Density Matters: A Complexity Dichotomy of Deleting Edges to Bound Subgraph Density

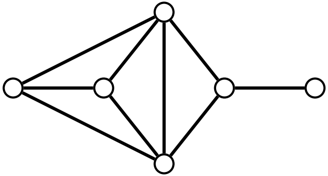
STACS 2026

Matthias Bentert[1], Tom-Lukas Breitkopf[2], Vincent Froese[2], Anton Herrmann[2], André Nichterlein[2]

[1] Technische Universität Berlin, Germany & University of Bergen, Norway
[2]Algorithmics and Computational Complexity, Technische Universität Berlin, Germany

11.03.2026

Our (Computational) Problem



Central notions:

graph density: $\rho(G) = \frac{|E(G)|}{|V(G)|} = \frac{m}{n}$

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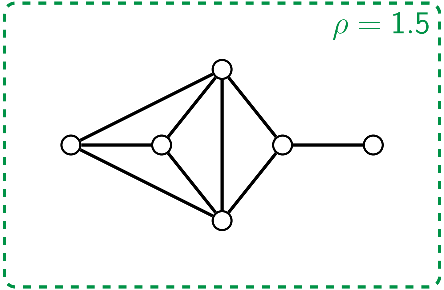

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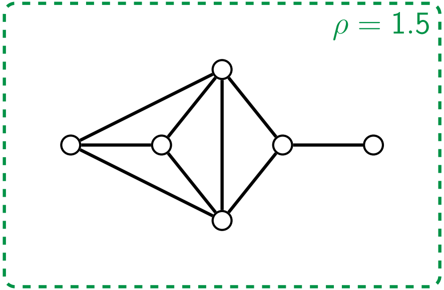

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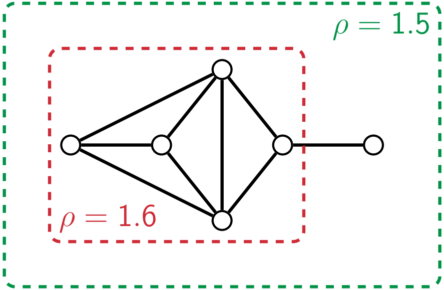

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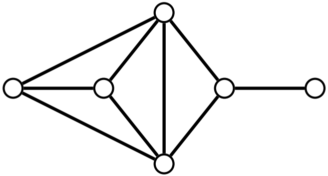

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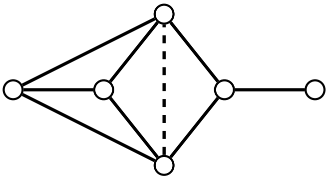

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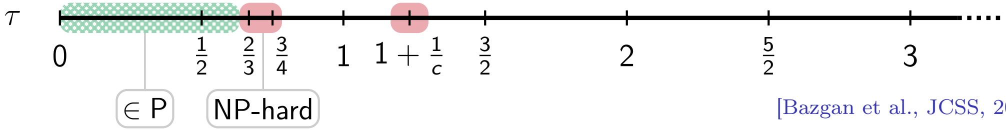
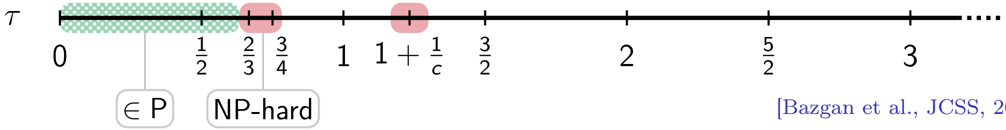
Known results



[Bazgan et al., JCSS, 2025]

Known results and Our Main Contribution



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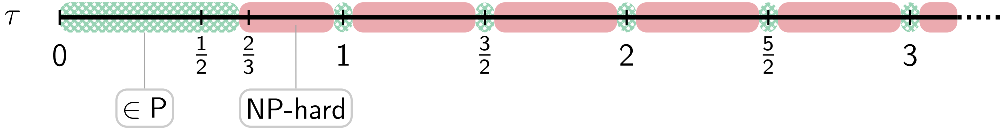
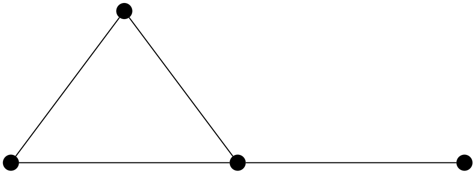


