Computing Twin-Width via Treedepth and Vertex Integrity

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TU Wien

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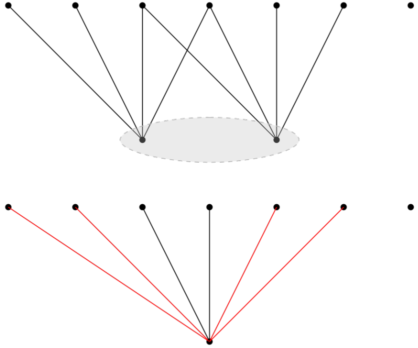
Contracting Vertices



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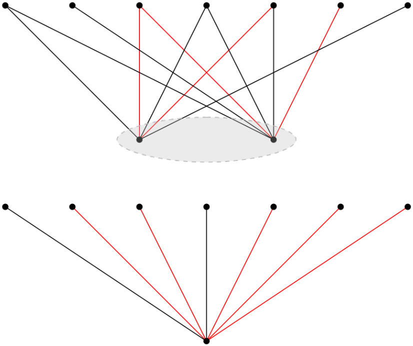

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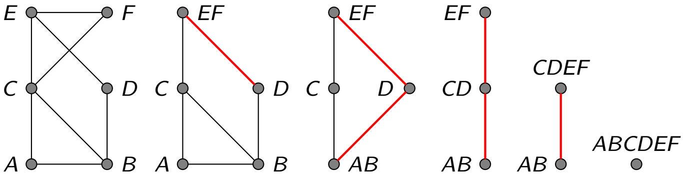

Figure by Jakub Balabán.

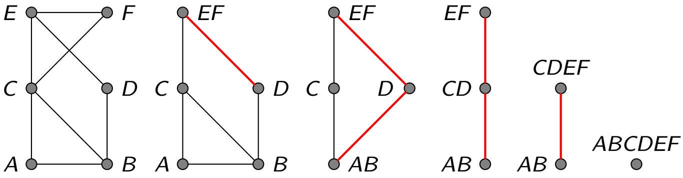

Figure by Jakub Balabán.

The *twin-width* (tww) of a graph G is the minimum width of a contraction sequence, over all valid contraction sequences from G to K_1.

Using Twin-Width to Solve FO

Theorem (Bonnet, Kim, Thomassé, Watrigant; 2020)

Provided a ***contraction sequence*** *of G of width d, evaluating a formula φ expressible in* ***First Order Logic (FO)*** *on G can be done in time $f(d, |\varphi|) \cdot |V(G)|$ for a computable function f.*

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Theorem (Bergé, Bonnet, Déprés; 2022)

Deciding whether the twin-width of a graph is at most 4 is NP-complete.

Computing Twin-Width

Is there an algorithm that given an n-vertex graph G and $k \in \mathbb{N}$, runs in time $f(k) \cdot n^{\mathcal{O}(1)}$ and either correctly reports that $\text{tww}(G) \geq k$ or outputs a contr. sequence of width at most $g(k)$?

- At this moment wide open!

Reminder: An algorithm is FPT parameterized by k if it runs in time $f(k) \cdot n^{\mathcal{O}(1)}$ for some computable $f(.)$.

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- What about using more restrictive parameters?
- Then maybe even *exact* FPT algorithms would be possible!

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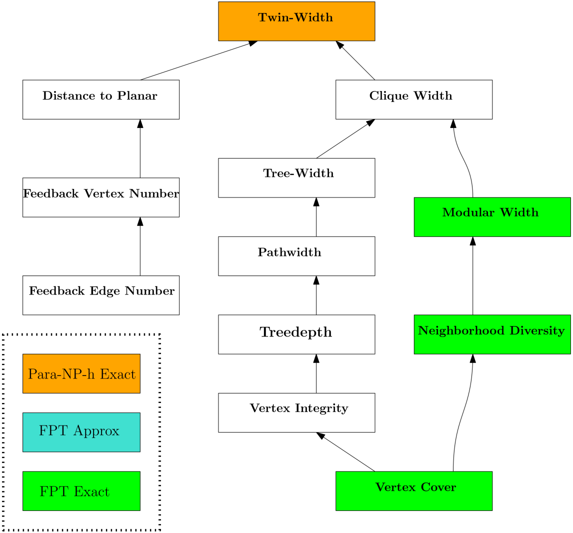
State of the Art



