Computational hardness of estimating quantum entropies via binary entropy bounds

Yupan Liu

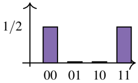
IC–QCC, École Polytechnique Fédérale de Lausanne

Available on arXiv:2601.03734

STACS 2026, Grenoble

Probability distribution vs. Quantum states

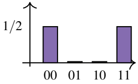
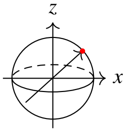
Probability distribution



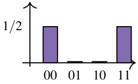
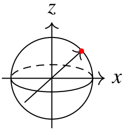
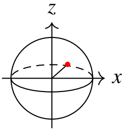
$\mathbf{p} = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$
over $\{00, 01, 10, 11\}$

A **distribution** over $\{0,1\}^n$
is a nonnegative vector
whose entries sum to 1.

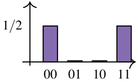
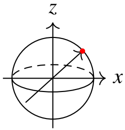
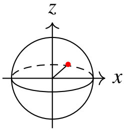
Probability distribution vs. Quantum states

Probability distribution	Pure state
	
$\mathbf{p} = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$ over $\{00, 01, 10, 11\}$	$\rho = \lvert\Psi\rangle\langle\Psi\rvert = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$, where $\lvert\Psi\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$
A **distribution** over $\{0,1\}^n$ is a nonnegative vector whose entries sum to 1.	An n-qubit **pure state** has the form $\rho = \lvert\psi\rangle\langle\psi\rvert$ (i.e., *rank*-1) for some unit vector $\lvert\psi\rangle \in \mathbb{C}^N$.

Probability distribution vs. Quantum states

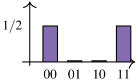
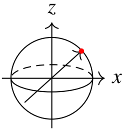
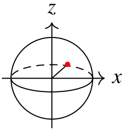
Probability distribution	Pure state	Quantum state (“quantum probability”)
		
$\mathbf{p} = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$ over $\{00, 01, 10, 11\}$	$\rho = \vert\Psi\rangle\langle\Psi\vert = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$, where $\vert\Psi\rangle = \frac{1}{\sqrt{2}}(\vert 00\rangle + \vert 11\rangle)$	$\rho = \frac{1}{2}\vert\Psi\rangle\langle\Psi\vert + \frac{1}{2}\vert 01\rangle\langle 01\vert$ $= \begin{pmatrix} 1/4 & 0 & 0 & 1/4 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 \end{pmatrix}$
A **distribution** over $\{0,1\}^n$ is a nonnegative vector whose entries sum to 1.	An n-qubit **pure state** has the form $\rho = \vert\psi\rangle\langle\psi\vert$ (i.e., *rank*-1) for some unit vector $\vert\psi\rangle \in \mathbb{C}^N$.	An n-qubit **quantum state** ρ is a **P**ositive **S**emi-**D**efinite matrix of size $N \times N$ ($N = 2^n$), such that $\mathrm{Tr}(\rho) = 1$.

Probability distribution vs. Quantum states

Probability distribution	Pure state	Quantum state ("quantum probability")
		
$\mathbf{p} = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$ over $\{00, 01, 10, 11\}$	$\rho = \|\Psi\rangle\langle\Psi\| = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$, where $\|\Psi\rangle = \frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\rho = \frac{1}{2}\|\Psi\rangle\langle\Psi\| + \frac{1}{2}\|01\rangle\langle 01\|$ $= \begin{pmatrix} 1/4 & 0 & 0 & 1/4 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 \end{pmatrix}$
A **distribution** over $\{0,1\}^n$ is a nonnegative vector whose entries sum to 1.	An n-qubit **pure state** has the form $\rho = \|\psi\rangle\langle\psi\|$ (i.e., *rank*-1) for some unit vector $\|\psi\rangle \in \mathbb{C}^N$.	An n-qubit **quantum state** ρ is a **P**ositive **S**emi-**D**efinite matrix of size $N \times N$ ($N = 2^n$), such that $\mathrm{Tr}(\rho) = 1$.

Purification. For any n-qubit quantum state ρ_A, there exists a $2n$-qubit pure state $|\Phi\rangle_{AB}$ (i.e., *on a larger system*) such that $\mathrm{Tr}_B(|\Phi\rangle\langle\Phi|) = \rho_A$.

Probability distribution vs. Quantum states

Probability distribution	Pure state	Quantum state ("quantum probability")
		
$\mathbf{p} = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$ over $\{00, 01, 10, 11\}$	$\rho = \lvert\Psi\rangle\langle\Psi\rvert = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$, where $\lvert\Psi\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$	$\rho = \frac{1}{2}\lvert\Psi\rangle\langle\Psi\rvert + \frac{1}{2}\lvert 01\rangle\langle 01\rvert = \begin{pmatrix} 1/4 & 0 & 0 & 1/4 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 \end{pmatrix}$
A **distribution** over $\{0,1\}^n$ is a nonnegative vector whose entries sum to 1.	An n-qubit **pure state** has the form $\rho = \lvert\psi\rangle\langle\psi\rvert$ (i.e., *rank*-1) for some unit vector $\lvert\psi\rangle \in \mathbb{C}^N$.	An n-qubit **quantum state** ρ is a **P**ositive **S**emi-**D**efinite matrix of size $N \times N$ ($N = 2^n$), such that $\mathrm{Tr}(\rho) = 1$.

Purification. For any n-qubit quantum state ρ_A, there exists a $2n$-qubit pure state $|\Phi\rangle_{AB}$ (i.e., *on a larger system*) such that $\mathrm{Tr}_B(|\Phi\rangle\langle\Phi|) = \rho_A$.

E.g., $|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \Rightarrow \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \frac{I}{2} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$.

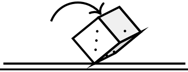
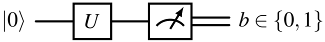
Probability distribution vs. Quantum states (Cont.)

	Probability distribution	Quantum state
Dynamics	Discrete-time Markov chain $\mathbf{p} \mapsto \mathbf{p}P$	Unitary transformation $\lvert\psi\rangle \mapsto U\lvert\psi\rangle$

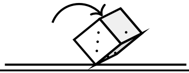
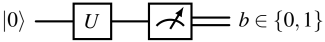
Probability distribution vs. Quantum states (Cont.)

	Probability distribution	Quantum state
Dynamics	Discrete-time Markov chain $\mathbf{p} \mapsto \mathbf{p}P$	Unitary transformation $\vert\psi\rangle \mapsto U\vert\psi\rangle$
Universal Gateset	Classical gates, e.g., AND, OR, NOT, together with random bits	Elementary quantum gates are *unitary* matrices acting on one or two qubits, e.g., $G_i \in \{\mathrm{CNOT}, \mathrm{Had}, \mathrm{T}\}$: $\vert 0\rangle^{\otimes n} \xrightarrow{G_1} G_1 \vert 0\rangle^{\otimes n} \xrightarrow{G_2} G_2 G_1 \vert 0\rangle^{\otimes n} \longrightarrow \cdots$

Probability distribution vs. Quantum states (Cont.)