Fractional orientation



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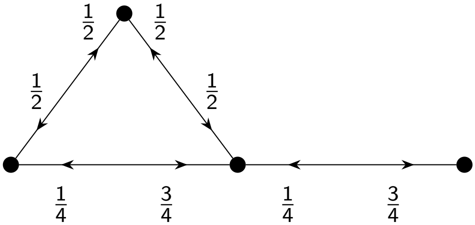

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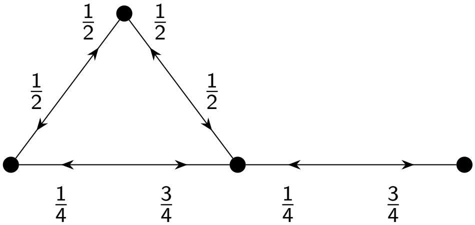

Δ^-_ϕ: maximum in-degree w.r.t. Φ

Lemma ([Charikar, APPROX, 2000])

For every graph G and every fractional orientation ϕ, it holds that $\rho^*(G) \leq \Delta^-_\phi$.

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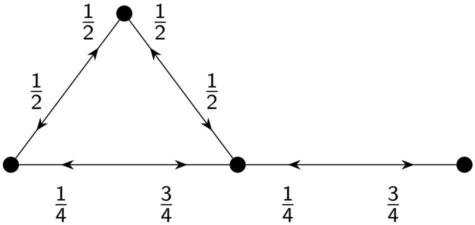

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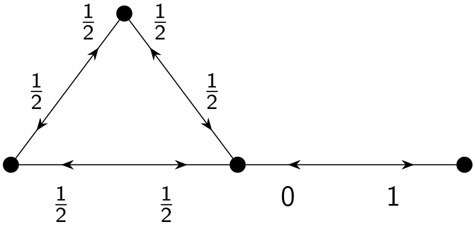

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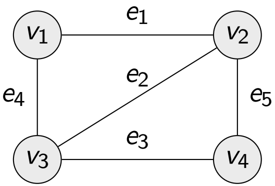
Solving τ-BDED for $\tau \in \mathbb{N}$

τ-BDED



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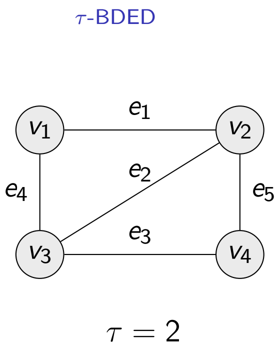
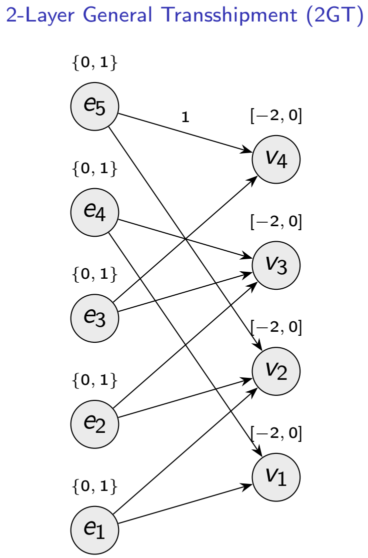


$\rightsquigarrow$



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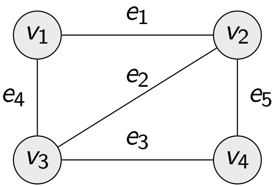

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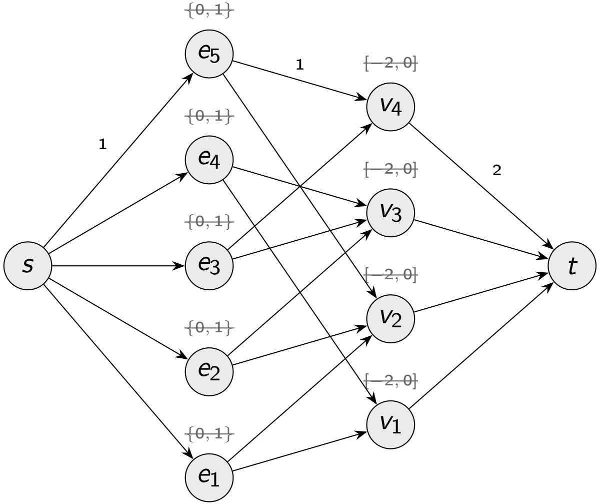
Maximum Flow



