RWTH AACHEN UNIVERSITY

The Complexity of Homomorphism Reconstruction

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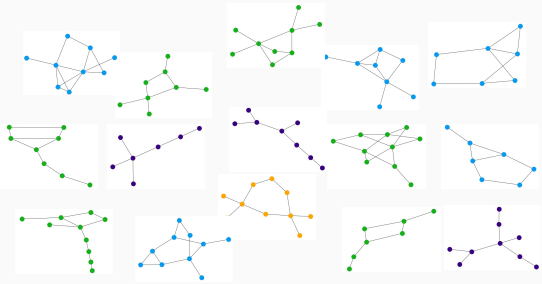
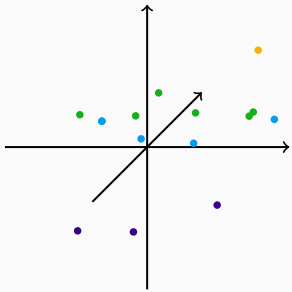
Homomorphism Embedding

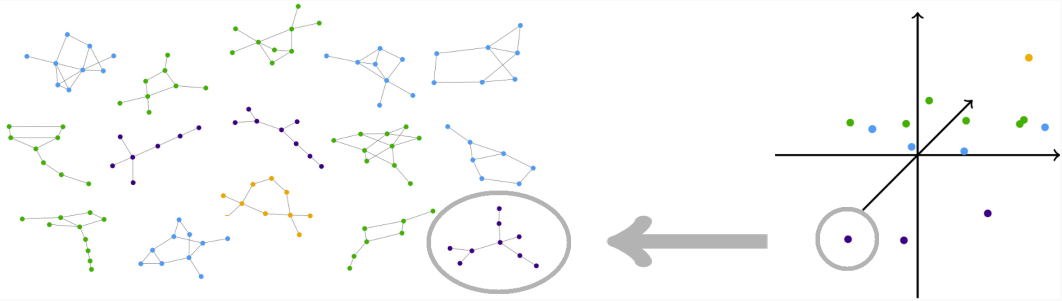
Illustration from Grohe, "X2vec: Towards a Theory of Vector Embeddings of Structured Data"





$$G \mapsto \begin{pmatrix} \hom(F_1, G) \\ \hom(F_2, G) \\ \hom(F_3, G) \end{pmatrix}$$

Homomorphism Reconstruction



$$\begin{pmatrix} \hom(F_1, G) \\ \hom(F_2, G) \\ \hom(F_3, G) \end{pmatrix} \mapsto G$$

Problem (HomRec)

INPUT: $(F_1, m_1), \ldots, (F_k, m_k)$, *where* $F_1, \ldots, F_k$ *are graphs and* $m_1, \ldots, m_k \in \mathbb{N}$ *(in binary encoding).*

QUESTION: *Is there a* G *such that* $\hom(F_i, G) = m_i$ *for all* i*?*

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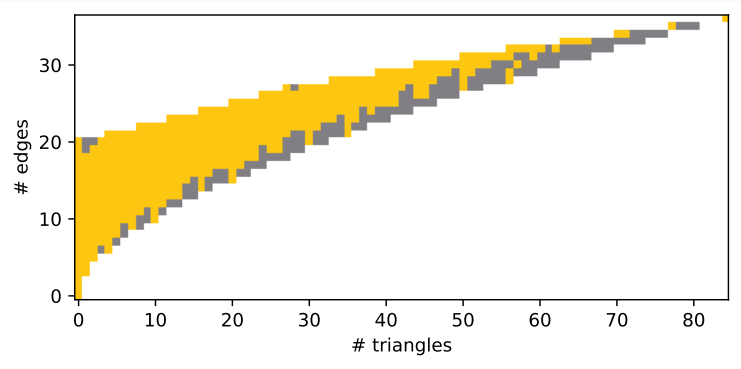
HOMREC is in NEXP:

- guess a graph G of size $N = \sum_{i=1}^{k} m_i |F_i|$
- for each F_i, enumerate all homomorphisms $F_i \to G$ in time $O(N^{|F_i|})$

Example for Edges and Triangles

INPUT: $(\triangle, t)$ and $(\bullet\!-\!\bullet, e)$.

