On the Complexity of Computing Strahler Numbers

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STACS 2026

Strahler numbers

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For a binary tree t, the Strahler number (aka Horton-Strahler number) $\mathrm{st}(t)$ is defined as follows:

$$\mathrm{st}(\bullet) = 0$$

$$\mathrm{st}(t_1 \overset{\bullet}{\wedge} t_2) = \begin{cases} \max\{\mathrm{st}(t_1), \mathrm{st}(t_2)\} & \text{if } \mathrm{st}(t_1) \neq \mathrm{st}(t_2) \\ \mathrm{st}(t_1) + 1 & \text{if } \mathrm{st}(t_1) = \mathrm{st}(t_2) \end{cases}$$

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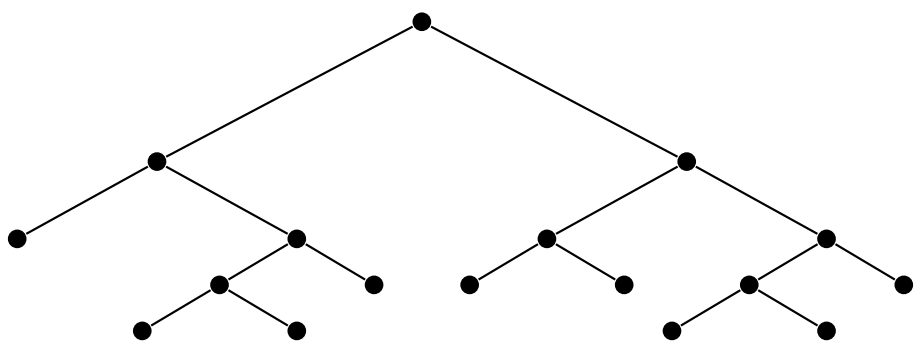

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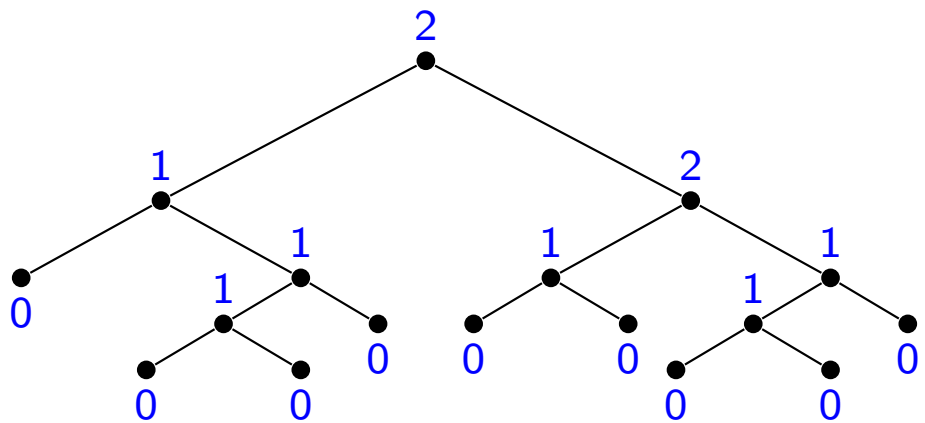

Origin of the Strahler number

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For more background, see the surveys by Viennot (1990) and Esparza, Luttenberger and Schlund (2014).

Applications of Strahler numbers

- Context-free grammars with finite index

  (Ginsburg, Spanier 1968)

- Parikh's theorem

  (Esparza, Ganty, Kiefer, Luttenberger 2011)

- Pushdown VASS

  (Atig, Ganty 2011; Bizière, Czerwiński 2025)

- Parity games

  (Daviaud, Jurdziński, Thejaswini 2020)

Computing Strahler numbers

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Observation

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Recursive algorithm where the larger subtree is visited first.
$\rightarrow$ stack of height $\mathcal{O}(\log n)$ with $\log\log n$ bit numbers.

