{\Omega(\sqrt{n}/\log n)}$.

Connections Between Lower and Upper Bounds

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If $\mathsf{P} = \mathsf{NP}$*, then* EXP *requires circuits of size* $\Omega(2^n/n)$.

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Theorem (Jahanjou, Miles and Viola, 2018 [11])
If k-SAT *can be solved in co-nondeterministic time* $O(2^{(1-\varepsilon)n})$*, then* E^{NP} *requires series-parallel Boolean circuits of size* $\omega(n)$.

Connections Between Lower and Upper Bounds (II)

Theorem (Williams, 2024 [12])
If k-SAT cannot be solved in time $O(2^{(1-\varepsilon)n})$, then $\underline{\mathsf{ETHR} \circ \mathsf{ETHR}}$ *circuit lower bound.*

Corollary (Williams, 2024 [12])
At least one of the following two circuit lower bounds holds:

1. E^{NP} *requires series-parallel circuits of size* $\omega(n)$*;*
2. *There exists* $\varepsilon > 0$ *such that the Boolean Inner Product on n-bit vectors does not have* $2^{\varepsilon n}$*-size* $\mathrm{ETHR} \circ \mathrm{ETHR}$ *circuits.*

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Theorem (Belova et al., 2024 [13])
If MAX-k-SAT *cannot be solved in co-nondeterministic time* $O(2^{(1-\varepsilon)n})$*, then arithmetic circuit lower bounds.*

Our Result on Monotone Circuits

Theorem (Cavalar, Kumar and Rossman, 2022 [1])
There is a monotone function $f \in \mathsf{P}$ with monotone circuit size $2^{\Omega(\sqrt{n}/\log n)}$.

Theorem
If, for some $\varepsilon > 0$ and $k \in \mathbb{Z}_{\geq 3}$, k-SAT cannot be solved in co-nondeterministic time $O(2^{(1/2+\varepsilon)n})$, then there exists a monotone Boolean function family in coNP of monotone circuit size $2^{\Omega(n/\log n)}$.

$$\phi(x_1, x_2, x_3, x_4) = (x_1 \vee \overline{x_3} \vee x_4) \wedge (\overline{x_2} \vee x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \\ \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$$

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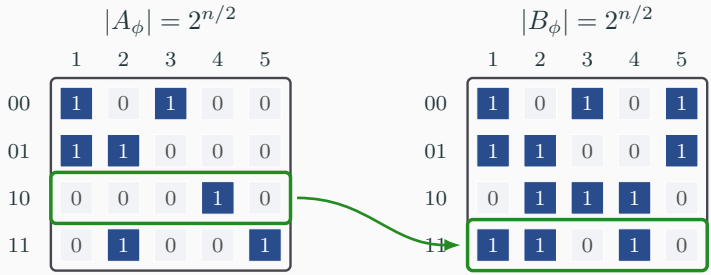
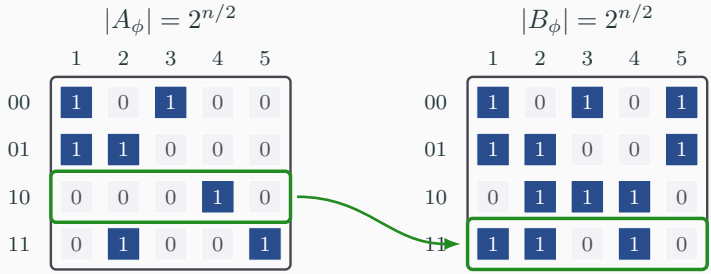


$A_\phi(a)[k] = 0$ iff a satisfies C_k on the first half,

$B_\phi(b)[k] = 1$ iff b satisfies C_k on the second half.



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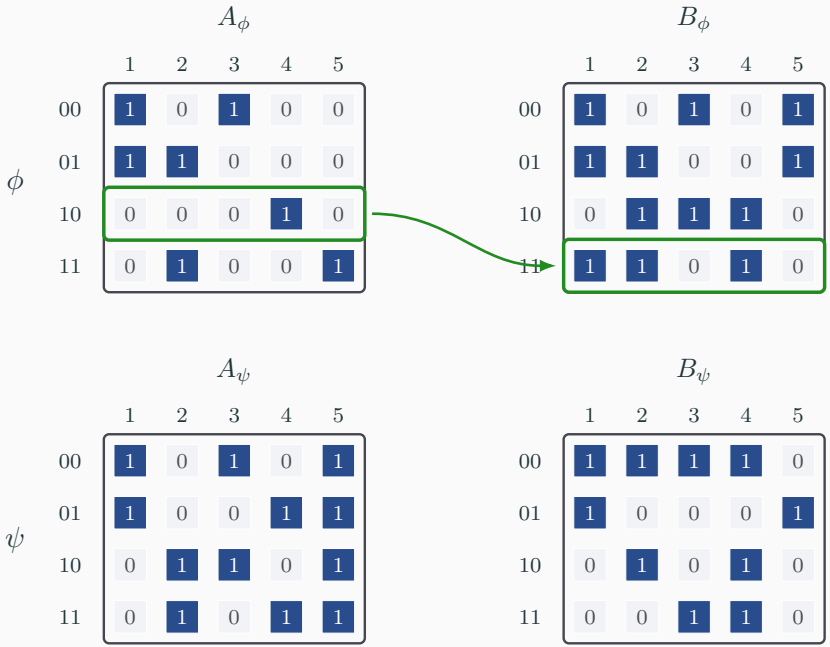
Claim

