Colouring Probe H-Free Graphs

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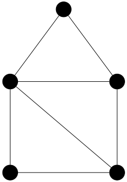
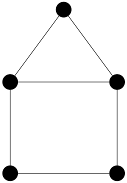
H-Free Graphs



C_4-free.



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Let H be a graph. A graph G is *H-free* if G does not contain an induced copy of H.

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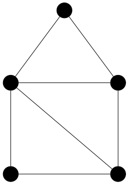

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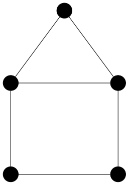

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Examples:

- C_t: cycle on t vertices.
- P_t: path on t vertices.
- $H_1 + H_2$: disjoint union of H_1 and H_2.

Colouring H-Free Graphs

COLOURING: minimum number of colours to colour a graph.

Theorem (Král et al. 2001)

On H-free graphs, COLOURING *is polynomial-time solvable if $H \subseteq P_4$ or $H \subseteq P_3 + P_1$ and NP-hard otherwise.*

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- quasi-polynomial on P_t-free graphs for $t \geq 8$ (Pilipczuk et al. 2021).

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Model for incomplete adjacency information: know only neighbourhoods of *probes*.

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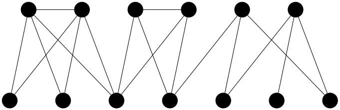

A probe P_5-free graph.

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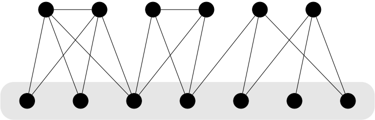

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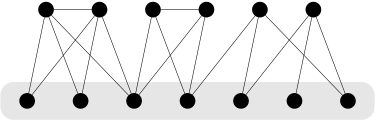

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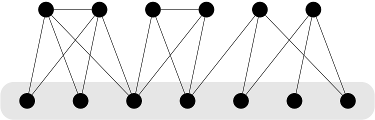

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A *partitioned* probe H-free graph (G, P, N):

- a graph G,
- a set $P \subseteq V(G)$,
- an independent set $N = V(G) \setminus P$,

such that $G + F$ is H-free for some $F \subseteq \binom{N}{2}$.

Optimisation Problems on Probe *H*-Free Graphs

Existing studies:

- VERTEX COVER (Brettell et al. 2025)
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	P_5-free	probe P_5-free
Vertex Cover	poly (Lokshtanov et al. 2014)	?
Matching Cut	poly (Feghali 2023)	NP-complete (Dabrowski et al. 2025)

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Our Research Question

What is the complexity of Colouring and k-Colouring on probe H-free graphs? Particularly when $H = P_t$ for some t.

Our Results

Theorem

For a graph H, COLOURING is polynomial-time solvable for probe H-free graphs if $H \subseteq_i P_4$, and else it is NP-hard even for partitioned probe H-free graphs.

Contrast: COLOURING is poly-time solvable on $3P_1$-free graphs (Král et al. 2001).

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For an integer $t \geq 1$, 3-COLOURING on partitioned probe P_t-free graphs is polynomial-time solvable if $t \leq 5$ and NP-complete if $t \geq 6$.

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3-COLOURING on partitioned probe P_5-free graphs is polynomial-time solvable.

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Main Result

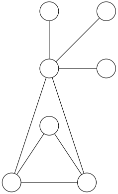
3-Colouring on partitioned probe P_5-free graphs is polynomial-time solvable.

- First problem for which polynomial-time solvability on P_5-free graphs is lifted to probe P_5-free graphs.
- Known algorithms for 3-Colouring on P_5-free graphs (Brettell et al. 2022 / Randerath et al. 2002 / Woeginger & Sgall 2001) do not directly generalize.

Subroutines for 3-COLOURING on Probe P_5-Free Graphs

Two important subroutines to the algorithm

Partial colour propagation



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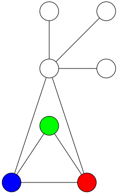

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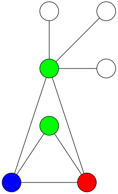

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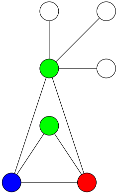
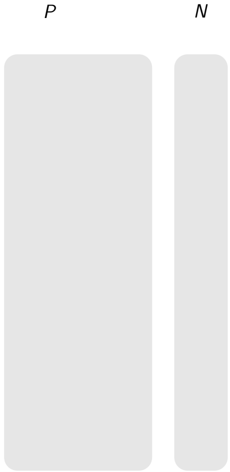


Pre-colouring extension

Theorem (Edwards 1986)

If, in a partial colouring of G, every vertex neighbours a pre-coloured vertex, then we can decide in polynomial time whether G admits a 3-colouring extending the partial colouring.