The stack can be stored with $\mathcal{O}(\log n)$ bits

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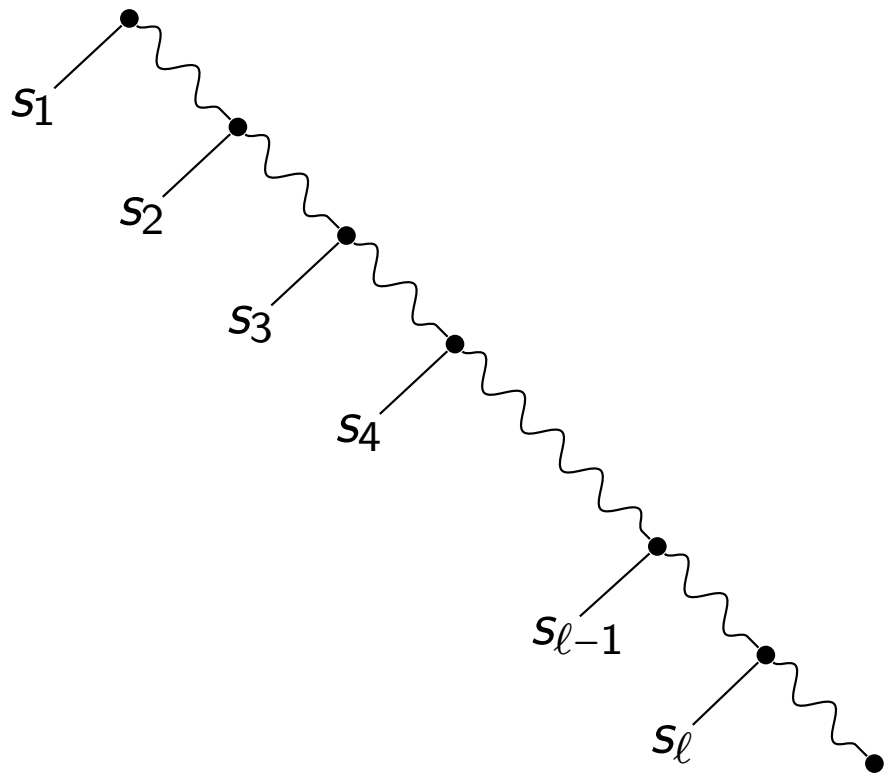

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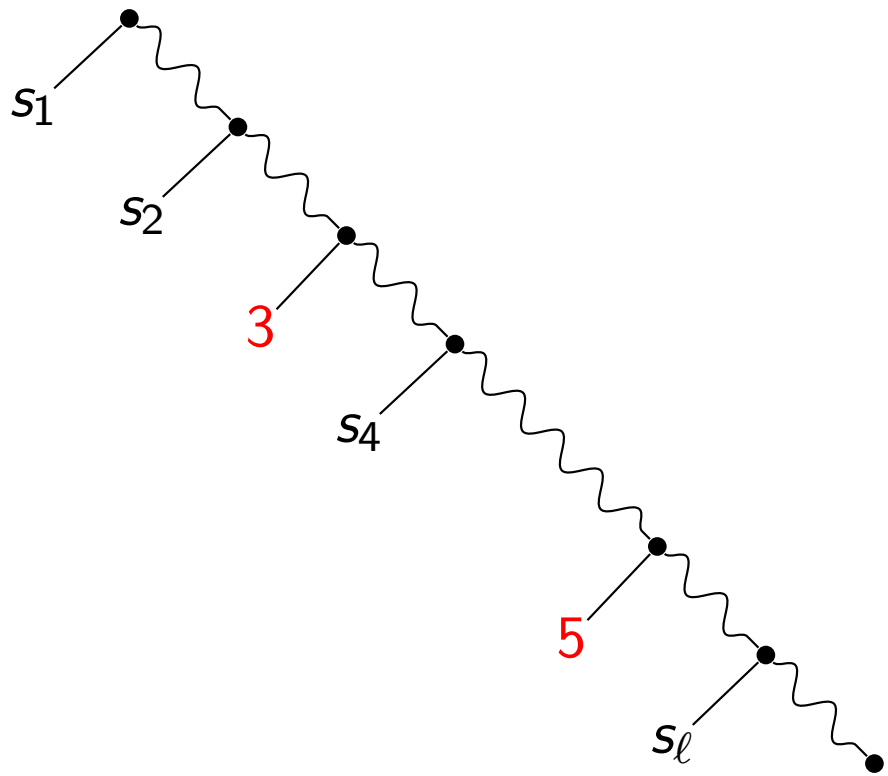