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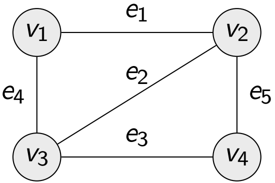

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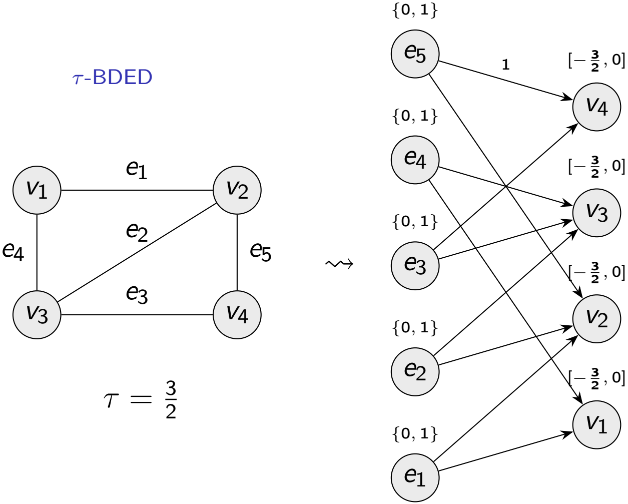
2-Layer General Transshipment (2GT)

τ-BDED



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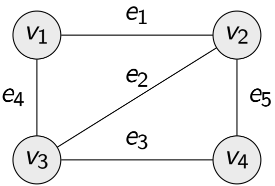

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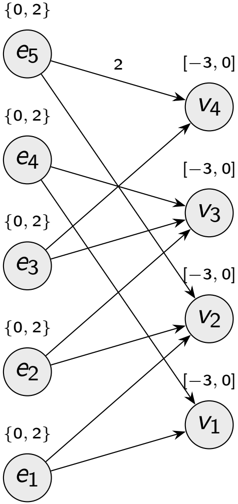

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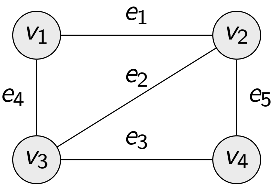

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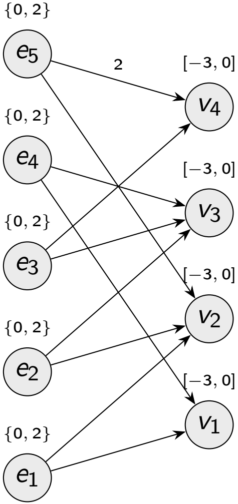

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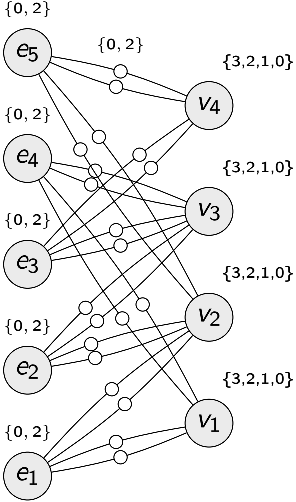
General Factors



NP-hardness for $1 < \tau = \frac{p}{q} < 2$

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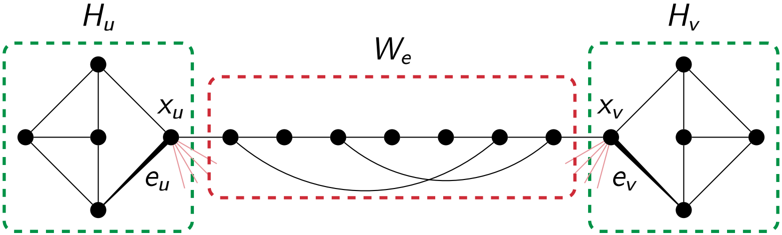
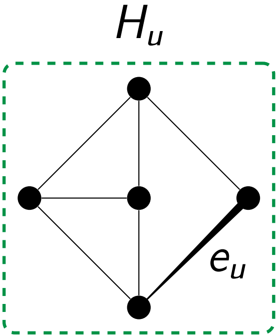
- For edge $e = \{u, v\}$, construct:



$k' := k$

Vertex gadget



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  [Ruciński & Vince, Journal of Graph Theory, 1986]

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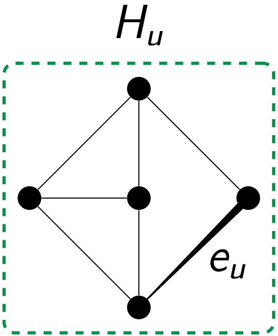

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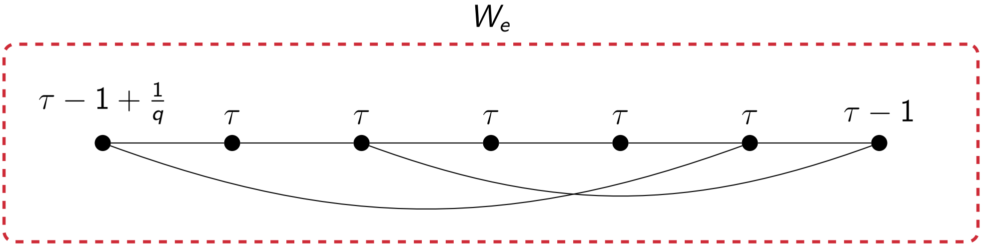
- H_u contains an edge that is completely assigned to one of its endpoints

Edge gadget



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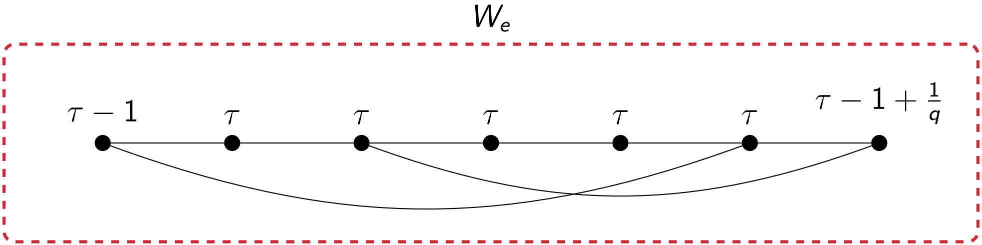

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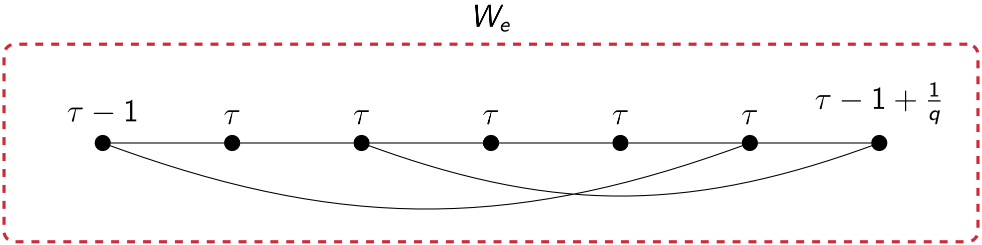

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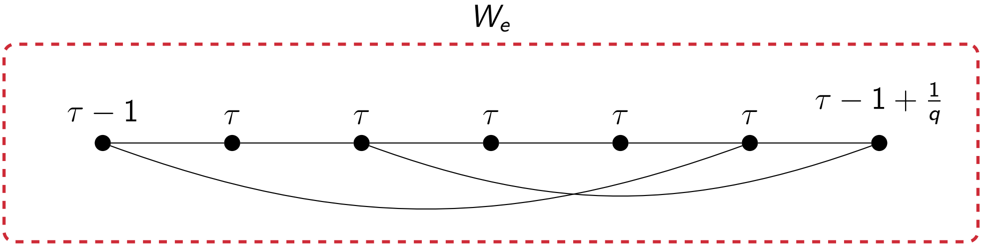