3-Colouring on Probe P_5-Free Graphs



- Recall: N is an independent set in G.

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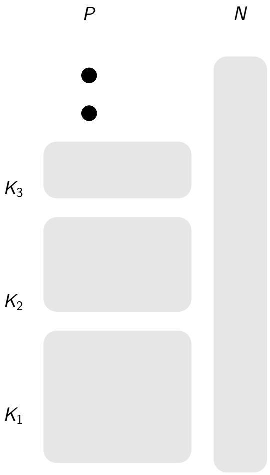

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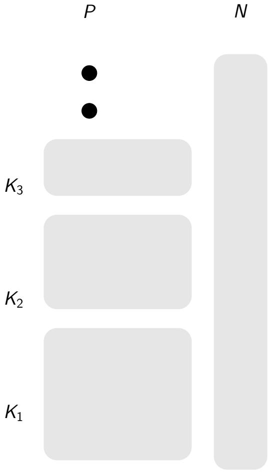

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- $K_1, \ldots, K_t$ bipartite: 3-colouring exists.

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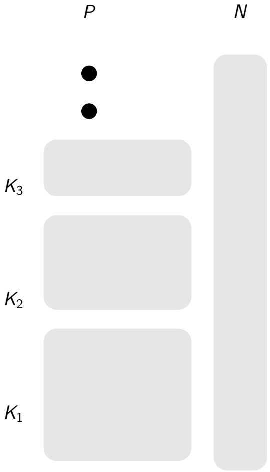

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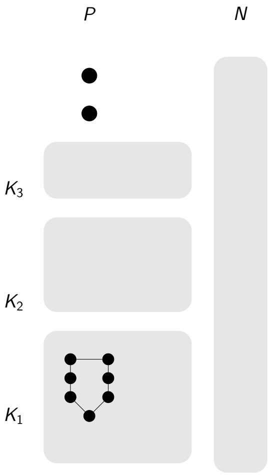

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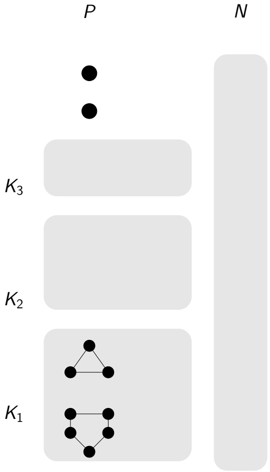

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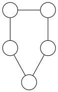
- $K_1, \ldots, K_t$: non-trivial comps of $G[P]$.
- $K_1, \ldots, K_t$ bipartite: 3-colouring exists.
- Assume K_1 is not bipartite.
- K_1 contains a C_3 or a C_5.

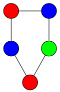
K_1 Contains a C_5

- $G[P]$ contains an induced 5-cycle C.



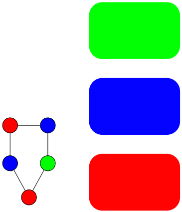
K_1 Contains a C_5

- $G[P]$ contains an induced 5-cycle C.
- Branch on all valid 3-colourings of C.



K_1 Contains a C_5

- $G[P]$ contains an induced 5-cycle C.
- Branch on all valid 3-colourings of C.
- For each branch, propagate the colours.



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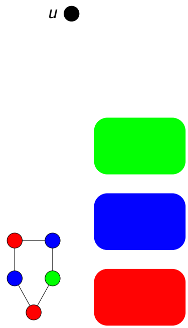

- $G[P]$ contains an induced 5-cycle C.
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Lemma

After propogation, every vertex u that is still uncoloured has a coloured neighbour.

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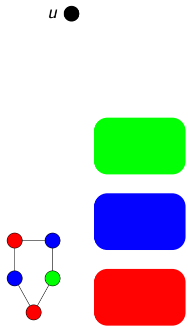

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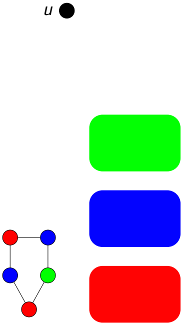

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Proof

- Suppose u has no coloured neighbours.
- Let $F \subseteq \binom{N}{2}$ s.t. $G + F$ is P_5-free.
- Find a shortest u-C path Q in $G + F$.
- Let v be the vertex of Q preceeding C.