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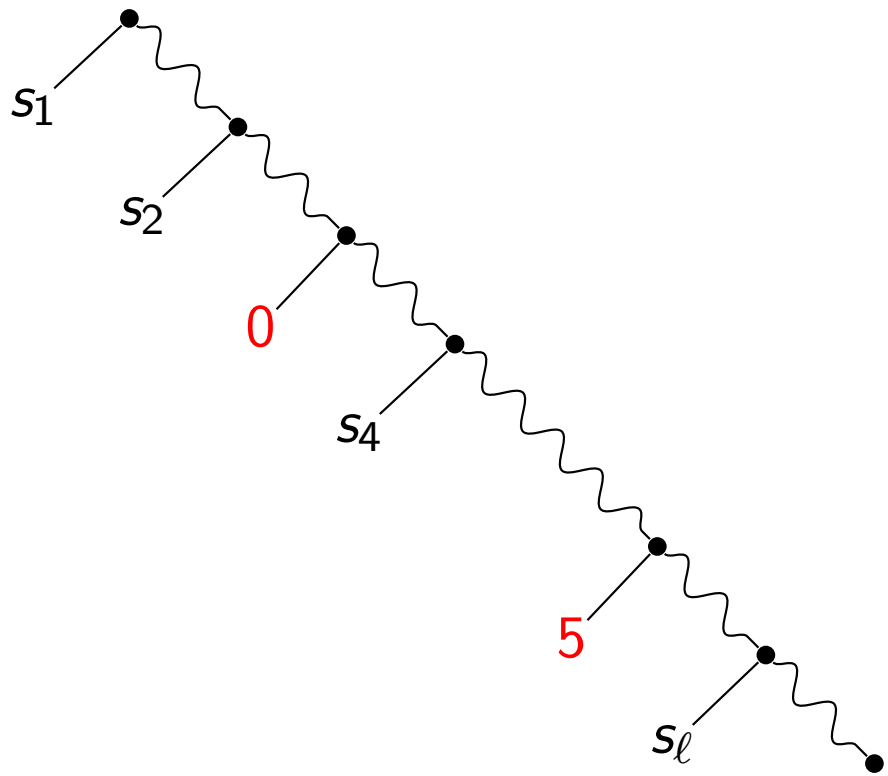

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Let the resulting sequence be

$$0, \ldots, 0, s_{i_1}, 0, \ldots, 0, s_{i_2}, \ldots, s_{i_{k-1}}, , 0, \ldots, 0, s_{i_k}, 0, \ldots, 0$$

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We now use a delta encoding:

$$0 \cdots 0\, 1^{s_{i_1} - s_{i_2}} \# 0 \cdots 0\, 1^{s_{i_2} - s_{i_3}} \# \cdots 1^{s_{i_{k-1}} - s_{i_k}} \# 0 \cdots 0\, 1^{s_{i_k}} \# 0 \cdots 0$$

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This is a word of length $\ell + s_{i_1} \leq \mathcal{O}(\log n)$. □

Broader context: Tree evaluation

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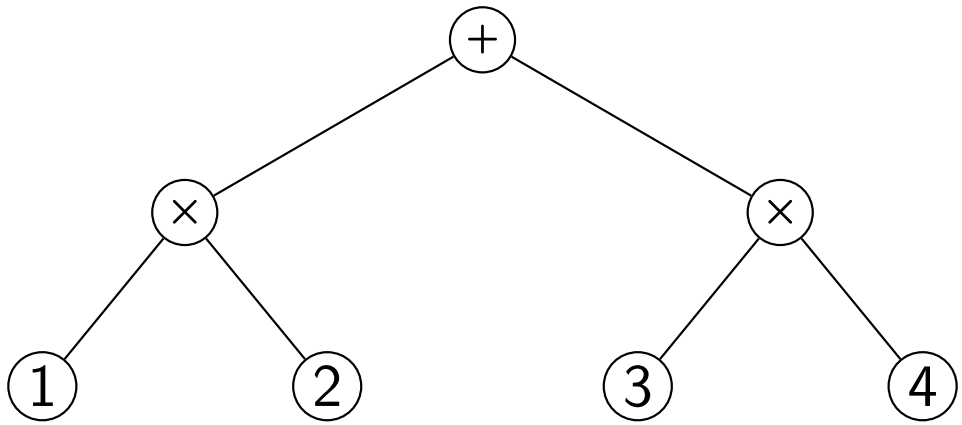

- Each leaf is labeled by an element $a \in A$.
- Each inner node carries a function $f\colon A^r \to A$, where r is the number of children.