State of the Art

Theorem (+1-approximation for FEN)

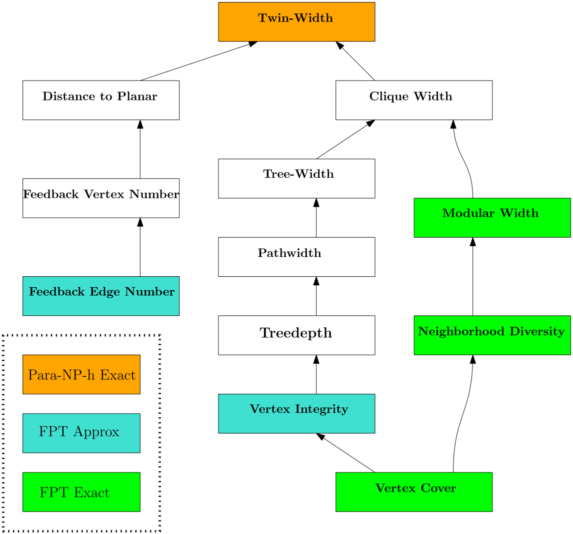
It is FPT to compute a contraction sequence for G of width at most $\text{tww}(G) + 1$*, parameterized by the Feedback Edge Number.*

Theorem (2-approximation for VI)

It is FPT to compute a contraction sequence for G of width at most $2 \cdot \text{tww}(G)$*, parameterized by the Vertex Integrity.*

Balabán, Ganian, R.; 2025

State of the Art



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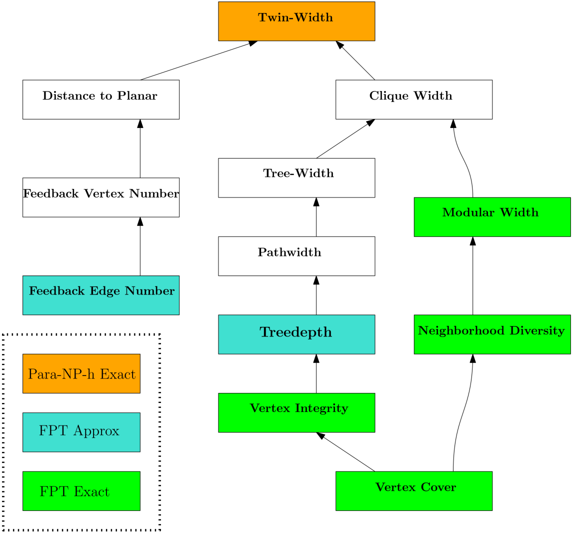


Improvements in Two Directions

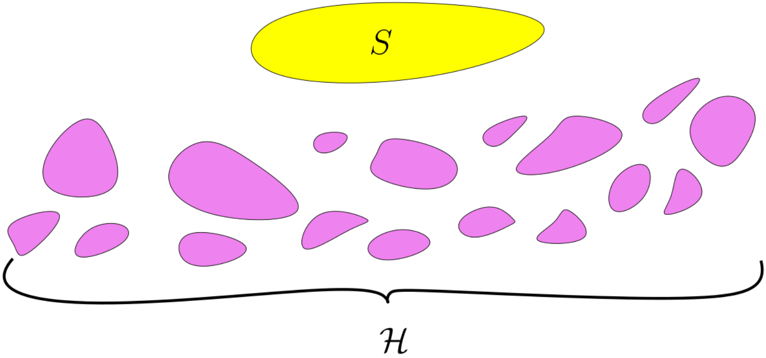
Theorem 1

Computing an **optimal** contraction sequence is FPT parameterized by the vertex integrity.

Theorem 2

For some computable function q, computing a contraction sequence of width at most $q(\text{tww})$ is FPT parameterized by treedepth.