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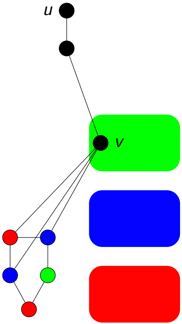

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- v is not complete to C.

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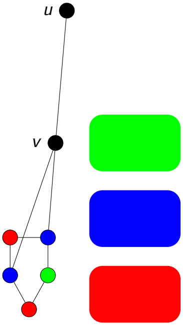

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- v sees only one colour.

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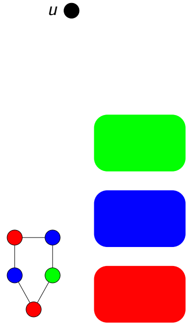

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- $G + F$ contains an induced P_5, contradiction.

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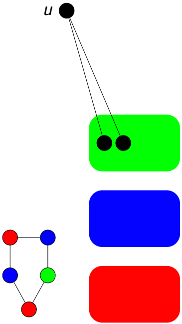

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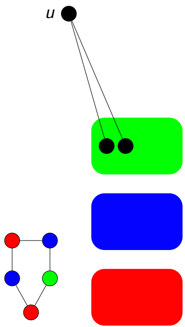

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Apply pre-colouring extension subroutine.

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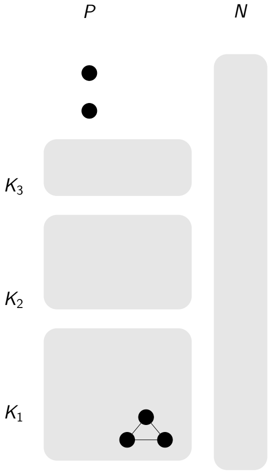

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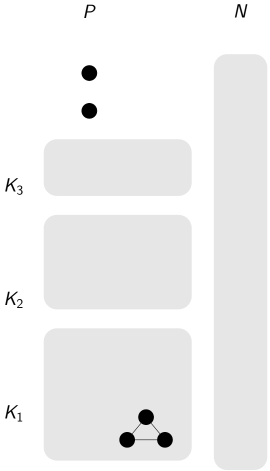
K_1 Contains a C_3



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Proposition

$K_2, \ldots, K_t$ are bipartite.

Proof.

□

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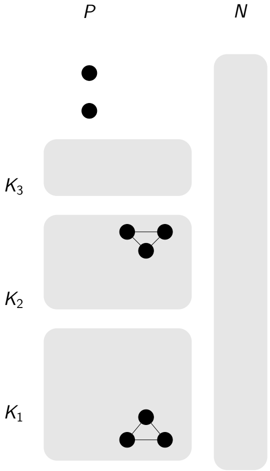

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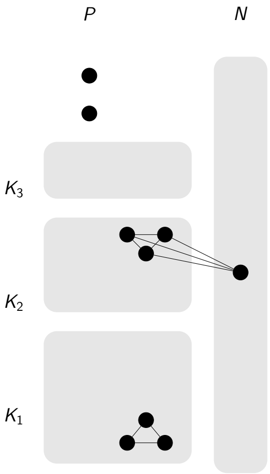
$K_2, \ldots, K_t$ are bipartite.

Proof.

- Suppose K_2 contains a C_3.

□

K_1 Contains a C_3



Recall:

- N is an independent set in G.
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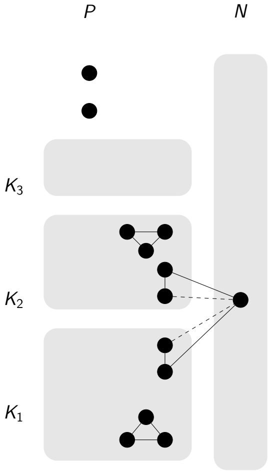

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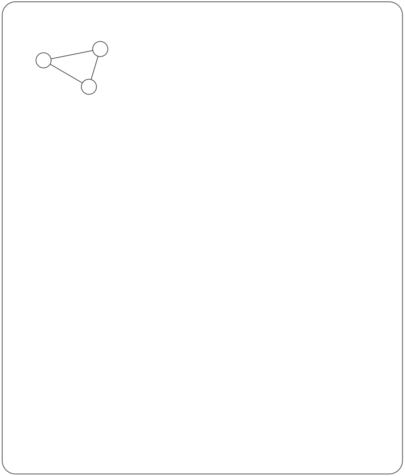