	Probability distribution	Quantum state
Dynamics	Discrete-time Markov chain $\mathbf{p} \mapsto \mathbf{p}P$	Unitary transformation $\|\psi\rangle \mapsto U\|\psi\rangle$
Universal Gateset	Classical gates, e.g., AND, OR, NOT, together with random bits	Elementary quantum gates are *unitary* matrices acting on one or two qubits, e.g., $G_i \in \{\mathrm{CNOT}, \mathrm{Had}, \mathrm{T}\}$: $\|0\rangle^{\otimes n} \xrightarrow{G_1} G_1\|0\rangle^{\otimes n} \xrightarrow{G_2} G_2G_1\|0\rangle^{\otimes n} \longrightarrow \cdots$
Readout	**Sampling**: Draw samples from the given distribution $x \sim \mathbf{p}$. 	**Measurement** in the computational basis $\{\|0\rangle\langle 0\|, \|1\rangle\langle 1\|\}$: 

Probability distribution vs. Quantum states (Cont.)

	Probability distribution	Quantum state
Dynamics	Discrete-time Markov chain $\mathbf{p} \mapsto \mathbf{p}P$	Unitary transformation $\lvert\psi\rangle \mapsto U\lvert\psi\rangle$
Universal Gateset	Classical gates, e.g., AND, OR, NOT, together with random bits	Elementary quantum gates are *unitary* matrices acting on one or two qubits, e.g., $G_i \in \{\mathrm{CNOT}, \mathrm{Had}, \mathrm{T}\}$: $\lvert 0\rangle^{\otimes n} \xrightarrow{G_1} G_1\lvert 0\rangle^{\otimes n} \xrightarrow{G_2} G_2G_1\lvert 0\rangle^{\otimes n} \longrightarrow \cdots$
Readout	**Sampling**: Draw samples from the given distribution $x \sim \mathbf{p}$. 	**Measurement** in the computational basis $\{\lvert 0\rangle\langle 0\rvert, \lvert 1\rangle\langle 1\rvert\}$: 

Task: Quantum state testing via entropy approximation

Given a state-preparation circuit Q ("quantum devices") that prepares (the purification of) n-qubit quantum states $\rho \in \mathbb{C}^{N\times N}$. Decide whether $\mathrm{Ent}(\rho) \geq \tau_0(n)$ or $\mathrm{Ent}(\rho) \leq \tau_1(n)$.

What is quantum state testing

Task: Quantum state testing via entropy approximation

Given a state-preparation circuit Q ("quantum devices") that prepares (the purification of) n-qubit quantum states $\rho \in \mathbb{C}^{N \times N}$. Decide whether $\mathrm{Ent}(\rho) \geq \tau_0(n)$ or $\mathrm{Ent}(\rho) \leq \tau_1(n)$.

What is quantum state testing

Task: Quantum state testing via entropy approximation

Given a state-preparation circuit Q ("quantum devices") that prepares (the purification of) n-qubit quantum states $\rho \in \mathbb{C}^{N \times N}$. Decide whether $\mathrm{Ent}(\rho) \geq \tau_0(n)$ or $\mathrm{Ent}(\rho) \leq \tau_1(n)$.

- Quantum devices Q can be given either as a query oracle (*black-box model*) or a sequence of $\mathrm{poly}(n)$ elementary quantum gates (*white-box model*).

What is quantum state testing

Task: Quantum state testing via entropy approximation

Given a state-preparation circuit Q ("quantum devices") that prepares (the purification of) n-qubit quantum states $\rho \in \mathbb{C}^{N \times N}$. Decide whether $\mathrm{Ent}(\rho) \geq \tau_0(n)$ or $\mathrm{Ent}(\rho) \leq \tau_1(n)$.

- Quantum devices Q can be given either as a query oracle (*black-box model*) or a sequence of $\mathrm{poly}(n)$ elementary quantum gates (*white-box model*).
- The most canonical choices of entropy measures are:
  - **Shannon entropy** $\mathrm{H}(D) := \sum_x -D(x) \ln D(x)$.
  - **von Neumann entropy** $\mathrm{S}(\rho) := -\mathrm{Tr}(\rho \ln \rho)$.

What is quantum state testing

Task: Quantum state testing via entropy approximation

Given a state-preparation circuit Q ("quantum devices") that prepares (the purification of) n-qubit quantum states $\rho \in \mathbb{C}^{N \times N}$. Decide whether $\mathrm{Ent}(\rho) \geq \tau_0(n)$ or $\mathrm{Ent}(\rho) \leq \tau_1(n)$.

- Quantum devices Q can be given either as a query oracle (*black-box model*) or a sequence of $\mathrm{poly}(n)$ elementary quantum gates (*white-box model*).
- The most canonical choices of entropy measures are:
  - **Shannon entropy** $\mathrm{H}(D) := \sum_x -D(x) \ln D(x)$.
  - **von Neumann entropy** $\mathrm{S}(\rho) := -\mathrm{Tr}(\rho \ln \rho)$.
- Entropy *difference* problems, defined by the quantity $\mathrm{Ent}(\rho_0) - \mathrm{Ent}(\rho_1)$, can be formulated similarly and ask which state has the *higher* entropy.

What is quantum state testing

Task: Quantum state testing via entropy approximation

Given a state-preparation circuit Q ("quantum devices") that prepares (the purification of) n-qubit quantum states $\rho \in \mathbb{C}^{N \times N}$. Decide whether $\mathrm{Ent}(\rho) \geq \tau_0(n)$ or $\mathrm{Ent}(\rho) \leq \tau_1(n)$.

- Quantum devices Q can be given either as a query oracle (*black-box model*) or a sequence of $\mathrm{poly}(n)$ elementary quantum gates (*white-box model*).
- The most canonical choices of entropy measures are:
  - **Shannon entropy** $\mathrm{H}(D) := \sum_x -D(x) \ln D(x)$.
  - **von Neumann entropy** $\mathrm{S}(\rho) := -\mathrm{Tr}(\rho \ln \rho)$.
- Entropy *difference* problems, defined by the quantity $\mathrm{Ent}(\rho_0) - \mathrm{Ent}(\rho_1)$, can be formulated similarly and ask which state has the *higher* entropy.

Typical goal. Minimize the "complexity" of ρ (or its corresponding Q):

What is quantum state testing

Task: Quantum state testing via entropy approximation

Given a state-preparation circuit Q ("quantum devices") that prepares (the purification of) n-qubit quantum states $\rho \in \mathbb{C}^{N\times N}$. Decide whether $\mathrm{Ent}(\rho) \geq \tau_0(n)$ or $\mathrm{Ent}(\rho) \leq \tau_1(n)$.

- Quantum devices Q can be given either as a query oracle (*black-box model*) or a sequence of $\mathrm{poly}(n)$ elementary quantum gates (*white-box model*).
- The most canonical choices of entropy measures are:
  - **Shannon entropy** $\mathrm{H}(D) := \sum_x -D(x)\ln D(x)$.
  - **von Neumann entropy** $\mathrm{S}(\rho) := -\mathrm{Tr}(\rho \ln \rho)$.
- Entropy *difference* problems, defined by the quantity $\mathrm{Ent}(\rho_0) - \mathrm{Ent}(\rho_1)$, can be formulated similarly and ask which state has the *higher* entropy.