Vertex Integrity



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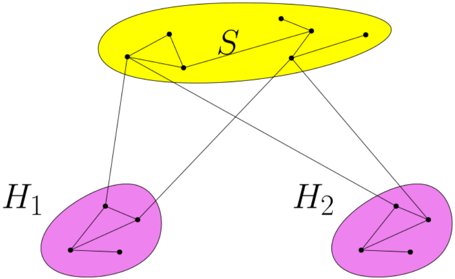
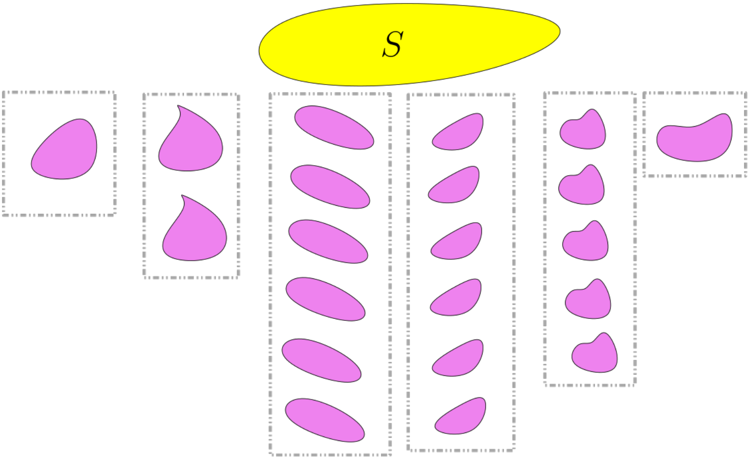
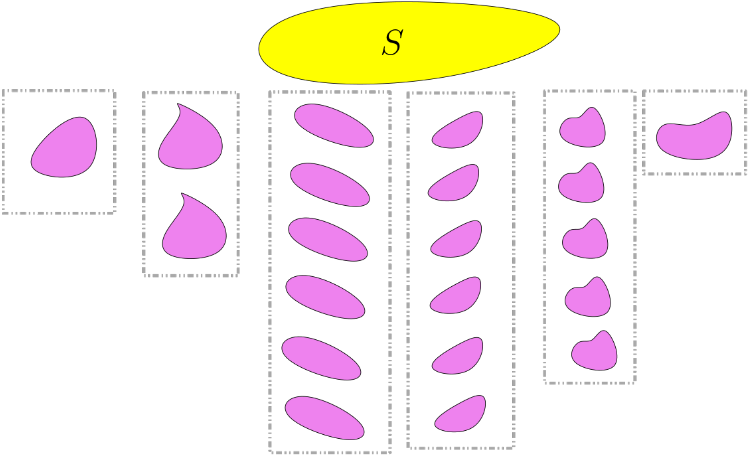


Classical Pruning



Technique introduced for minimum-page stack and queue layouts

[Depian, Fink, Ganian, Surianarayanan 25']



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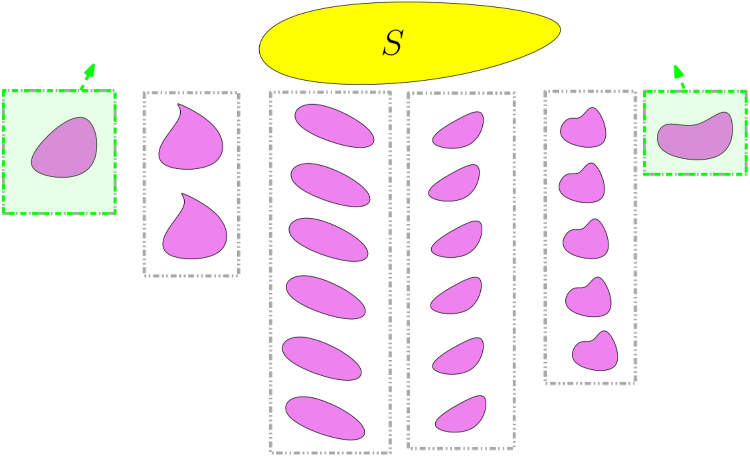