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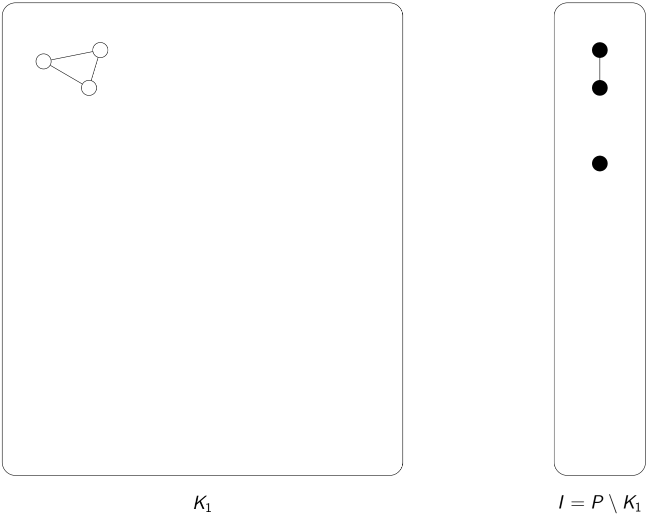

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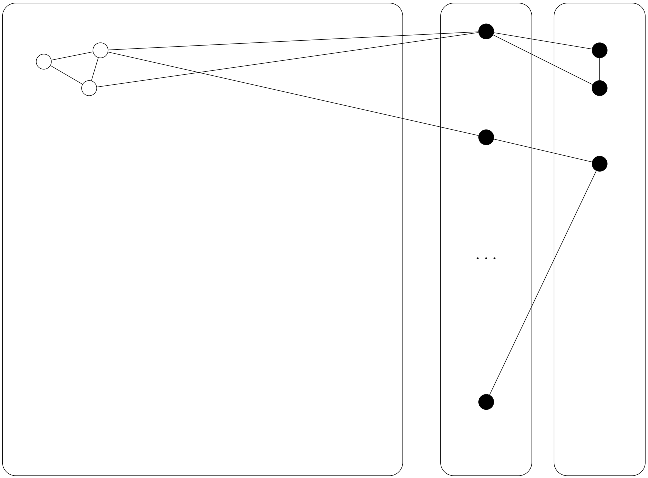

K_1 $M = N(I)$ $I = P \setminus K_1$

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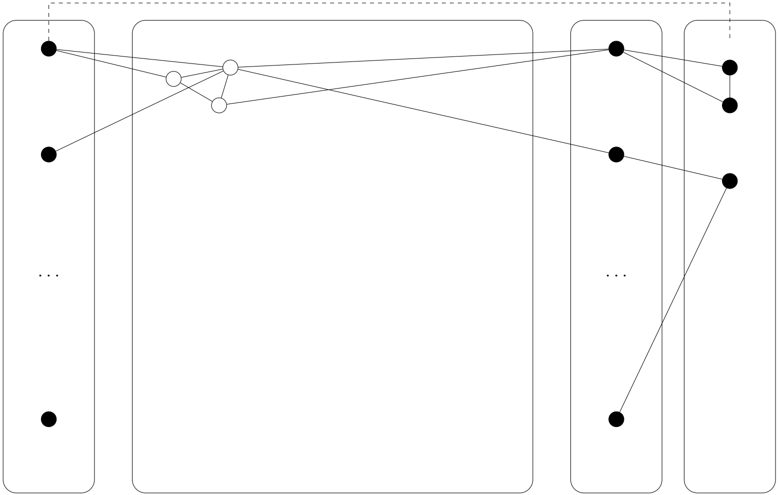

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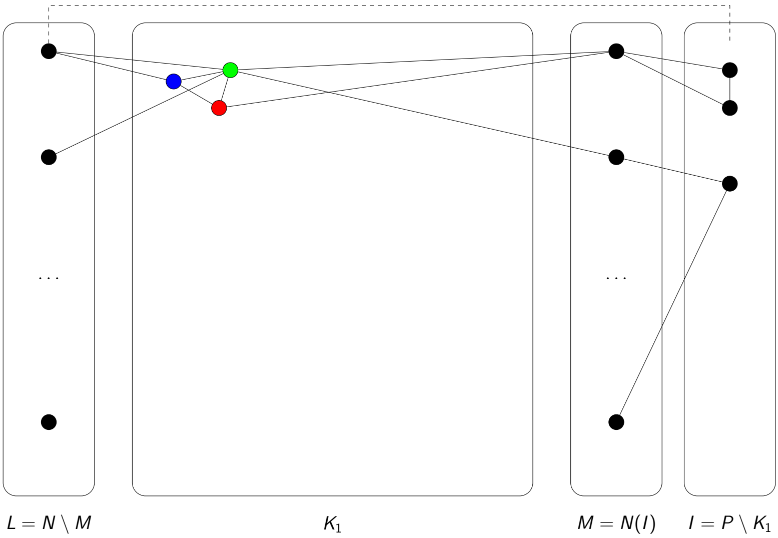