Typical goal. Minimize the "complexity" of ρ (or its corresponding Q):

Type of query access	Complexity measure
Black-box model	Query complexity (the number of queries to Q)
White-box model	Complexity class

Generalizations of the von Neumann entropy

Generalizations. There are two families of generalizations of the von Neumann entropy $S(\rho)$, namely, the α-Rényi entropy $S^{R}_{\alpha}(\rho)$ and the q-Tsallis entropy $S^{T}_{q}(\rho)$:

$$S^{R}_{\alpha}(\rho) := \frac{\ln \mathrm{Tr}(\rho^{\alpha})}{1-\alpha} \quad \text{and} \quad S^{T}_{q}(\rho) := \frac{1-\mathrm{Tr}(\rho^{q})}{q-1}.$$

Generalizations of the von Neumann entropy

<u>**Generalizations**</u>. There are two families of generalizations of the von Neumann entropy $\mathrm{S}(\rho)$, namely, the α-Rényi entropy $\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho)$ and the q-Tsallis entropy $\mathrm{S}^{\mathtt{T}}_{q}(\rho)$:

$$\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho) := \frac{\ln \mathrm{Tr}(\rho^{\alpha})}{1-\alpha} \quad \text{and} \quad \mathrm{S}^{\mathtt{T}}_{q}(\rho) := \frac{1-\mathrm{Tr}(\rho^{q})}{q-1}.$$

As the order approaches 1, these two generalizations converge to $\mathrm{S}(\rho)$.

Generalizations of the von Neumann entropy

Generalizations. There are two families of generalizations of the von Neumann entropy $\mathrm{S}(\rho)$, namely, the α-Rényi entropy $\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho)$ and the q-Tsallis entropy $\mathrm{S}^{\mathtt{T}}_{q}(\rho)$:

$$\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho) := \frac{\ln \mathrm{Tr}(\rho^{\alpha})}{1-\alpha} \quad \text{and} \quad \mathrm{S}^{\mathtt{T}}_{q}(\rho) := \frac{1-\mathrm{Tr}(\rho^{q})}{q-1}.$$

As the order approaches 1, these two generalizations converge to $\mathrm{S}(\rho)$.

von Neumann entropy (order 1). The entropy approximation problem in this case is *hard*, with complexity depending (polynomially) on the rank r of the state:

- The query complexity for estimating $\mathrm{S}(\rho)$ to within *constant* additive error is $\widetilde{O}(r)$ [Wang–Guan–Liu–Zhang–Ying'22] and $\widetilde{\Omega}(\sqrt{r})$ [Bun–Kothari–Thaler'17].

Generalizations of the von Neumann entropy

Generalizations. There are two families of generalizations of the von Neumann entropy $\mathrm{S}(\rho)$, namely, the α-Rényi entropy $\mathrm{S}^{\mathrm{R}}_{\alpha}(\rho)$ and the q-Tsallis entropy $\mathrm{S}^{\mathrm{T}}_{q}(\rho)$:

$$\mathrm{S}^{\mathrm{R}}_{\alpha}(\rho) := \frac{\ln \mathrm{Tr}(\rho^{\alpha})}{1-\alpha} \quad \text{and} \quad \mathrm{S}^{\mathrm{T}}_{q}(\rho) := \frac{1-\mathrm{Tr}(\rho^{q})}{q-1}.$$

As the order approaches 1, these two generalizations converge to $\mathrm{S}(\rho)$.

von Neumann entropy (order 1). The entropy approximation problem in this case is *hard*, with complexity depending (polynomially) on the rank r of the state:

- The query complexity for estimating $\mathrm{S}(\rho)$ to within *constant* additive error is $\widetilde{O}(r)$ [Wang–Guan–Liu–Zhang–Ying'22] and $\widetilde{\Omega}(\sqrt{r})$ [Bun–Kothari–Thaler'17].
- The promise problem Quantum Entropy Approximation (QEA), with respect to $\mathrm{S}(\rho)$, is NIQSZK-complete [Kobayashi'02, Chailloux–Ciocan–Kerenidis–Vadhan'07].

Generalizations of the von Neumann entropy

Generalizations. There are two families of generalizations of the von Neumann entropy $\mathrm{S}(\rho)$, namely, the α-Rényi entropy $\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho)$ and the q-Tsallis entropy $\mathrm{S}^{\mathtt{T}}_{q}(\rho)$:

$$\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho) := \frac{\ln \mathrm{Tr}(\rho^{\alpha})}{1-\alpha} \quad \text{and} \quad \mathrm{S}^{\mathtt{T}}_{q}(\rho) := \frac{1-\mathrm{Tr}(\rho^{q})}{q-1}.$$

As the order approaches 1, these two generalizations converge to $\mathrm{S}(\rho)$.

von Neumann entropy (order 1). The entropy approximation problem in this case is *hard*, with complexity depending (polynomially) on the rank r of the state:

- The query complexity for estimating $\mathrm{S}(\rho)$ to within *constant* additive error is $\widetilde{O}(r)$ [Wang–Guan–Liu–Zhang–Ying'22] and $\widetilde{\Omega}(\sqrt{r})$ [Bun–Kothari–Thaler'17].
- The promise problem Quantum Entropy Approximation (QEA), with respect to $\mathrm{S}(\rho)$, is NIQSZK-complete [Kobayashi'02, Chailloux–Ciocan–Kerenidis–Vadhan'07].
  - It is widely believed that BQP $\subsetneq$ NIQSZK.

Generalizations of the von Neumann entropy

Generalizations. There are two families of generalizations of the von Neumann entropy $\mathrm{S}(\rho)$, namely, the α-Rényi entropy $\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho)$ and the q-Tsallis entropy $\mathrm{S}^{\mathtt{T}}_{q}(\rho)$:

$$\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho) := \frac{\ln \mathrm{Tr}(\rho^{\alpha})}{1-\alpha} \quad \text{and} \quad \mathrm{S}^{\mathtt{T}}_{q}(\rho) := \frac{1-\mathrm{Tr}(\rho^{q})}{q-1}.$$

As the order approaches 1, these two generalizations converge to $\mathrm{S}(\rho)$.

von Neumann entropy (order 1). The entropy approximation problem in this case is *hard*, with complexity depending (polynomially) on the rank r of the state:

- The query complexity for estimating $\mathrm{S}(\rho)$ to within *constant* additive error is $\widetilde{O}(r)$ [Wang–Guan–Liu–Zhang–Ying'22] and $\widetilde{\Omega}(\sqrt{r})$ [Bun–Kothari–Thaler'17].
- The promise problem Quantum Entropy Approximation (QEA), with respect to $\mathrm{S}(\rho)$, is NIQSZK-complete [Kobayashi'02, Chailloux–Ciocan–Kerenidis–Vadhan'07].
  - It is widely believed that BQP $\subsetneq$ NIQSZK.
- The *poly(n)-rank* variant LowRankQEA is BQP-complete: containment in [Wang–Guan–Liu–Zhang–Ying'22] and hardness in [**L.**–Wang'24].

Generalizations of the von Neumann entropy

Generalizations. There are two families of generalizations of the von Neumann entropy $\mathrm{S}(\rho)$, namely, the α-Rényi entropy $\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho)$ and the q-Tsallis entropy $\mathrm{S}^{\mathtt{T}}_{q}(\rho)$:

$$\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho) := \frac{\ln \mathrm{Tr}(\rho^{\alpha})}{1-\alpha} \quad \text{and} \quad \mathrm{S}^{\mathtt{T}}_{q}(\rho) := \frac{1-\mathrm{Tr}(\rho^{q})}{q-1}.$$

As the order approaches 1, these two generalizations converge to $\mathrm{S}(\rho)$.

von Neumann entropy (order 1). The entropy approximation problem in this case is *hard*, with complexity depending (polynomially) on the rank r of the state:

- The query complexity for estimating $\mathrm{S}(\rho)$ to within *constant* additive error is $\widetilde{O}(r)$ [Wang–Guan–Liu–Zhang–Ying'22] and $\widetilde{\Omega}(\sqrt{r})$ [Bun–Kothari–Thaler'17].
- The promise problem Quantum Entropy Approximation (QEA), with respect to $\mathrm{S}(\rho)$, is NIQSZK-complete [Kobayashi'02, Chailloux–Ciocan–Kerenidis–Vadhan'07].
  - It is widely believed that BQP $\subsetneq$ NIQSZK.
- The *poly(n)-rank* variant LowRankQEA is BQP-complete: containment in [Wang–Guan–Liu–Zhang–Ying'22] and hardness in [**L.**–Wang'24].
  - Containment: a polynomial-time ("efficient") quantum algorithm that solves the problem.