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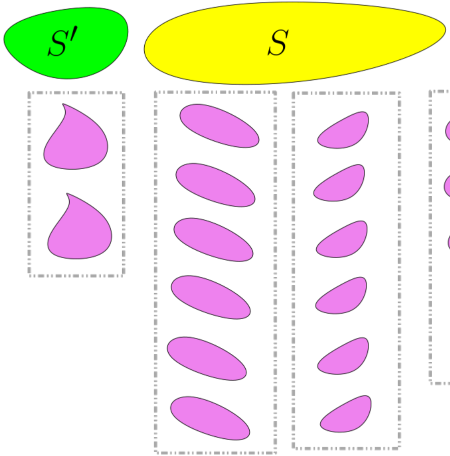

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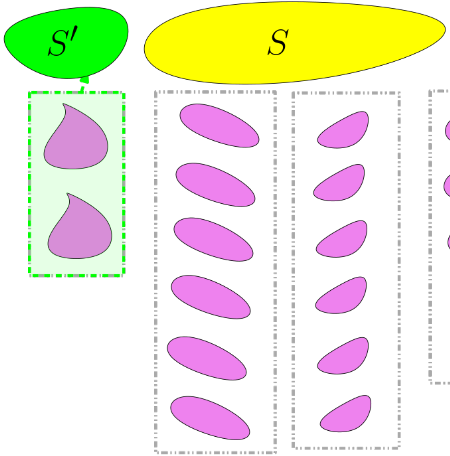

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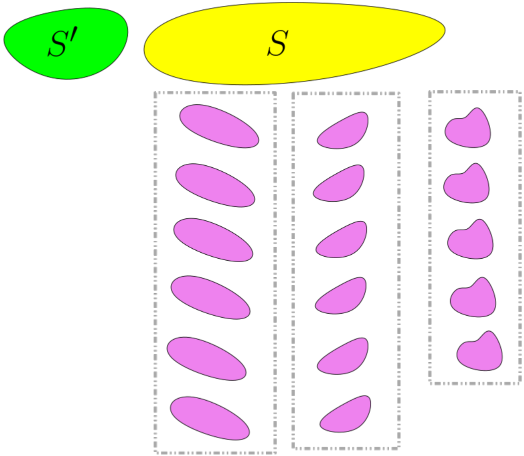

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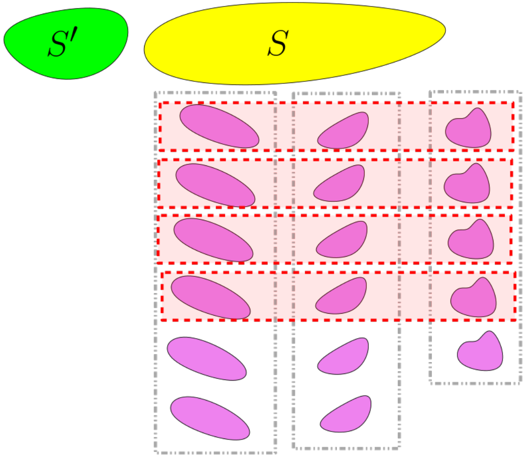

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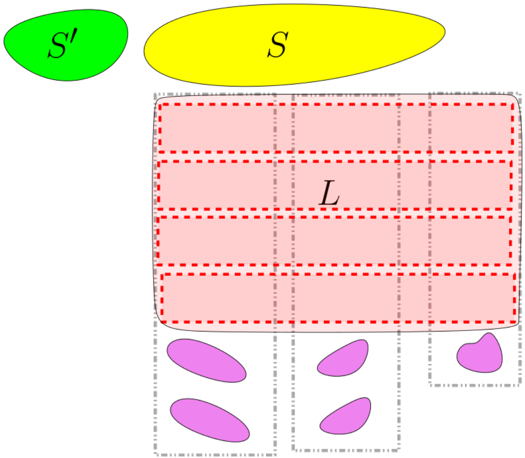