QUESTION: Is there a G with $\hom(\triangle, G) = t$ and $\hom(\bullet\!-\!\bullet, G) = e$?



For graphs G on 9 vertices, all possible $\triangle$ and $\bullet\!-\!\bullet$ subgraph counts in yellow.

Böker, H., Runde, Seppelt, Standke showed (STACS 2024)

- HomRec is NP$^{\#P}$-hard.
- There is a set of 9 coloured stars $S_1, \ldots, S_9$ for which deciding if there is a G such that $\hom(S_1, G) = m_1, \ldots, \hom(S_9, G) = m_9$ is NP-hard.
- For a single connected graph F and $m \in \mathbb{N}$, deciding if there is a G such that $\hom(F, G) = m$ is fpt in $|F|$.

Theorem

HomRec *is* NEXP*-complete.*

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UnHomRec *is* Σ_2^p*-complete.*

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HomRec *for stars* $F_1, \ldots, F_k$ *is solvable in (deterministic) time* $m^{O(\ell^2)}$*, where* $\ell := \max\{|F_i| \mid i \in [k]\}$ *and* $m := \max\{m_i \mid i \in [k]\}$.

Proof Outline

Theorem

HomRec *is* NEXP*-complete.*

We give a reduction from NEXP-complete problem SuccinctClique.

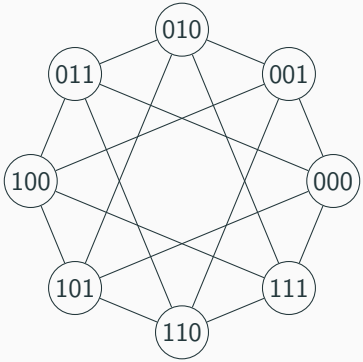
The main ingredients:

- succinct input representation,
- evaluation of Boolean circuits, and
- gadgets for replacing colours.

An Introduction to Succinct Input Representations

- Problems on graphs G given as (small) Boolean circuits C_G computing the entry (i, j) of their adjacency matrices
- C_G is the "SCR of G"
- Most objects do not admit succinct representations of much smaller size
- Examples often have very regular structure

TikZ “One”-Liner



$$C_G(i_0, i_1, i_2, j_0, j_1, j_2) = \boxed{\neg i_2 \wedge j_2}$$

The Increase of Complexity

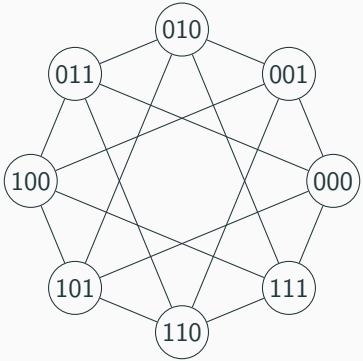
Theorem (Papadimitriou, Yannakakis 1986)

Let $\mathcal{G}$ be any property of graphs such that deciding whether $G \in \mathcal{G}$ holds is NP*-hard under projection reductions.*

Then, given SCR C_G, deciding whether $G \in \mathcal{G}$ holds is NEXP*-hard.*

Corollary

SuccinctClique *is* NEXP*-complete.*



$(G, 3) \in \textsc{Clique}?$

$$C_G := \boxed{\neg i_2 \wedge j_2}$$

$$\left(\boxed{\neg i_2 \wedge j_2}, 3\right) \in \textsc{SuccinctClique}?$$

$$(\boxed{\neg i_2 \wedge j_2}, 3) \longrightarrow \textsc{HomRec}$$

Encoding Boolean Circuits

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$$\textsc{CircuitSat} \leq_p \textsc{ColHomRec}.$$

$\textsc{ColHomRec}$ allows F_i and the target graph G to be coloured graphs.

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$\textsc{ColHomRec}$ allows F_i and the target graph G to be coloured graphs.

The reconstructed graph will be a copy of the input circuit C.