Generalizations of the von Neumann entropy

Generalizations. There are two families of generalizations of the von Neumann entropy $\mathrm{S}(\rho)$, namely, the α-Rényi entropy $\mathrm{S}^{\mathrm{R}}_{\alpha}(\rho)$ and the q-Tsallis entropy $\mathrm{S}^{\mathrm{T}}_{q}(\rho)$:

$$\mathrm{S}^{\mathrm{R}}_{\alpha}(\rho) := \frac{\ln \mathrm{Tr}(\rho^{\alpha})}{1-\alpha} \quad \text{and} \quad \mathrm{S}^{\mathrm{T}}_{q}(\rho) := \frac{1-\mathrm{Tr}(\rho^{q})}{q-1}.$$

As the order approaches 1, these two generalizations converge to $\mathrm{S}(\rho)$.

von Neumann entropy (order 1). The entropy approximation problem in this case is *hard*, with complexity depending (polynomially) on the rank r of the state:

- The query complexity for estimating $\mathrm{S}(\rho)$ to within *constant* additive error is $\widetilde{O}(r)$ [Wang–Guan–Liu–Zhang–Ying'22] and $\widetilde{\Omega}(\sqrt{r})$ [Bun–Kothari–Thaler'17].
- The promise problem Quantum Entropy Approximation (QEA), with respect to $\mathrm{S}(\rho)$, is NIQSZK-complete [Kobayashi'02, Chailloux–Ciocan–Kerenidis–Vadhan'07].
  - It is widely believed that BQP $\subsetneq$ NIQSZK.
- The *poly(n)-rank* variant LowRankQEA is BQP-complete: containment in [Wang–Guan–Liu–Zhang–Ying'22] and hardness in [L.–Wang'24].
  - Containment: a polynomial-time ("efficient") quantum algorithm that solves the problem.
  - Hardness: if you can solve this problem, you can solve all problems in BQP.

Generalizations of the von Neumann entropy (Cont.)

Prior quantum query complexity upper bounds are summarized as follows:

Order (α or q)	Quantum α-Rényi entropy	Quantum q-Tsallis entropy
$(0,1)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]
1	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]	
$(1,\infty)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(1/\varepsilon)$ [**L.**–Wang'24]

Generalizations of the von Neumann entropy (Cont.)

Prior quantum query complexity upper bounds are summarized as follows:

Order (α or q)	Quantum α-Rényi entropy	Quantum q-Tsallis entropy
$(0,1)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]
1	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]	
$(1,\infty)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(1/\varepsilon)$ [**L.**–Wang'24]

We also summarize the prior work in terms of complexity classes:

- ⋆ For all $\alpha > 0$, LowRankRényiQEA$_\alpha$ is in BQP [Wang–Zhang–Li'22].
- ⋆ For all $q \in (0,1)$, LowRankTsallisQEA$_q$ is in BQP [WGLZY22].

Generalizations of the von Neumann entropy (Cont.)

Prior quantum query complexity upper bounds are summarized as follows:

Order (α or q)	Quantum α-Rényi entropy	Quantum q-Tsallis entropy
$(0,1)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]
1	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]	
$(1,\infty)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(1/\varepsilon)$ [**L.**–Wang'24]

We also summarize the prior work in terms of complexity classes:

- ⋆ For all $\alpha > 0$, LowRankRényiQEA$_\alpha$ is in BQP [Wang–Zhang–Li'22].
- ⋆ For all $q \in (0,1)$, LowRankTsallisQEA$_q$ is in BQP [WGLZY22].
- ⋆ For the order-1 case (von Neumann entropy), LowRankQEA is BQP-complete; containment follows from [WGLZY22], and hardness from [**L.**–Wang'24].
- ⋆ For all $q \in (1,2]$, TsallisQEA$_q$ is BQP-complete [**L.**–Wang'24].

Generalizations of the von Neumann entropy (Cont.)

Prior quantum query complexity upper bounds are summarized as follows:

Order (α or q)	Quantum α-Rényi entropy	Quantum q-Tsallis entropy
$(0,1)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]
1	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]	
$(1,\infty)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(1/\varepsilon)$ [**L.**–Wang'24]

We also summarize the prior work in terms of complexity classes:

- ⋆ For all $\alpha > 0$, LowRankRényiQEA$_\alpha$ is in BQP [Wang–Zhang–Li'22].
- ⋆ For all $q \in (0,1)$, LowRankTsallisQEA$_q$ is in BQP [WGLZY22].
- ⋆ For the order-1 case (von Neumann entropy), LowRankQEA is BQP-complete; containment follows from [WGLZY22], and hardness from [**L.**–Wang'24].
- ⋆ For all $q \in (1,2]$, TsallisQEA$_q$ is BQP-complete [**L.**–Wang'24].
- ⋆ For all $q > 2$, TsallisQEA$_q$ is in BQP [**L.**–Wang'24].

Generalizations of the von Neumann entropy (Cont.)

Prior quantum query complexity upper bounds are summarized as follows:

Order (α or q)	Quantum α-Rényi entropy	Quantum q-Tsallis entropy
$(0,1)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]
1	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]	
$(1,\infty)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(1/\varepsilon)$ [**L.**–Wang'24]

We also summarize the prior work in terms of complexity classes:

- ⋆ For all $\alpha > 0$, LowRankRényiQEA$_\alpha$ is in BQP [Wang–Zhang–Li'22].
- ⋆ For all $q \in (0,1)$, LowRankTsallisQEA$_q$ is in BQP [WGLZY22].
- ⋆ For the order-1 case (von Neumann entropy), LowRankQEA is BQP-complete; containment follows from [WGLZY22], and hardness from [**L.**–Wang'24].
- ⋆ For all $q \in (1,2]$, TsallisQEA$_q$ is BQP-complete [**L.**–Wang'24].
- ⋆ For all $q > 2$, TsallisQEA$_q$ is in BQP [**L.**–Wang'24].

🔔 These results lead to the following questions:

Generalizations of the von Neumann entropy (Cont.)

Prior quantum query complexity upper bounds are summarized as follows:

Order (α or q)	Quantum α-Rényi entropy	Quantum q-Tsallis entropy
$(0,1)$	$\mathrm{poly}(r,1/\varepsilon)$ [WZL24]	$\mathrm{poly}(r,1/\varepsilon)$ [WGLZY22]
1	$\mathrm{poly}(r,1/\varepsilon)$ [WGLZY22]	
$(1,\infty)$	$\mathrm{poly}(r,1/\varepsilon)$ [WZL24]	$\mathrm{poly}(1/\varepsilon)$ [**L.**–Wang'24]

We also summarize the prior work in terms of complexity classes:

- ⋆ For all $\alpha > 0$, LowRankRényiQEA$_\alpha$ is in BQP [Wang–Zhang–Li'22].
- ⋆ For all $q \in (0,1)$, LowRankTsallisQEA$_q$ is in BQP [WGLZY22].
- ⋆ For the order-1 case (von Neumann entropy), LowRankQEA is BQP-complete; containment follows from [WGLZY22], and hardness from [**L.**–Wang'24].
- ⋆ For all $q \in (1,2]$, TsallisQEA$_q$ is BQP-complete [**L.**–Wang'24].
- ⋆ For all $q > 2$, TsallisQEA$_q$ is in BQP [**L.**–Wang'24].