Ramsey Pruning

Technique introduced for minimum-page stack and queue layouts

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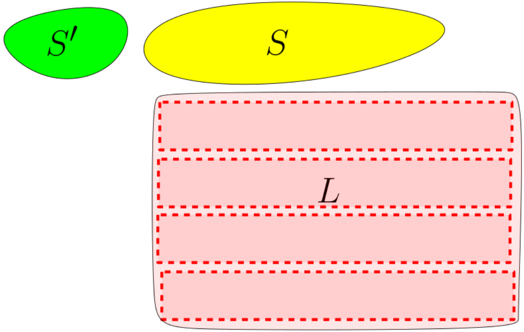

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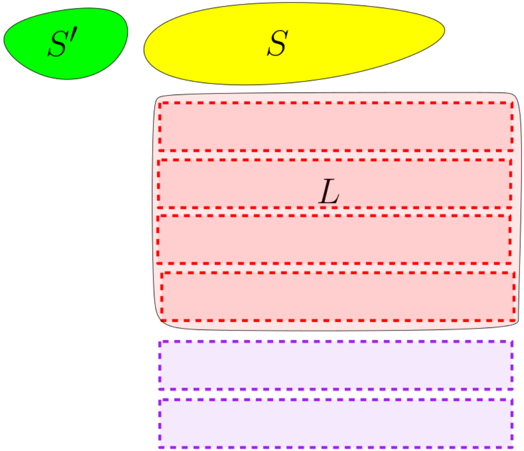

Hypergraph Ramsey

- We want to capture the trace of a potential solution on the *red blocks* . . .
- . . . but also the interaction between blocks . . .
- . . . and the order of these interactions.

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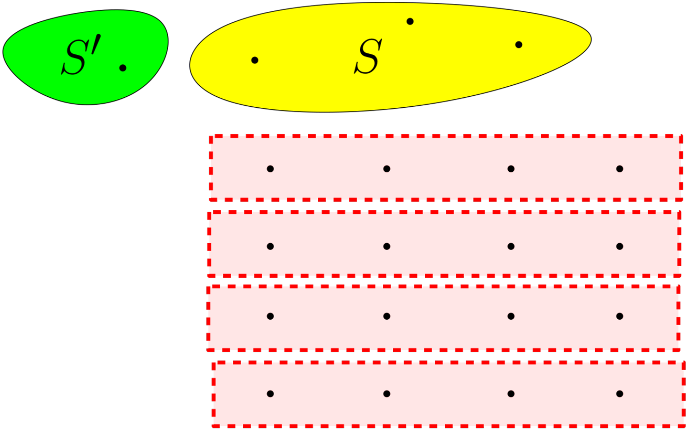
$\rightarrow$ We label **triplets** (L_1, L_2, L_3) with the restriction of the solution to $G[S \cup S' \cup L_1 \cup L_2 \cup L_3]$, and Hypergraph Ramsey gives us a family $\mathcal{L}' \subset L$ in which all triplets have the same label.

Tame Solution



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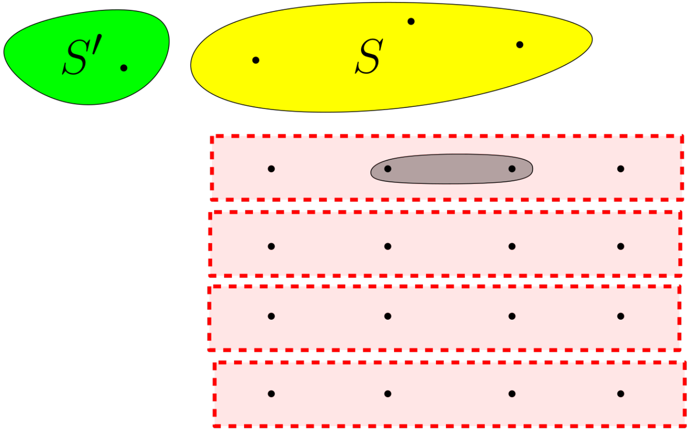