Encoding Boolean Circuits

Problem (CircuitSat)

INPUT: *Boolean circuit* $C : \{0,1\}^n \to \{0,1\}$

QUESTION: *Does there exist an input* $x \in \{0,1\}^n$ *such that* $C(x) = 1$*?*

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The coloured graph $G = (V(G), E(G), c : V(G) \to \mathcal{C})$ we will reconstruct, if an input x with $C(x) = 1$ exists, should contain an explicit representation of the computation of C on x.

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and output at gate $o =$ edges: either $(o, \bot)$ or $(o, \top)$.

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$$\mathcal{F}_\alpha \coloneqq ((\bullet\alpha, 2), (\bullet\bot, 1), (\bullet\top, 1), (\ \overset{\bot\ \alpha\ \alpha\ \top}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}\ , 1), (\alpha\bullet\!\!-\!\!\bullet\bot, 1), (\alpha\bullet\!\!-\!\!\bullet\top, 1)).$$

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Lemma

Any graph G that satisfies $\mathcal{F}_\alpha$ contains a single copy of the coloured graph $\overset{\bot\ \alpha\ \alpha\ \top}{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}$.

HomRec Constraints

$$\mathcal{F}_{\equiv}(\mathsf{A}, \mathsf{B}, n, m) \coloneqq \big((\bullet\mathsf{A}, n), (\mathsf{A}\!\bullet\!\!-\!\!\bullet\mathsf{B}, nm), (\ \overset{\mathsf{B}\ \mathsf{A}\ \mathsf{B}}{\bullet\!-\!\bullet\!-\!\bullet}\ , nm^2)\big).$$

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Lemma

Any graph G that satisfies $\mathcal{F}_{\equiv}(\mathsf{A},\mathsf{B},n,m)$ contains a colour class A^G of size n. Each vertex coloured A has exactly m neighbours coloured B.

CircuitSat $\leq_p$ ColHomRec

Let $C = (V(C), E(C), \mathsf{In}^C, \vee^C, \wedge^C, \neg^C)$ be a circuit with n input gates $v_1, \ldots, v_n$ and 1 output gate o.

For any input $x \in \{0,1\}^n$ we define the graph $G_{C(x)}$:

- A copy of the "value gadget" 

,
- for each $v \in V(C)$: a node with colour C_v
- the C_v node is connected to either α next to $\bot$ or $\top$, depending on its truth value.

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We can construct list of constraints $\mathcal{F}$ such that

$$\exists x \in \{0,1\}^n : C(x) = 1 \Leftrightarrow \mathcal{F} \in \textsc{HomRec}.$$

CircuitSat $\leq_p$ ColHomRec

Correct evaluation of $C(x)$ in $G_{C(x)}$ is guaranteed by the following set of constraints.

$$\mathcal{F}_C(v) \coloneqq \begin{cases} ((\ \bot\ \alpha\ \alpha\ \top\ \text{over } C_u C_w C_v\ , 0), (\ \bot\ \alpha\ \alpha\ \top\ \text{over } C_u C_w C_v\ , 0), (\ \bot\ \alpha\ \alpha\ \top\ \text{over } C_u C_w C_v\ , 0), (\ \bot\ \alpha\ \alpha\ \top\ \text{over } C_u C_w C_v\ , 0)), & v = \wedge(u, w), \\ ((\ \bot\ \alpha\ \alpha\ \top\ \text{over } C_u C_w C_v\ , 0), (\ \bot\ \alpha\ \alpha\ \top\ \text{over } C_u C_w C_v\ , 0), (\ \bot\ \alpha\ \alpha\ \top\ \text{over } C_u C_w C_v\ , 0), (\ \bot\ \alpha\ \alpha\ \top\ \text{over } C_u C_w C_v\ , 0)), & v = \vee(u, w), \\ ((\ \bot\ \alpha\ \alpha\ \top\ \text{over } C_w\ C_v\ , 0), (\ \bot\ \alpha\ \alpha\ \top\ \text{over } C_w\ C_v\ , 0)), & v = \neg(w). \end{cases}$$

SuccinctClique Reduction

Lemma

$$\textsc{SuccinctClique} \leq_p \textsc{ColHomRec}$$

Given input (C_G, k), instead of one copy of C_G, create $\binom{k}{2}$ copies.