These results lead to the following questions:

1. How hard is the task of estimating α-Rényi or q-Tsallis entropy of quantum states for *all* positive order α or q?

Generalizations of the von Neumann entropy (Cont.)

Prior quantum query complexity upper bounds are summarized as follows:

Order (α or q)	Quantum α-Rényi entropy	Quantum q-Tsallis entropy
$(0,1)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]
1	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]	
$(1,\infty)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(1/\varepsilon)$ [**L.**–Wang'24]

We also summarize the prior work in terms of complexity classes:

- ⋆ For all $\alpha > 0$, LowRankRényiQEA$_\alpha$ is in BQP [Wang–Zhang–Li'22].
- ⋆ For all $q \in (0,1)$, LowRankTsallisQEA$_q$ is in BQP [WGLZY22].
- ⋆ For the order-1 case (von Neumann entropy), LowRankQEA is BQP-complete; containment follows from [WGLZY22], and hardness from [**L.**–Wang'24].
- ⋆ For all $q \in (1,2]$, TsallisQEA$_q$ is BQP-complete [**L.**–Wang'24].
- ⋆ For all $q > 2$, TsallisQEA$_q$ is in BQP [**L.**–Wang'24].

These results lead to the following questions:

1. How hard is the task of estimating α-Rényi or q-Tsallis entropy of quantum states for *all* positive order α or q?
2. Could LowRankRényiQEA$_\alpha$ $(\alpha > 0)$ and LowRankTsallisQEA$_q$ $(q > 0)$ both be BQP-hard, thus capturing the full power of quantum computation?

Main results

Let $\textsc{Rank2RényiQEA}_\alpha$ and $\textsc{Rank2TsallisQEA}_q$ denote the restricted versions of $\textsc{RényiQEA}_\alpha$ and $\textsc{TsallisQEA}_q$, where the state ρ has rank 2.

Main results

Let $\textsc{Rank2RényiQEA}_\alpha$ and $\textsc{Rank2TsallisQEA}_q$ denote the restricted versions of $\textsc{RényiQEA}_\alpha$ and $\textsc{TsallisQEA}_q$, where the state ρ has rank 2.

Theorem 1 (Hardness of estimating quantum entropies with positive orders).

1. For all real-valued $\alpha > 0$ and $\alpha = \infty$, $\textsc{Rank2RényiQEA}_\alpha$ is BQP-hard.
2. For all real-valued $q > 0$, $\textsc{Rank2TsallisQEA}_q$ is BQP-hard.

Main results

Let $\textsc{Rank2RényiQEA}_\alpha$ and $\textsc{Rank2TsallisQEA}_q$ denote the restricted versions of $\textsc{RényiQEA}_\alpha$ and $\textsc{TsallisQEA}_q$, where the state ρ has rank 2.

Theorem 1 (Hardness of estimating quantum entropies with positive orders).

1. For all real-valued $\alpha > 0$ and $\alpha = \infty$, $\textsc{Rank2RényiQEA}_\alpha$ is BQP-hard.
2. For all real-valued $q > 0$, $\textsc{Rank2TsallisQEA}_q$ is BQP-hard.

Combining Theorem 1 with prior quantum query complexity upper bounds implies:

Corollary 2. For all real-valued $\alpha > 0$, $\textsc{LowRankRényiQEA}_\alpha$ is BQP-complete.

Main results

Let $\textsc{Rank2RényiQEA}_\alpha$ and $\textsc{Rank2TsallisQEA}_q$ denote the restricted versions of $\textsc{RényiQEA}_\alpha$ and $\textsc{TsallisQEA}_q$, where the state ρ has rank 2.

Theorem 1 (Hardness of estimating quantum entropies with positive orders).

1. For all real-valued $\alpha > 0$ and $\alpha = \infty$, $\textsc{Rank2RényiQEA}_\alpha$ is BQP-hard.
2. For all real-valued $q > 0$, $\textsc{Rank2TsallisQEA}_q$ is BQP-hard.

Combining Theorem 1 with prior quantum query complexity upper bounds implies:

Corollary 2. For all real-valued $\alpha > 0$, $\textsc{LowRankRényiQEA}_\alpha$ is BQP-complete.

Corollary 3. The following holds:

1. For all real-valued $q \in (0, 1]$, $\textsc{LowRankTsallisQEA}_q$ is BQP-complete.
2. For all real-valued $q > 1$, $\textsc{TsallisQEA}_q$ is BQP-complete.

Main results

Let $\textsc{Rank2RényiQEA}_\alpha$ and $\textsc{Rank2TsallisQEA}_q$ denote the restricted versions of $\textsc{RényiQEA}_\alpha$ and $\textsc{TsallisQEA}_q$, where the state ρ has rank 2.

Theorem 1 (Hardness of estimating quantum entropies with positive orders).

1. For all real-valued $\alpha > 0$ and $\alpha = \infty$, $\textsc{Rank2RényiQEA}_\alpha$ is BQP-hard.
2. For all real-valued $q > 0$, $\textsc{Rank2TsallisQEA}_q$ is BQP-hard.

Combining Theorem 1 with prior quantum query complexity upper bounds implies:

Corollary 2. For all real-valued $\alpha > 0$, $\textsc{LowRankRényiQEA}_\alpha$ is BQP-complete.

Corollary 3. The following holds:

1. For all real-valued $q \in (0,1]$, $\textsc{LowRankTsallisQEA}_q$ is BQP-complete.
2. For all real-valued $q > 1$, $\textsc{TsallisQEA}_q$ is BQP-complete.

Theorem 4. $\textsc{Rank2RényiQEA}_0$ and $\textsc{Rank2TsallisQEA}_0$ are NQP-complete.

- NP $\subseteq$ NQP by comparing definitions, while it is widely believed that NP $\not\subseteq$ BQP.

Main results

Let $\textsc{Rank2RényiQEA}_\alpha$ and $\textsc{Rank2TsallisQEA}_q$ denote the restricted versions of $\textsc{RényiQEA}_\alpha$ and $\textsc{TsallisQEA}_q$, where the state ρ has rank 2.

Theorem 1 (Hardness of estimating quantum entropies with positive orders).

1. For all real-valued $\alpha > 0$ and $\alpha = \infty$, $\textsc{Rank2RényiQEA}_\alpha$ is BQP-hard.
2. For all real-valued $q > 0$, $\textsc{Rank2TsallisQEA}_q$ is BQP-hard.

Combining Theorem 1 with prior quantum query complexity upper bounds implies:

Corollary 2. For all real-valued $\alpha > 0$, $\textsc{LowRankRényiQEA}_\alpha$ is BQP-complete.

Corollary 3. The following holds:

1. For all real-valued $q \in (0,1]$, $\textsc{LowRankTsallisQEA}_q$ is BQP-complete.
2. For all real-valued $q > 1$, $\textsc{TsallisQEA}_q$ is BQP-complete.

Theorem 4. $\textsc{Rank2RényiQEA}_0$ and $\textsc{Rank2TsallisQEA}_0$ are NQP-complete.

- NP $\subseteq$ NQP by comparing definitions, while it is widely believed that NP $\not\subseteq$ BQP.

The BQP-hardness in Theorem 1 holds for the *smallest non-trivial* rank, as rank-1 states are pure states whose entropies are 0 for all orders.