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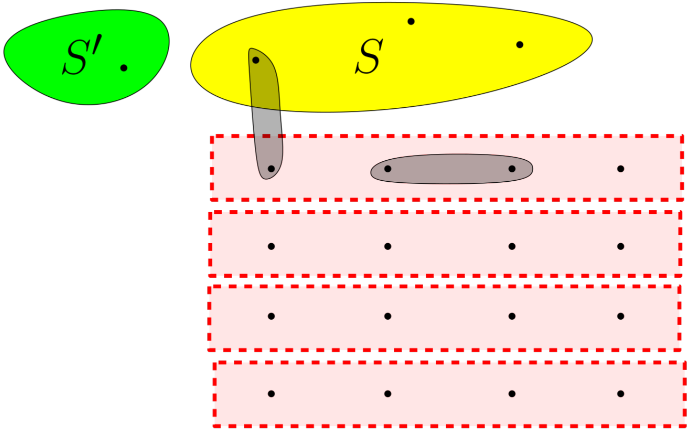

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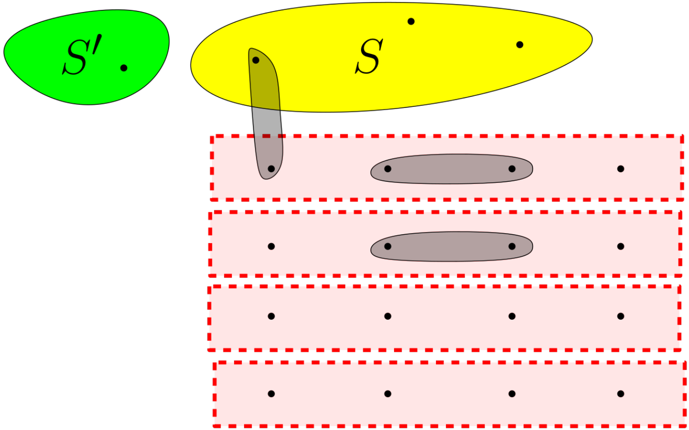

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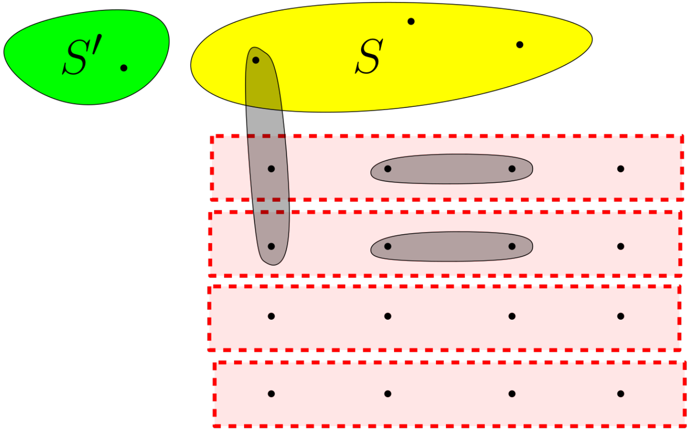

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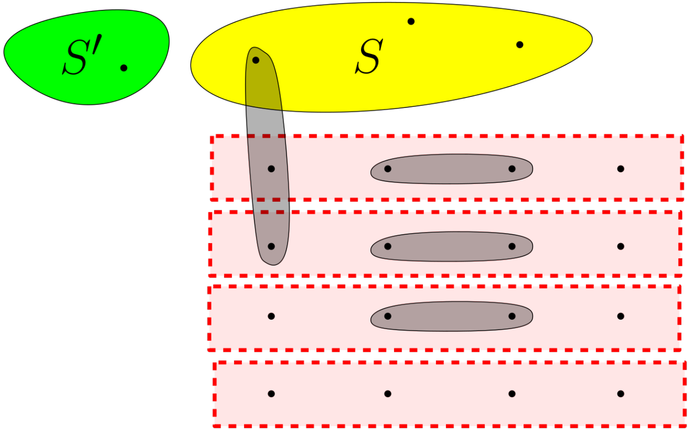