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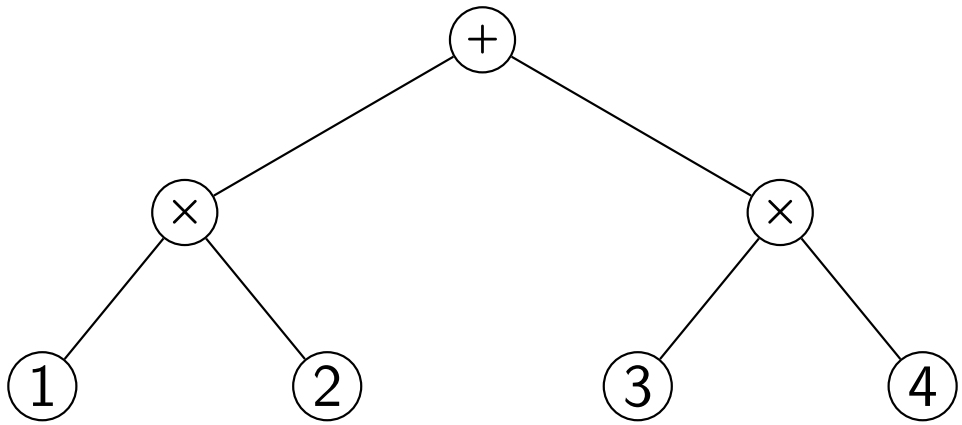

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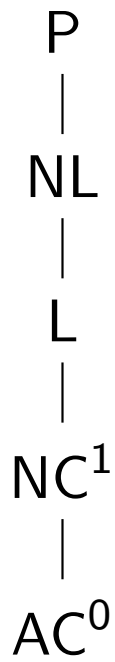
Goal: Understand the complexity of tree evaluation for various algebras.

Small complexity classes

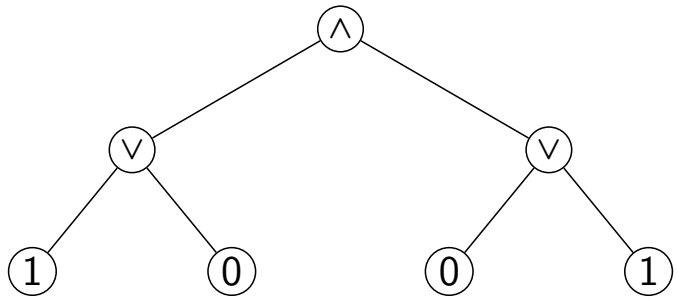


NC^1 = class of all problems that can be solved by a uniform family $(C_n)_{n\geq 0}$ of fan-in-2 boolean circuits such that C_n (= circuit for input length n) has

- size $\mathrm{poly}(n)$ and
- depth $\mathcal{O}(\log n)$.

Broader context: Tree evaluation

Boolean formulas



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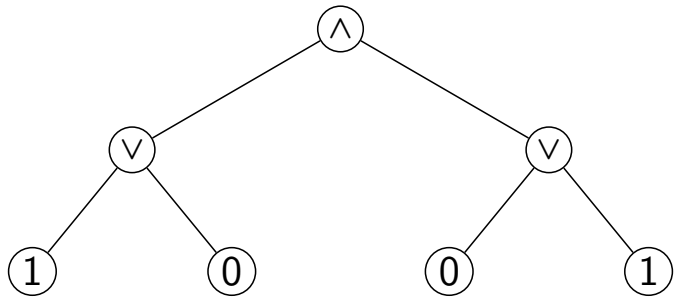
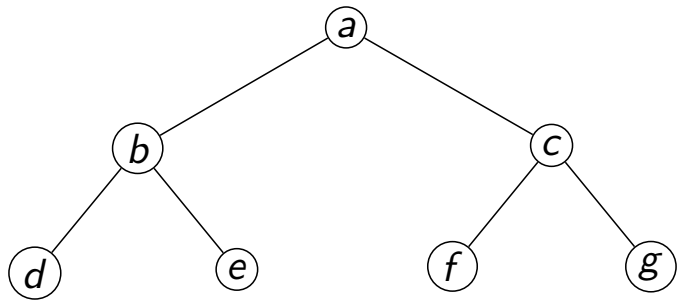


Acceptance of a (fixed) tree automaton



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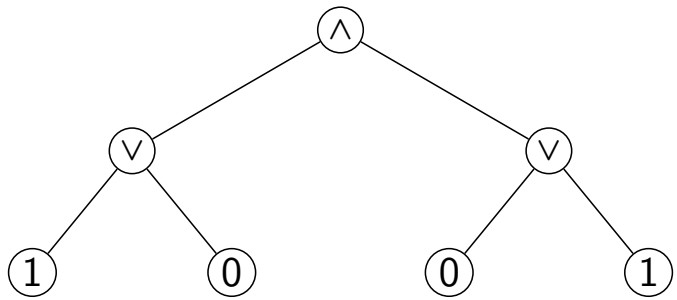