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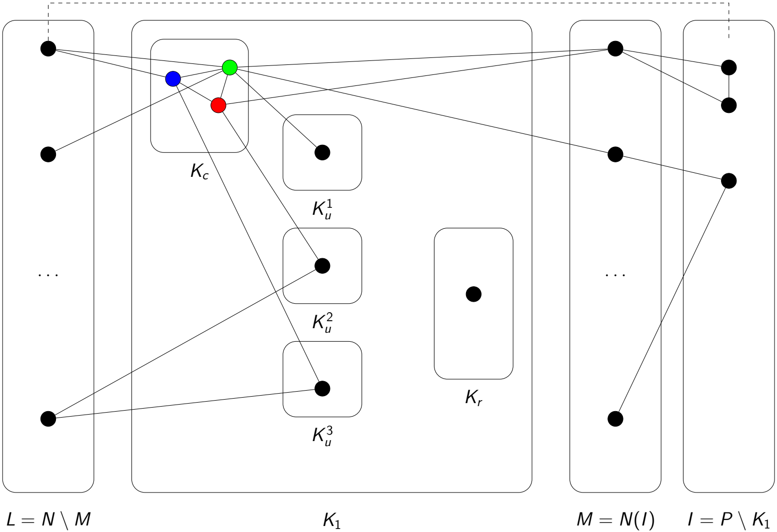

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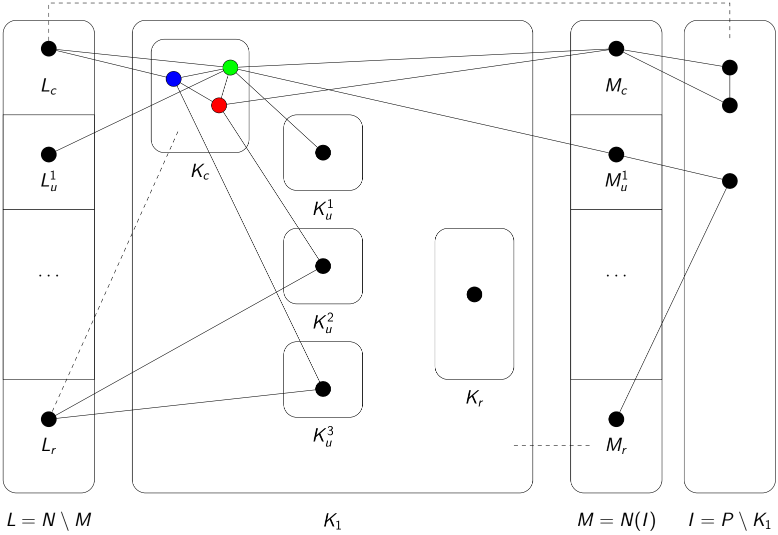
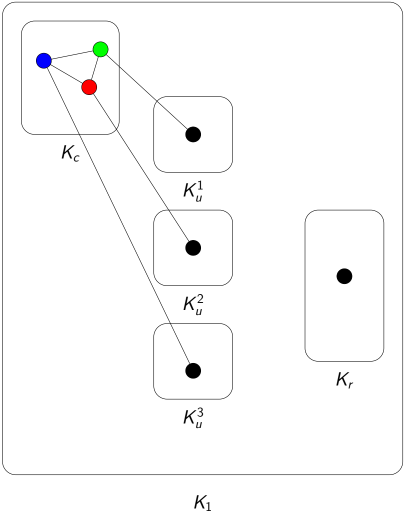


A Single Vertex Dominates K_r



- $G[K_1]$ is a connected P_5-free graph.
- It has dominating clique or C_5.