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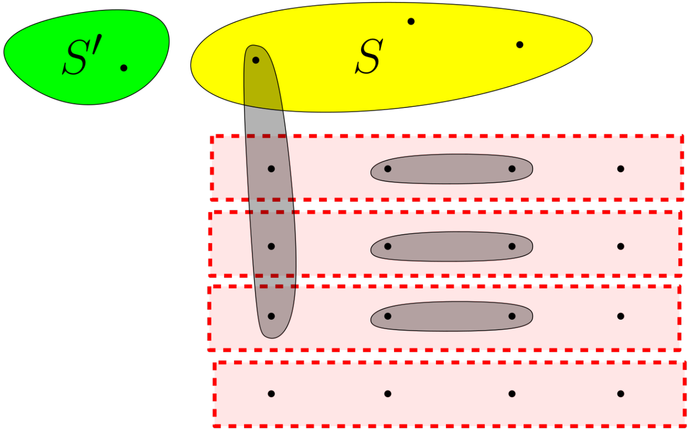

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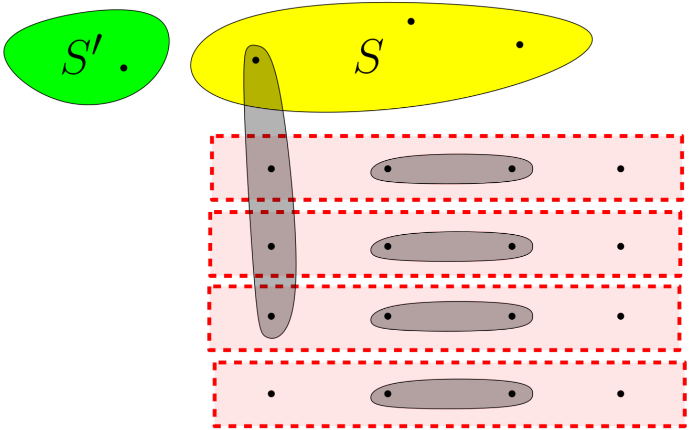

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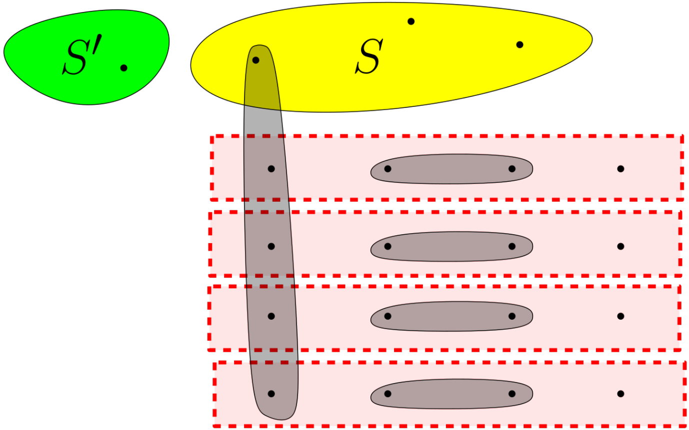


The Cost of Ramsey

The size of the kernel is:

$$2 \uparrow\uparrow ((p \cdot 2^{2p^2} \cdot 6 + 6)^p + 10)$$

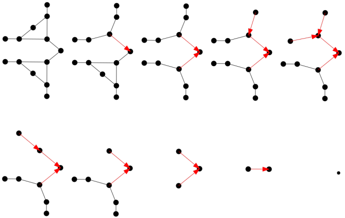
...definitely not tight

From Vertex Integrity to Treedepth

- Vertex Integrity is too restrictive: Treedepth is more general.
- No known equivalent to the Ramsey Pruning technique.
- Classical pruning approach for Vertex Integrity (2-approx.) can be adapted for Treedepth ...

  ... but yields a 2^{td}-approximation.
- **Workaround:** the *oriented* twin-width (otww).

Making Twin-Width Oriented



- Refines the error edges with orientation
- Only red out-degree is taken into account
- Functionally equivalent to Twin-Width

→ Near-optimal contractions for otww yield near-optimal contractions for tww in FPT time!

(combining results from Twin-Width I, IV and VI)

Open directions

- Ramsey Pruning for other problems
- Ramsey Pruning generalization to Treedepth
- Is oriented twin-width the "right" parameter to crack twin-width approximability?
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Thank you for your attention!
Questions?