The formula $\phi \in \mathsf{SAT}$ iff there is a pair $x \in A_\phi, y \in B_\phi \colon x \leq y$.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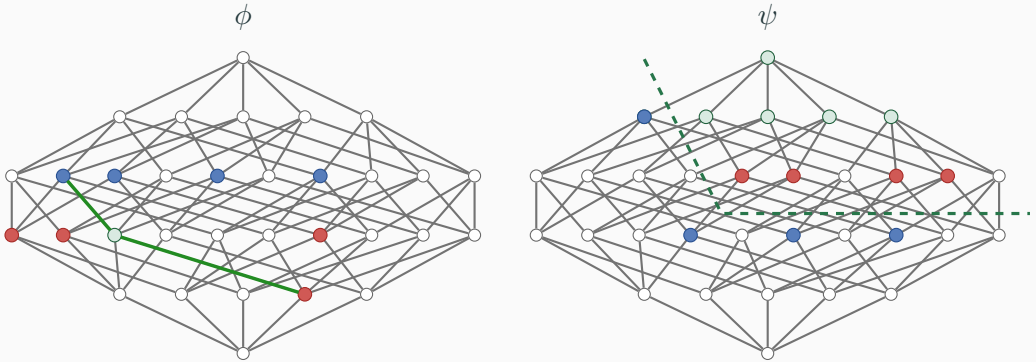
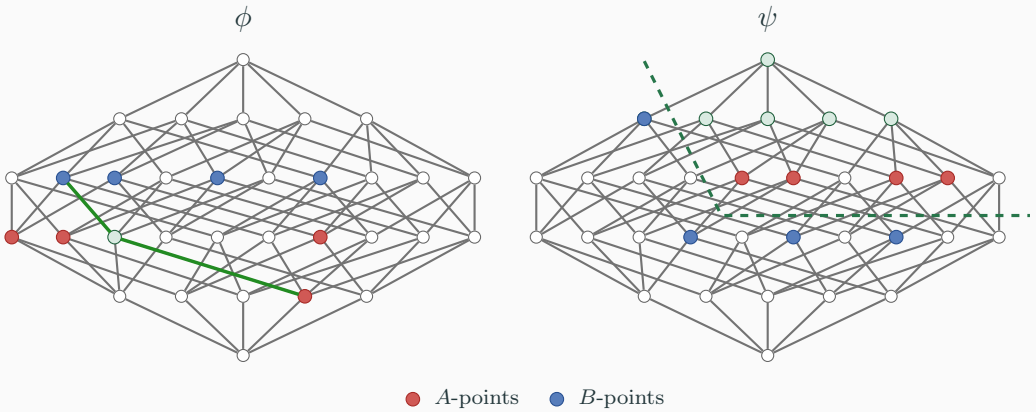

ϕ
ψ
A-points
B-points



Claim

The formula $\phi \in \mathsf{UNSAT}$ iff there exists a monotone $f_\phi : \{0,1\}^{\#\text{clauses}} \to \{0,1\}$ such that $f_\phi(A_\phi) = 1$ and $f_\phi(B_\phi) = 0$.

Outline

Given k-SAT formula ϕ:

- Construct A_ϕ, B_ϕ.
- Guess monotone circuit computing f_ϕ.
- Check that $f_\phi(a) = 1 \; \forall a \in A_\phi$ and $f_\phi(b) = 0 \; \forall b \in B_\phi$.

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Corollary
At least one of the following two circuit lower bounds holds:

1. E^{NP} *requires series-parallel circuits of size* $\omega(n)$*;*
2. *There is an explicit monotone function* $f \in \mathrm{coNP}$ *that requires monotone circuits of size* $2^{\Omega(n/\log n)}$.

Matrix Rigidity and Tensor Rank

Definition
For a matrix $M \in \mathbb{F}^{a \times b}$, we say that it has r-rigidity s if it is necessary to change at least s entries of M to reduce its rank to r. That is, for each decomposition $M = R + S$ such that $\text{rank}(R) \leq r$, it follows that $|S| \geq s$.