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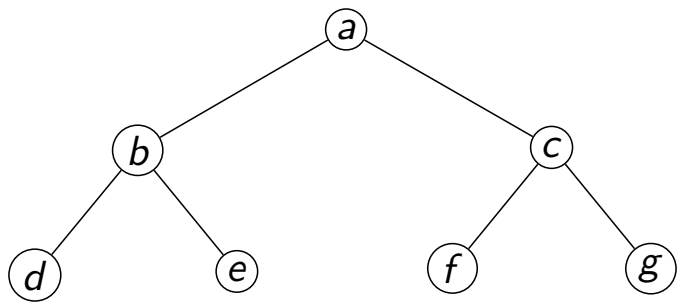


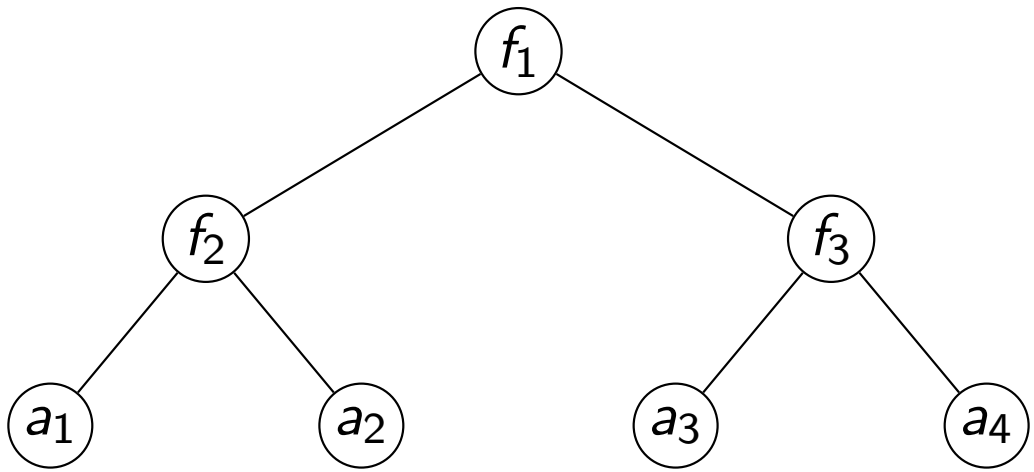
→ NC^1-complete assuming parenthesis representations $(1 \vee 0) \wedge (0 \vee 1)$.

(Buss 1987, Lohrey 2001)

Broader context: Tree evaluation

"The tree evaluation" problem

Evaluate a complete binary tree of height h over a domain of size k



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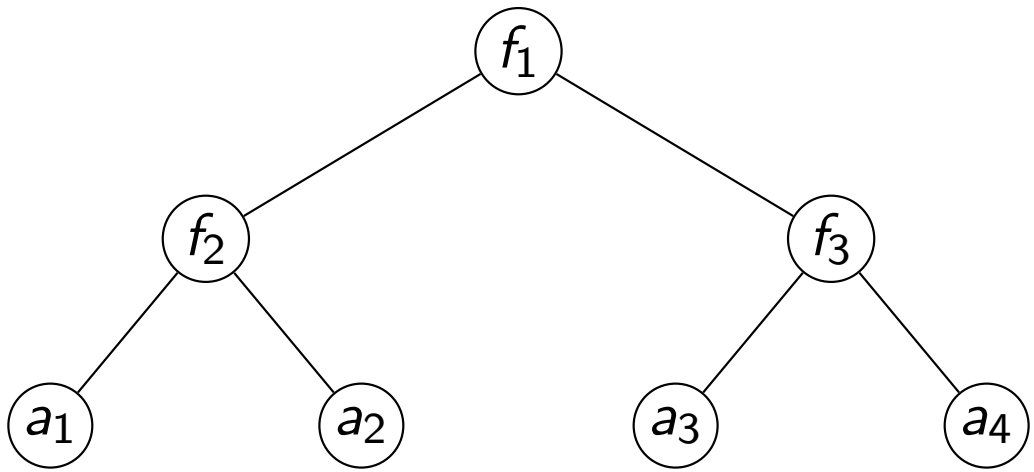