If these evaluate to true, the inputs encode $\binom{k}{2}$ edges in G.

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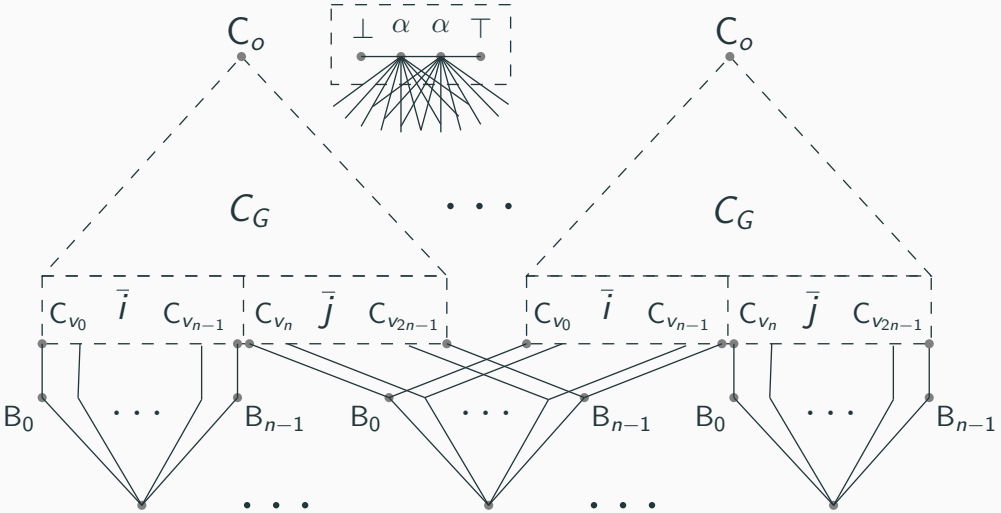
Getting the inputs consistent requires a little bit of effort (set of k vertices, correct bit order).


C_o
$\bot$ α α $\top$
C_o
C_G
C_G
C_{v_0} $\bar{i}$ $C_{v_{n-1}}$
C_{v_n} $\bar{j}$ $C_{v_{2n-1}}$
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For any instance of ColHomRec *with at most* 4 *colour classes, we can compute an equivalent instance of* HomRec *with uncoloured simple graphs in polynomial time.*

Main Result

Theorem

HomRec *is* NEXP*-complete.* ✓

Reconstructing From Unary Counts

UnHomRec is in $\Sigma_2^p = \mathsf{NP}^{\mathsf{NP}}$:

- guess a graph G of (polynomial) size $N = \sum_{i=1}^{k} m_i |F_i|$
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Idea: $G =$  is the only graph that satisfies $\hom(\bullet, G) = 3$ and $\hom(\bullet\!-\!\bullet, G) = 6$.

Reconstructing From Stars

Hard Example with Easy Counting

INPUT: ( , m_1) and ( , m_2).

QUESTION: Is there a G with hom(, G) $= m_1$ and hom(, G) $= m_2$?

Answers is YES if and only if G exists with

$$\sum_{v \in V(G)} \deg(v) = m_1, \text{ and}$$

$$\sum_{v \in V(G)} \deg(v)^2 = m_2.$$

Reconstructing From Star Counts

Theorem

HomRec *for stars* $F_1, \ldots, F_k$ *is solvable in (deterministic) time* $m^{O(\ell^2)}$*, where* $\ell := \max\{|F_i| \mid i \in [k]\}$ *and* $m := \max\{m_i \mid i \in [k]\}$.

Idea: use dynamic programming to recurse on graphic degree sequences.

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EXP, for stars F_i and m_i in binary.

Theorem

HomRec *is* NEXP*-complete.*

Theorem

UnHomRec *is* Σ_2^p*-complete.*

Corollary

HomRec *for stars* $F_1, \ldots, F_k$ *is in* EXP*.*

It is open if HomRec is in EXP for other graph classes.