Definition
For a tensor $\mathcal{A} \in \mathbb{F}^{n \times n \times n}$, we define its rank, $\text{rank}(\mathcal{A})$, as the smallest integer r such that there exist r tuples of vectors $a_l, b_l, c_l \in \mathbb{F}^n$ for which

$$\mathcal{A} = \sum_{l \in [r]} a_l \otimes b_l \otimes c_l,$$

or equivalently,

$$\mathcal{A}[i, j, k] = \sum_{l \in [r]} a_l[i] b_l[j] c_l[k].$$

Theorem (Shokrollahi, Spielman and Stemann, 1997 [14])
Polynomial-time construction of r-rigidity $n^2/(r\log(n/r))$ matrices for every r.

Theorem (Goldreich and Tal, 2018 [15])
A random $n \times n$ Toeplitz matrix over $\mathbb{F}_2$ has r-rigidity $n^3/(r^2 \log n)$ for $r \geq \sqrt{n}$.

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Theorem
If MAX-3-SAT *cannot be solved in co-nondeterministic time $O(2^{(1-\varepsilon)n})$ for any $\varepsilon > 0$, then for all $\delta > 0$ there exists an efficient generator for matrices of $k^{\frac{1}{2}-\delta}$-rigidity $k^{2-\delta}$.*

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Theorem
If MAX-3-SAT *cannot be solved in co-nondeterministic time* $O(2^{(1-\varepsilon)n})$ *for any* $\varepsilon > 0$*, then for all* $\delta > 0$ *and some* $\Delta > 0$ *there exists either an efficient generator for:*

- *matrices of* $k^{1-\delta}$*-rigidity* $k^{2-\delta}$*, or*
- 3*-dimensional tensors of rank* $k^{1+\Delta}$*.*

Proof Idea

Given 3-SAT formula ϕ, $t \in \mathbb{N}$, construct a 4-partite 3-uniform hypergraph having a 4-clique iff one can satisfy exactly t clauses in ϕ [16, 17].

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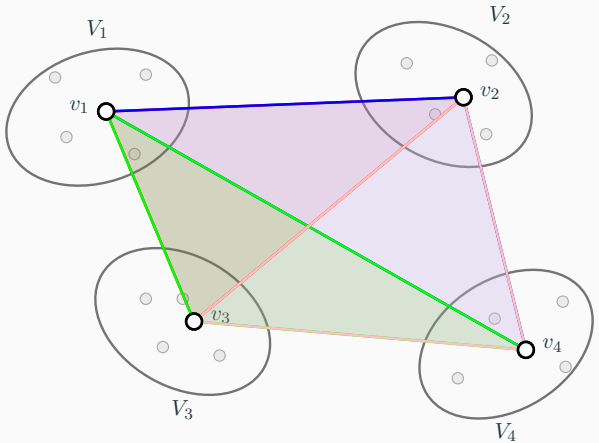
$$|V_1| = |V_2| = |V_3| = |V_4| = 2^{n/4}$$



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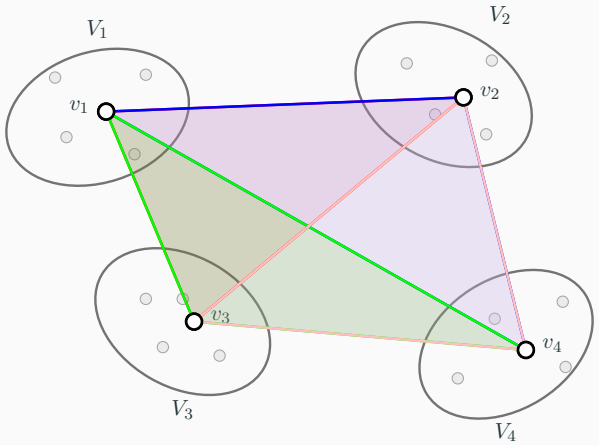

$$\sum_{v_1,v_2,v_3,v_4 \in \{0,1\}^{n/4}} A_0[v_1, v_2, v_3] \cdot A_1[v_2, v_3, v_4] \cdot A_2[v_3, v_4, v_1] \cdot A_3[v_4, v_1, v_2] > 0.$$

Contributions

We show how to construct hard objects from lower bounds for algorithms:

- Hardness of satisfiability algorithms yields explicit monotone functions with large monotone circuit complexity.
- Hardness of MAX-3-SAT yields explicit matrices with high rigidity, and possibly tensors with high rank.

More broadly: algorithmic lower bounds imply nonuniform lower bounds.

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