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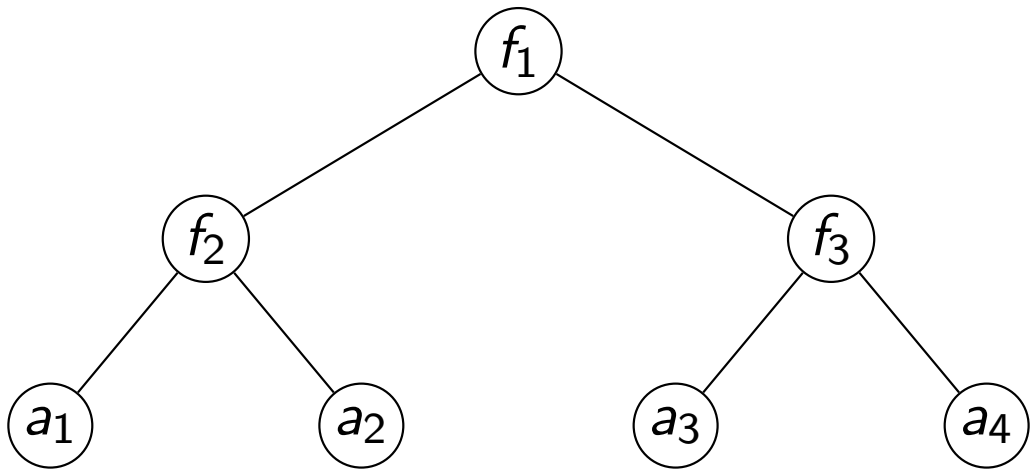

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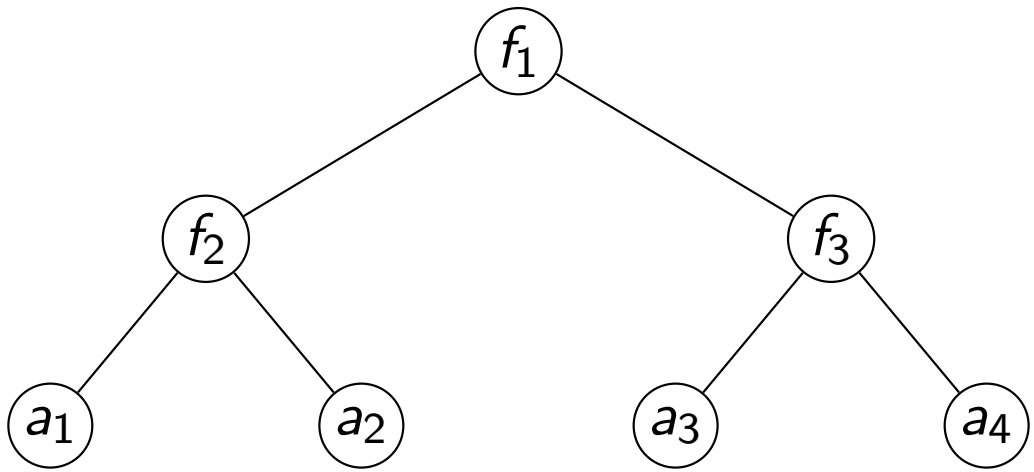

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Williams (2025):	$\mathrm{TIME}[t] \subseteq \mathrm{SPACE}[\sqrt{t \log t}]$

Our contributions

The complexity of computing the Strahler number

Theorem

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NC^1 = Fan-in-2 boolean circuits of poly-size and log-depth

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Further results on Strahler number computations on context-free grammars.

Proof idea for the NC^1 upper bound

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This function f_c can be described by two numbers $\ell_c \leq h_c$:

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3. Evaluate this circuit in NC^1.

Summary

Theorem

Computing the Strahler number of a binary tree t is NC^1-complete if the tree t is given by a parenthesis expression.

Open questions:

- Evaluating $\{\max, +\}$-trees with binary encoded numbers
- Shortest path problem in directed graphs with binary encoded weights
- Strahler numbers for unranked trees
- Pathwidth of trees

Strahler numbers and context-free grammars

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Gruska 1971

The following problem is undecidable:

- Input: A context-free grammar G in Chomsky normal form and $k \in \mathbb{N}$
- Question: Does every word in $L(G)$ have a derivation tree of Strahler number $\leq k$?

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The following problem is P-complete:

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Why are acyclic derivation trees interesting?

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Fix a constant $k \geq 1$. Intersection non-emptiness for group DFAs and a single CFG for which all acyclic derivation trees have Strahler number $\leq k$ is NP-complete.