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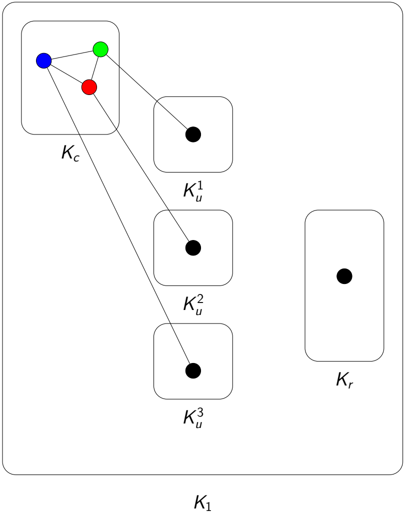

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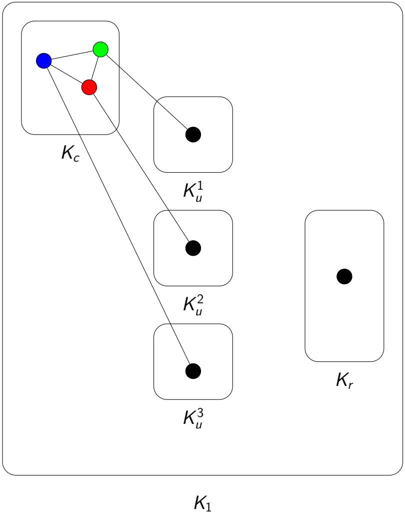

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- $|D| \leq 3$, or no 3-colouring.

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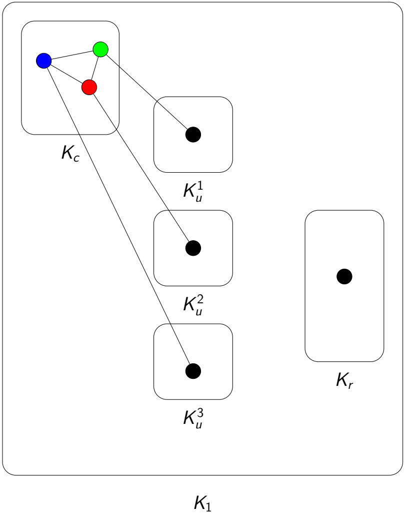

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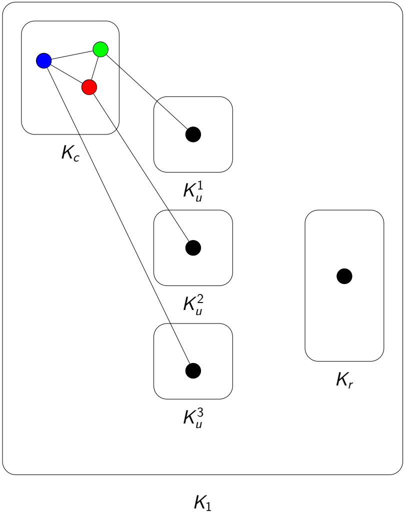

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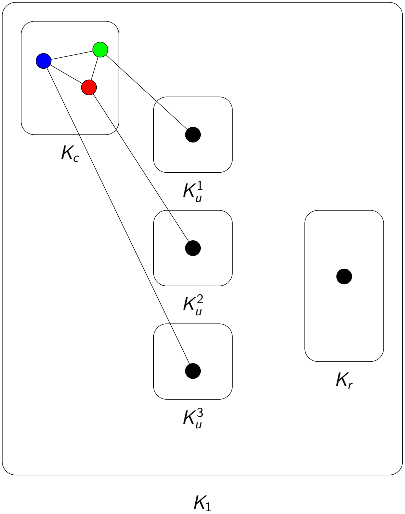

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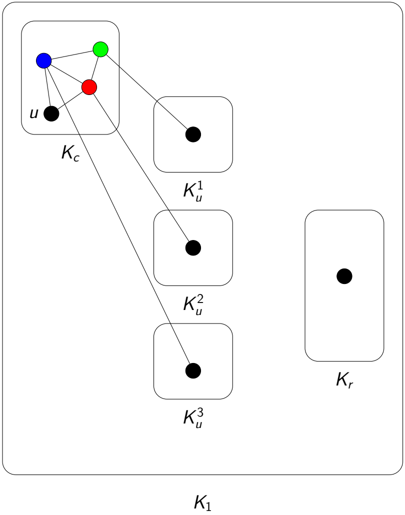