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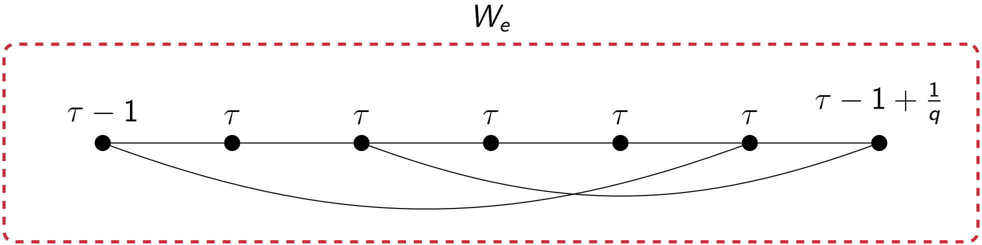

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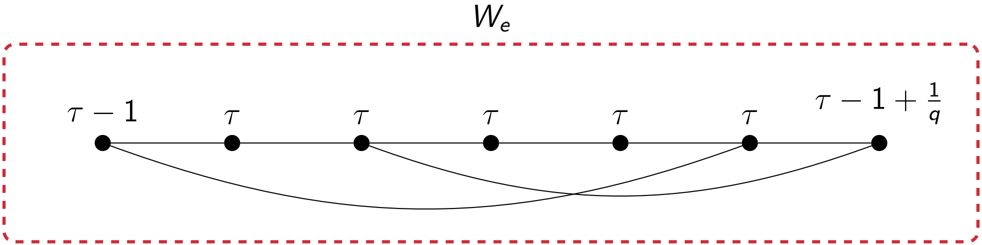

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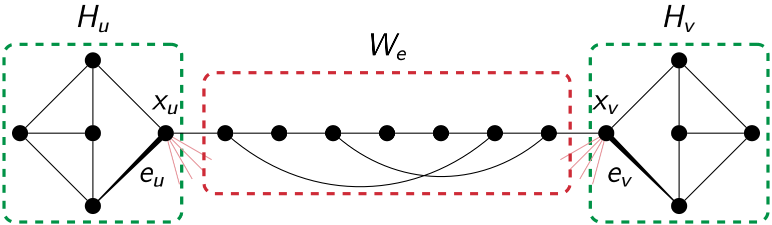

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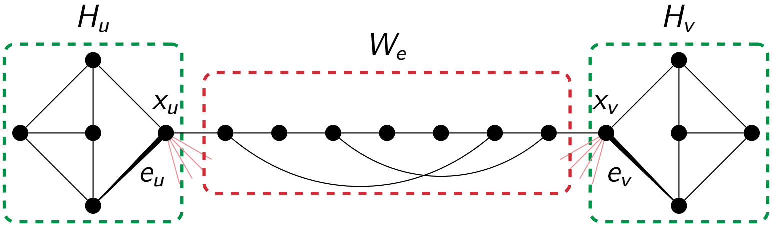

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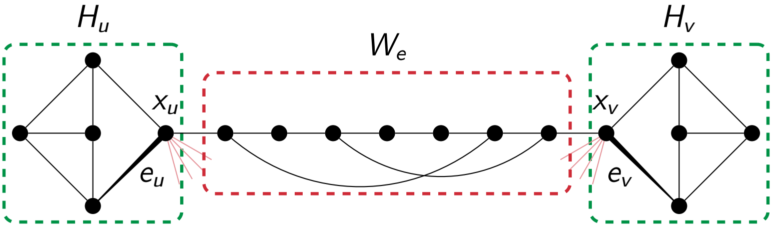

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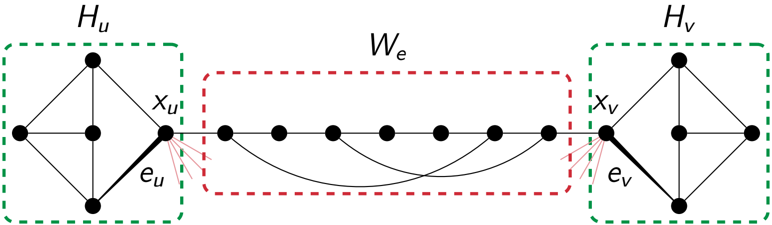