Prior approaches via quantum entropy difference

The key quantity behind the prior approaches in [Ben-Aroya–Ta-Shma'07, **L.**'23] is the *quantum Jensen–Shannon divergence* $\mathrm{QJS}(\rho_0, \rho_1)$, introduced in [Majtey–Lamberti–Prato'05]:

$$\mathrm{QJS}(\rho_0,\rho_1) := \mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}(\rho_0)+\mathrm{S}(\rho_1)}{2} = \frac{1}{2}\cdot\left(\mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\otimes\frac{\rho_0+\rho_1}{2}\right) - \mathrm{S}(\rho_0\otimes\rho_1)\right).$$

Prior approaches via quantum entropy difference

The key quantity behind the prior approaches in [Ben-Aroya–Ta-Shma'07, **L.**'23] is the *quantum Jensen–Shannon divergence* $\mathrm{QJS}(\rho_0, \rho_1)$, introduced in [Majtey–Lamberti–Prato'05]:

$$\mathrm{QJS}(\rho_0,\rho_1) := \mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}(\rho_0)+\mathrm{S}(\rho_1)}{2} = \frac{1}{2}\cdot\left(\mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\otimes\frac{\rho_0+\rho_1}{2}\right) - \mathrm{S}(\rho_0\otimes\rho_1)\right).$$

The BQP-hardness of LowRankQED (and LowRankQEA) follows from the facts:

Prior approaches via quantum entropy difference

The key quantity behind the prior approaches in [Ben-Aroya–Ta-Shma'07, **L.**'23] is the *quantum Jensen–Shannon divergence* $\mathrm{QJS}(\rho_0,\rho_1)$, introduced in [Majtey–Lamberti–Prato'05]:

$$\mathrm{QJS}(\rho_0,\rho_1) := \mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}(\rho_0)+\mathrm{S}(\rho_1)}{2} = \frac{1}{2}\cdot\left(\mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\otimes\frac{\rho_0+\rho_1}{2}\right) - \mathrm{S}(\rho_0\otimes\rho_1)\right).$$

The BQP-hardness of LowRankQED (and LowRankQEA) follows from the facts:

1. The *pure-state variant* of the Quantum State Distinguishability Problem (PureQSD$[2/3,1/3]$), deciding whether $\mathrm{T}(|\psi_0\rangle\langle\psi_0|,|\psi_1\rangle\langle\psi_1|)$ is at least $2/3$ or at most $1/3$, is BQP-hard [Rethinasamy–Agarwal–Sharma–Wilde'21, Wang–Zhang'23].

Prior approaches via quantum entropy difference

The key quantity behind the prior approaches in [Ben-Aroya–Ta-Shma'07, **L.**'23] is the *quantum Jensen–Shannon divergence* $\mathrm{QJS}(\rho_0,\rho_1)$, introduced in [Majtey–Lamberti–Prato'05]:

$$\mathrm{QJS}(\rho_0,\rho_1) := \mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}(\rho_0)+\mathrm{S}(\rho_1)}{2} = \frac{1}{2}\cdot\left(\mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\otimes\frac{\rho_0+\rho_1}{2}\right) - \mathrm{S}(\rho_0\otimes\rho_1)\right).$$

The BQP-hardness of LowRankQED (and LowRankQEA) follows from the facts:

1. The *pure-state variant* of the Quantum State Distinguishability Problem (PureQSD$[2/3,1/3]$), deciding whether $\mathrm{T}(|\psi_0\rangle\langle\psi_0|,|\psi_1\rangle\langle\psi_1|)$ is at least $2/3$ or at most $1/3$, is BQP-hard [Rethinasamy–Agarwal–Sharma–Wilde'21, Wang–Zhang'23].
2. The inequalities connecting QJS to T [Fuchs–van de Graaf'99, Briët–Harremoës'09]:
$$\mathrm{H}\left(\frac{1}{2}\right) - \mathrm{H}\left(\frac{1-\mathrm{T}(\rho_0,\rho_1)}{2}\right) \le \mathrm{QJS}(\rho_0,\rho_1) \le \mathrm{H}\left(\frac{1}{2}\right)\cdot\mathrm{T}(\rho_0,\rho_1).$$

Prior approaches via quantum entropy difference

The key quantity behind the prior approaches in [Ben-Aroya–Ta-Shma'07, **L.**'23] is the *quantum Jensen–Shannon divergence* $\mathrm{QJS}(\rho_0,\rho_1)$, introduced in [Majtey–Lamberti–Prato'05]:

$$\mathrm{QJS}(\rho_0,\rho_1) := \mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}(\rho_0)+\mathrm{S}(\rho_1)}{2} = \frac{1}{2}\cdot\left(\mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\otimes\frac{\rho_0+\rho_1}{2}\right) - \mathrm{S}(\rho_0\otimes\rho_1)\right).$$

The BQP-hardness of LOWRANKQED (and LOWRANKQEA) follows from the facts:

1. The *pure-state variant* of the QUANTUM STATE DISTINGUISHABILITY PROBLEM (PUREQSD$[2/3,1/3]$), deciding whether $\mathrm{T}(|\psi_0\rangle\langle\psi_0|,|\psi_1\rangle\langle\psi_1|)$ is at least $2/3$ or at most $1/3$, is BQP-hard [Rethinasamy–Agarwal–Sharma–Wilde'21, Wang–Zhang'23].
2. The inequalities connecting QJS to T [Fuchs–van de Graaf'99, Briët–Harremoës'09]:
$$\mathrm{H}\left(\frac{1}{2}\right) - \mathrm{H}\left(\frac{1-\mathrm{T}(\rho_0,\rho_1)}{2}\right) \leq \mathrm{QJS}(\rho_0,\rho_1) \leq \mathrm{H}\left(\frac{1}{2}\right)\cdot\mathrm{T}(\rho_0,\rho_1).$$

This approach can be generalized to TSALLISQED$_q$ ($1\leq q\leq 2$). Considering the *quantum Jensen–Tsallis divergence* $\mathrm{QJT}_q(\rho_0,\rho_1)$ introduced in [Briët–Harremoës'09]:

$$\mathrm{QJT}_q(\rho_0,\rho_1) := \mathrm{S}_q^{\mathtt{T}}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}_q^{\mathtt{T}}(\rho_0)+\mathrm{S}_q^{\mathtt{T}}(\rho_1)}{2}.$$

Prior approaches via quantum entropy difference

The key quantity behind the prior approaches in [Ben-Aroya–Ta-Shma'07, **L.**'23] is the *quantum Jensen–Shannon divergence* $\mathrm{QJS}(\rho_0,\rho_1)$, introduced in [Majtey–Lamberti–Prato'05]:

$$\mathrm{QJS}(\rho_0,\rho_1) := \mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}(\rho_0)+\mathrm{S}(\rho_1)}{2} = \frac{1}{2}\cdot\left(\mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\otimes\frac{\rho_0+\rho_1}{2}\right) - \mathrm{S}(\rho_0\otimes\rho_1)\right).$$

The BQP-hardness of LowRankQED (and LowRankQEA) follows from the facts:

1. The *pure-state variant* of the Quantum State Distinguishability Problem (PureQSD$[2/3,1/3]$), deciding whether $\mathrm{T}(|\psi_0\rangle\langle\psi_0|,|\psi_1\rangle\langle\psi_1|)$ is at least $2/3$ or at most $1/3$, is BQP-hard [Rethinasamy–Agarwal–Sharma–Wilde'21, Wang–Zhang'23].
2. The inequalities connecting QJS to T [Fuchs–van de Graaf'99, Briët–Harremoës'09]:

$$\mathrm{H}\left(\frac{1}{2}\right) - \mathrm{H}\left(\frac{1-\mathrm{T}(\rho_0,\rho_1)}{2}\right) \le \mathrm{QJS}(\rho_0,\rho_1) \le \mathrm{H}\left(\frac{1}{2}\right)\cdot\mathrm{T}(\rho_0,\rho_1).$$

This approach can be generalized to TsallisQED$_q$ ($1\le q\le 2$). Considering the *quantum Jensen–Tsallis divergence* $\mathrm{QJT}_q(\rho_0,\rho_1)$ introduced in [Briët–Harremoës'09]:

$$\mathrm{QJT}_q(\rho_0,\rho_1) := \mathrm{S}_q^{\mathtt{T}}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}_q^{\mathtt{T}}(\rho_0)+\mathrm{S}_q^{\mathtt{T}}(\rho_1)}{2}.$$

The inequalities connecting QJT_q to T were established in [**L.**–Wang'24] via the *joint convexity* of QJT_q [Chen–Tropp'13, Virosztek'19]. These inequalities, together with several other known ingredients, yield the corresponding hardness result.

Establishing the hardness via binary entropy bounds

We start with a new approach that establishes the BQP-hardness of RANK2QEA. This approach is based on two key observations:

Establishing the hardness via binary entropy bounds

We start with a new approach that establishes the BQP-hardness of RANK2QEA. This approach is based on two key observations:

1. The 2-Tsallis entropy of a rank-2 state $\frac{1}{2}(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|)$ can be expressed as the 2-Tsallis binary entropy, and this coincidence naturally extends to all $q > 0$:

$$\mathrm{S}_2^{\mathtt{T}}\Big(\frac{|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|}{2}\Big) = \frac{1 - |\langle\psi_0|\psi_1\rangle|^2}{2} = \mathrm{H}_2^{\mathtt{T}}\Big(\frac{1 - \langle\psi_0|\psi_1\rangle}{2}\Big).$$

Establishing the hardness via binary entropy bounds

We start with a new approach that establishes the BQP-hardness of RANK2QEA. This approach is based on two key observations:

1. The 2-Tsallis entropy of a rank-2 state $\frac{1}{2}(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|)$ can be expressed as the 2-Tsallis binary entropy, and this coincidence naturally extends to all $q > 0$:
$$\mathrm{S}_2^{\mathtt{T}}\left(\frac{|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|}{2}\right) = \frac{1 - |\langle\psi_0|\psi_1\rangle|^2}{2} = \mathrm{H}_2^{\mathtt{T}}\left(\frac{1 - \langle\psi_0|\psi_1\rangle}{2}\right).$$
2. The following bounds on the Shannon binary entropy in [Lin'91, Topsøe'01]:
$$2\mathrm{H}\left(\frac{1}{2}\right) \cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}(x) \leq \sqrt{2}\mathrm{H}\left(\frac{1}{2}\right) \cdot \sqrt{\mathrm{H}_2^{\mathtt{T}}(x)}.$$

Establishing the hardness via binary entropy bounds

We start with a new approach that establishes the BQP-hardness of RANK2QEA. This approach is based on two key observations:

1. The 2-Tsallis entropy of a rank-2 state $\frac{1}{2}(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|)$ can be expressed as the 2-Tsallis binary entropy, and this coincidence naturally extends to all $q > 0$:

$$\mathrm{S}_2^{\mathtt{T}}\left(\frac{|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|}{2}\right) = \frac{1 - |\langle\psi_0|\psi_1\rangle|^2}{2} = \mathrm{H}_2^{\mathtt{T}}\left(\frac{1 - \langle\psi_0|\psi_1\rangle}{2}\right).$$

2. The following bounds on the Shannon binary entropy in [Lin'91, Topsøe'01]:

$$2\mathrm{H}\left(\frac{1}{2}\right) \cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}(x) \leq \sqrt{2}\mathrm{H}\left(\frac{1}{2}\right) \cdot \sqrt{\mathrm{H}_2^{\mathtt{T}}(x)}.$$

Since PURE-STATE INFIDELITY ESTIMATION (PUREINFIDELITY$\left[\frac{2}{3}, \frac{1}{3}\right]$), deciding whether $1 - |\langle\psi_0|\psi_1\rangle|^2$ is at least $2/3$ or at most $1/3$, is BQP-hard [Rethinasamy–Agarwal–Sharma–Wilde'21], it follows that both RANK2TSALLISQEA$_2$ and RANK2RÉNYIQEA$_2$ are BQP-hard.

Establishing the hardness via binary entropy bounds

We start with a new approach that establishes the BQP-hardness of RANK2QEA. This approach is based on two key observations:

1. The 2-Tsallis entropy of a rank-2 state $\frac{1}{2}(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|)$ can be expressed as the 2-Tsallis binary entropy, and this coincidence naturally extends to all $q > 0$:

$$\mathrm{S}_2^{\mathtt{T}}\left(\frac{|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|}{2}\right) = \frac{1 - |\langle\psi_0|\psi_1\rangle|^2}{2} = \mathrm{H}_2^{\mathtt{T}}\left(\frac{1 - \langle\psi_0|\psi_1\rangle}{2}\right).$$

2. The following bounds on the Shannon binary entropy in [Lin'91, Topsøe'01]:

$$2\mathrm{H}\left(\frac{1}{2}\right) \cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}(x) \leq \sqrt{2}\mathrm{H}\left(\frac{1}{2}\right) \cdot \sqrt{\mathrm{H}_2^{\mathtt{T}}(x)}.$$

Since PURE-STATE INFIDELITY ESTIMATION (PUREINFIDELITY$\left[\frac{2}{3}, \frac{1}{3}\right]$), deciding whether $1 - |\langle\psi_0|\psi_1\rangle|^2$ is at least $2/3$ or at most $1/3$, is BQP-hard [Rethinasamy–Agarwal–Sharma–Wilde'21], it follows that both RANK2TSALLISQEA$_2$ and RANK2RÉNYIQEA$_2$ are BQP-hard.

🔔 The inequalities relating the order-2 binary entropy to Shannon binary entropy (Step 2) can be extended to *all* positive orders!

Establishing the hardness via binary entropy bounds (Cont.)

The BQP-hardness of RANK2RÉNYIQEA$_\alpha$ can then be established via the following inequalities that relate $\mathrm{H}_2^{\mathtt{R}}(x)$ to $\mathrm{H}_\alpha^{\mathtt{R}}(x)$:

Range of α	Hardness	Reduction from	New inequalities
$0 < \alpha < 1$	BQP-hard Theorem 1(1)	RANK2RÉNYIQEA$_2$	$\mathrm{H}_2^{\mathtt{R}}(x) \leq \mathrm{H}_\alpha^{\mathtt{R}}(x)$ $\mathrm{H}_\alpha^{\mathtt{R}}(x) \leq \ln(2)^{1-\frac{\alpha}{2}} \cdot \mathrm{H}_2^{\mathtt{R}}(x)^{\frac{\alpha}{2}}$ [Beck–Schögl'93, Sec 5.3]
$1 \leq \alpha < 2$	BQP-hard Theorem 1(1)	RANK2RÉNYIQEA$_2$	
$\alpha = 2$	BQP-hard Theorem 1(1)	PUREINFIDELITY [RASW'21]	None
$\alpha \in (2, \infty]$	BQP-hard Theorem 1(1)	RANK2RÉNYIQEA$_2$	$\frac{\alpha}{2(\alpha-1)} \cdot \mathrm{H}_2^{\mathtt{R}}(x) \leq \mathrm{H}_\alpha^{\mathtt{R}}(x) \leq \mathrm{H}_2^{\mathtt{R}}(x)$ [Beck–Schögl'93, Sec 5.3]