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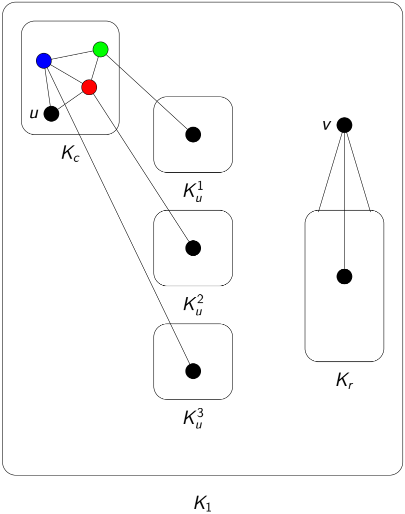

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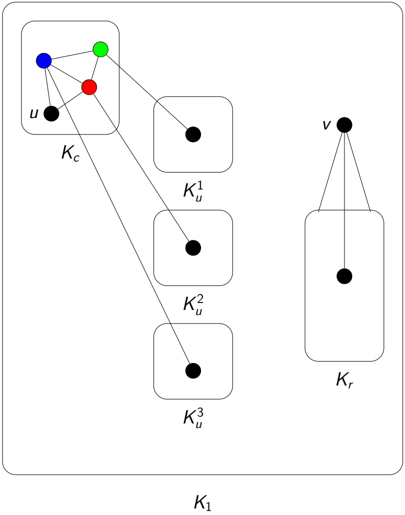

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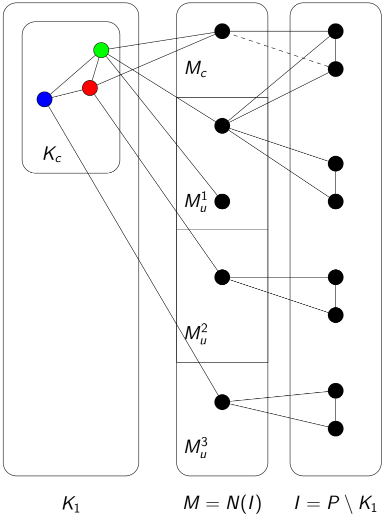
Dealing with I



- Separate argument: wlog $M_r = \emptyset$.

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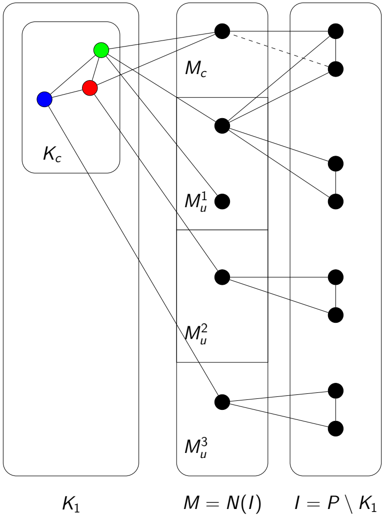

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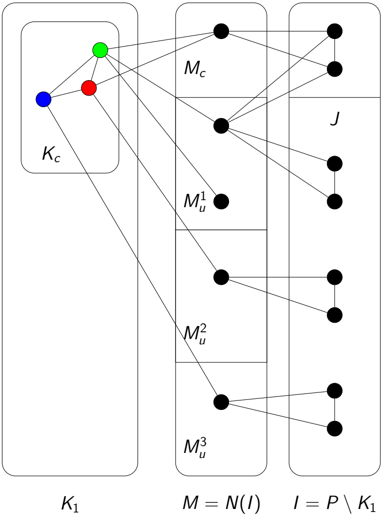

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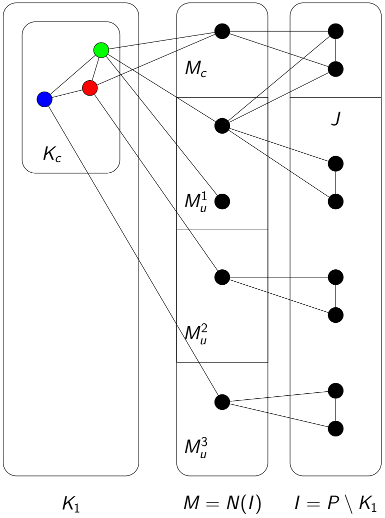

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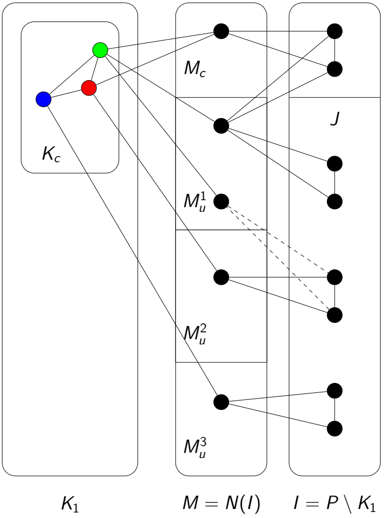