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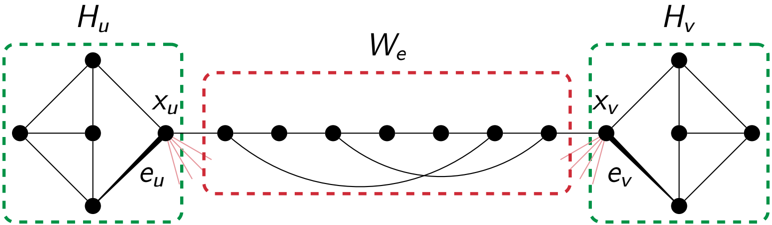

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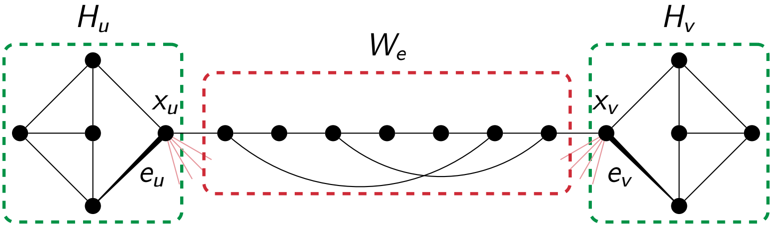

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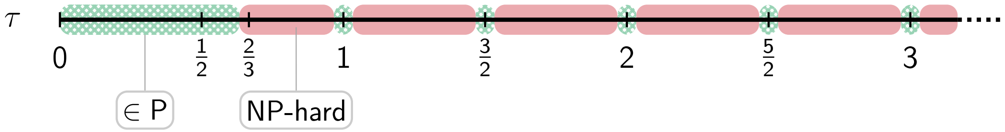
- For $\tau > 2$ "padding" with balanced graphs

Concluding remarks



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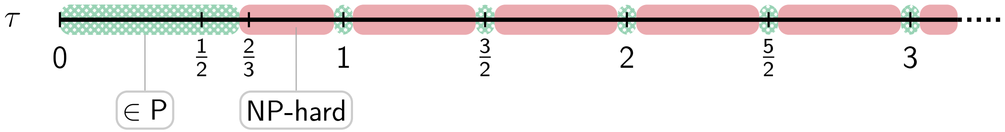

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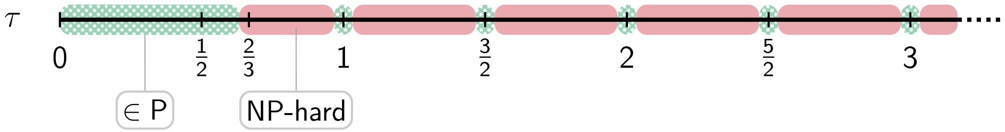

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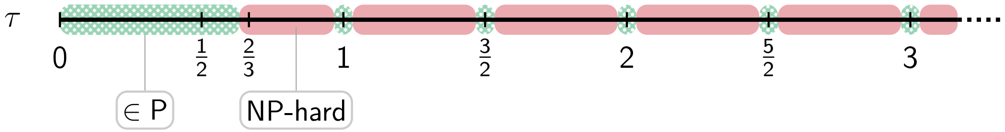

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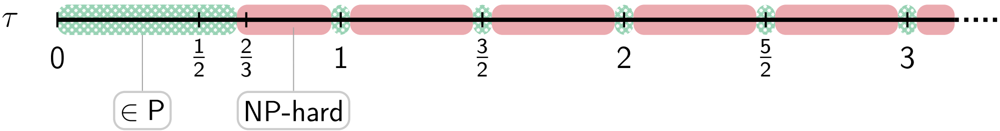

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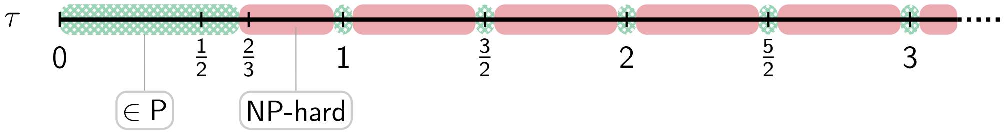

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- There is a direct reduction from τ-BDED with half-integral τ to MATCHING
- Polynomial-time algorithms cover weighted problem variants
- What about non-constant τ?
- Can the efficiently solvable cases of 2GT be extended to cover larger classes of networks?