Establishing the hardness via binary entropy bounds (Cont.^{2})

The BQP-hardness of RANK2TSALLISQEA$_q$ can then be established via the following inequalities that relate $\mathrm{H}_2^{\mathtt{T}}(x)$ to $\mathrm{H}_q^{\mathtt{T}}(x)$:

Range of q	Hardness	Reduction from	New inequalities
$0 < q < 1$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	$2\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right) \cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}_q^{\mathtt{T}}(x)$ $\mathrm{H}_q^{\mathtt{T}}(x) \leq 2^{\frac{q}{2}}\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right) \cdot \left(\mathrm{H}_2^{\mathtt{T}}(x)\right)^{\frac{q}{2}}$ [L.–Wang'24]
$1 \leq q < 2$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	
$q = 2$	BQP-hard Theorem 1(2)	PUREINFIDELITY [RASW'21]	None
$2 < q \leq 3$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	$\frac{q}{2(q-1)} \cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}_q^{\mathtt{T}}(x) \leq 2\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right) \cdot \mathrm{H}_2^{\mathtt{T}}(x)$
$q \in (3, \infty)$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	$2\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right) \cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}_q^{\mathtt{T}}(x)$ $\mathrm{H}_q^{\mathtt{T}}(x) \leq \frac{q}{2(q-1)} \cdot \mathrm{H}_2^{\mathtt{T}}(x)$ [L.–Wang'24]

Establishing the hardness via binary entropy bounds (Cont.^{2})

The BQP-hardness of RANK2TSALLISQEA$_q$ can then be established via the following inequalities that relate $\mathrm{H}_2^{\mathtt{T}}(x)$ to $\mathrm{H}_q^{\mathtt{T}}(x)$:

Range of q	Hardness	Reduction from	New inequalities
$0 < q < 1$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	$2\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right)\cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}_q^{\mathtt{T}}(x)$ $\mathrm{H}_q^{\mathtt{T}}(x) \leq 2^{\frac{q}{2}}\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right)\cdot\left(\mathrm{H}_2^{\mathtt{T}}(x)\right)^{\frac{q}{2}}$ [L.–Wang'24]
$1 \leq q < 2$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	
$q = 2$	BQP-hard Theorem 1(2)	PUREINFIDELITY [RASW'21]	None
$2 < q \leq 3$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	$\frac{q}{2(q-1)}\cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}_q^{\mathtt{T}}(x) \leq 2\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right)\cdot \mathrm{H}_2^{\mathtt{T}}(x)$
$q \in (3,\infty)$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	$2\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right)\cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}_q^{\mathtt{T}}(x)$ $\mathrm{H}_q^{\mathtt{T}}(x) \leq \frac{q}{2(q-1)}\cdot \mathrm{H}_2^{\mathtt{T}}(x)$ [L.–Wang'24]

The additional row links to the *normalized* q-Tsallis entropy $\widetilde{H}_q^{\mathtt{T}}(x) := \mathrm{H}_q^{\mathtt{T}}(x)/\mathrm{H}_q^{\mathtt{T}}(1/2)$ (cf. [Daróczy'70]), whose monotonicity *changes* at some point $q^*(x) \in [2,3]$.

Conclusions and open problems

Take-home messages on our work

For ***all*** **positive** orders, estimating α-Rényi or q-Tsallis entropies of **rank-**2 quantum states, *the smallest non-trivial rank*, is BQP-hard.

Conclusions and open problems

Take-home messages on our work

For ***all*** **positive** orders, estimating α-Rényi or q-Tsallis entropies of **rank-**2 quantum states, *the smallest non-trivial rank*, is BQP-hard.

Discussion and open problems

Two limitations of our new approach are as follows:

Conclusions and open problems

Take-home messages on our work

For ***all*** **positive** orders, estimating α-Rényi or q-Tsallis entropies of **rank-**2 quantum states, *the smallest non-trivial rank*, is BQP-hard.

Discussion and open problems

Two limitations of our new approach are as follows:

- Our approach works only when quantum entropy values and the promise gap $\tau_0 - \tau_1$ are both *constant*. Otherwise, reductions based on inequalities, such as $\mathrm{S}^{\mathtt{R}}_{\infty}(\rho) \leq \mathrm{S}^{\mathtt{R}}_{\alpha}(\rho) \leq \frac{\alpha}{\alpha-1}\mathrm{S}^{\mathtt{R}}_{\infty}(\rho)$ for all $\alpha > 1$, break down for sufficiently large n.

Conclusions and open problems

Take-home messages on our work

For ***all*** **positive** orders, estimating α-Rényi or q-Tsallis entropies of **rank-**2 quantum states, *the smallest non-trivial rank*, is BQP-hard.

Discussion and open problems

Two limitations of our new approach are as follows:

- Our approach works only when quantum entropy values and the promise gap $\tau_0 - \tau_1$ are both *constant*. Otherwise, reductions based on inequalities, such as $\mathrm{S}^{\mathtt{R}}_{\infty}(\rho) \leq \mathrm{S}^{\mathtt{R}}_{\alpha}(\rho) \leq \frac{\alpha}{\alpha-1}\mathrm{S}^{\mathtt{R}}_{\infty}(\rho)$ for all $\alpha > 1$, break down for sufficiently large n.
- A complexity-theoretic barrier is that reductions between different orders do not hold in general. Noting that RÉNYIQEA$_\infty$ is coSBP-complete [Watson'12] and EA is NISZK-complete [Goldreich–Vadhan'99], such reductions would yield

$$\text{coNP} \subseteq \text{coSBP} \subseteq \text{NISZK} \subseteq \text{SZK} \subseteq \text{AM} \cap \text{coAM}.$$

These inclusions would collapse PH to its second level [Boppana–Håstad–Zachos'87].

Conclusions and open problems

Take-home messages on our work

For ***all* positive** orders, estimating α-Rényi or q-Tsallis entropies of **rank-**2 quantum states, *the smallest non-trivial rank*, is BQP-hard.

Discussion and open problems

Two limitations of our new approach are as follows:

- Our approach works only when quantum entropy values and the promise gap $\tau_0 - \tau_1$ are both *constant*. Otherwise, reductions based on inequalities, such as $\mathrm{S}^{\mathrm{R}}_{\infty}(\rho) \leq \mathrm{S}^{\mathrm{R}}_{\alpha}(\rho) \leq \frac{\alpha}{\alpha-1}\mathrm{S}^{\mathrm{R}}_{\infty}(\rho)$ for all $\alpha > 1$, break down for sufficiently large n.
- A complexity-theoretic barrier is that reductions between different orders do not hold in general. Noting that RÉNYIQEA$_\infty$ is coSBP-complete [Watson'12] and EA is NISZK-complete [Goldreich–Vadhan'99], such reductions would yield

  coNP $\subseteq$ coSBP $\subseteq$ NISZK $\subseteq$ SZK $\subseteq$ AM $\cap$ coAM.

  These inclusions would collapse PH to its second level [Boppana–Håstad–Zachos'87].

Question: Is it possible to establish a complexity-theoretic classification theorem for estimating the quantum (α-Rényi or q-Tsallis) entropies?

Thanks!