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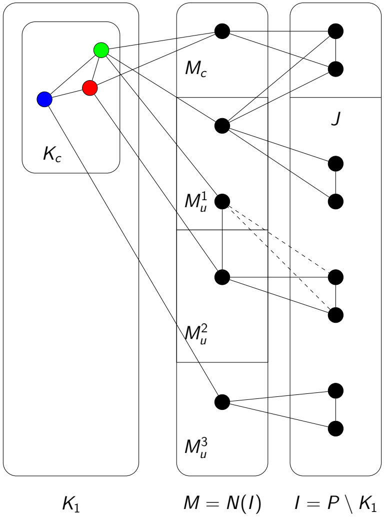

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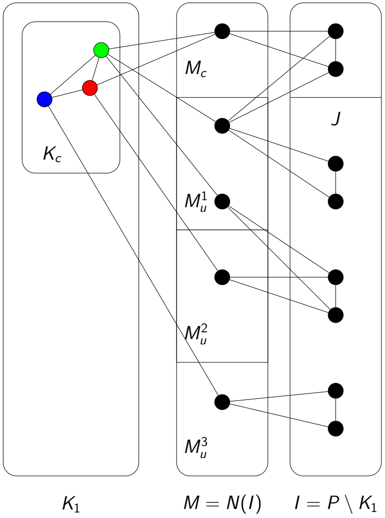

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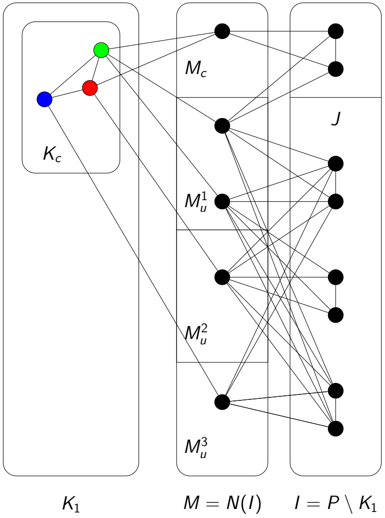

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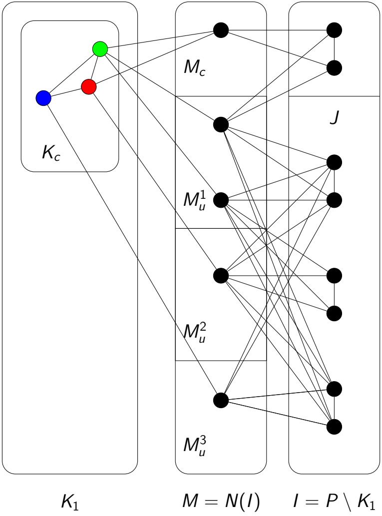

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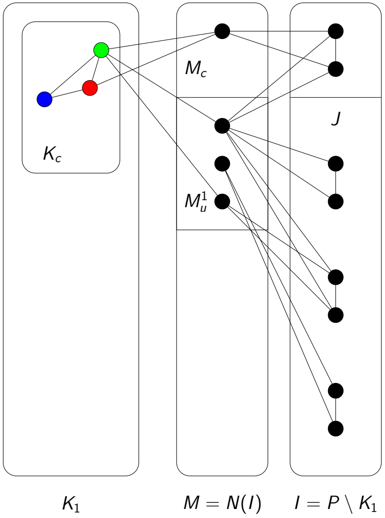

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  - In each $K' \in I$, some partite set gets colour 1.
  - $N(K')$ must receive same colour.
  - Add these constraints to 2-SAT, solve for $G - J$, and extend this colouring to J.

Discussion

Open Problems

- 3-COLOURING on probe H-free graphs when H is a linear forest. Five open cases, e.g. $H = P_5 + sP_1$, $P_4 + P_2 + sP_1$. We show $P_3 + sP_1$ is polynomial.

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- For a graph H, Colouring is polynomial-time solvable for probe H-free graphs if $H \subseteq_i P_4$; else, is NP-hard even for partitioned probe H-free graphs.
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- **Main Result:** 3-Colouring on partitioned probe P_5-free graphs is polynomial-time solvable.

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More open problems and proofs: arXiv:2505.20784 and STACS proceedings.