Approximate Cartesian Tree Matching with Substitutions

Panagiotis Charalampopoulos, *Jonas Ellert*, Manal Mohamed

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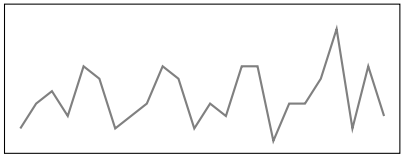

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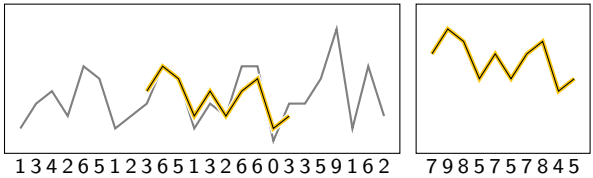

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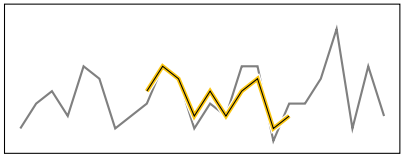

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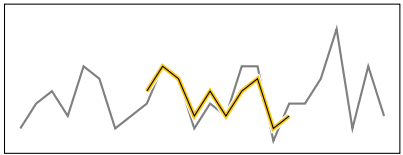

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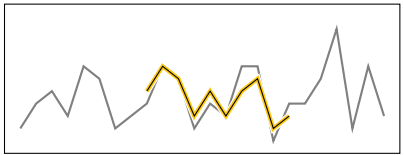

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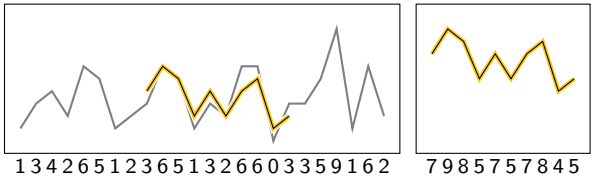

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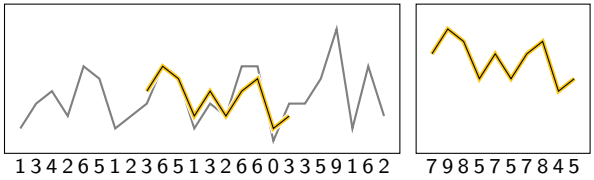

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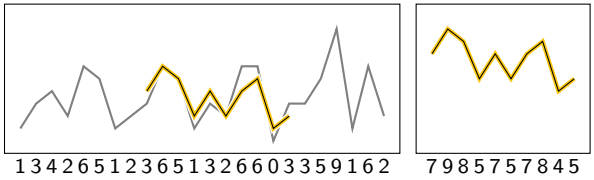

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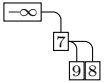

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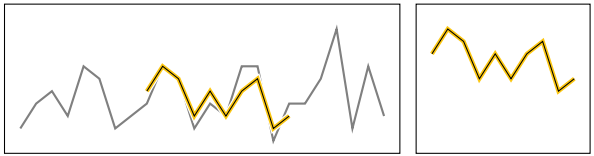

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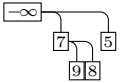

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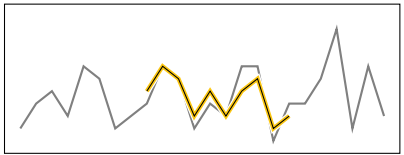

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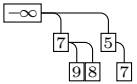

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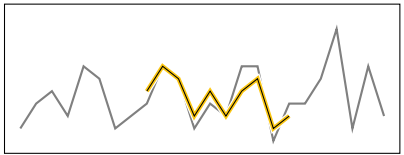

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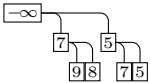

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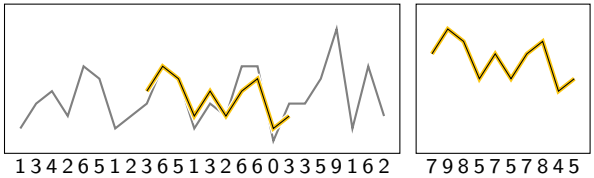

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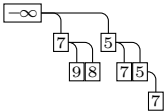

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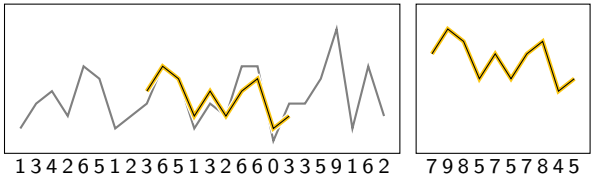

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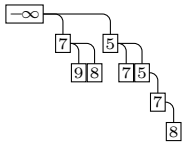

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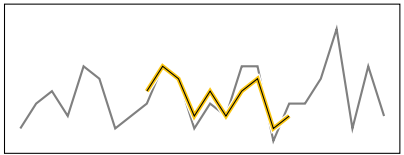

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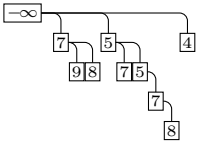

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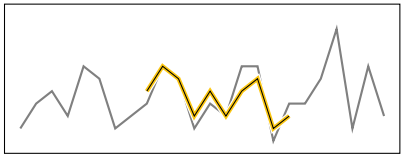

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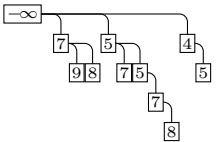

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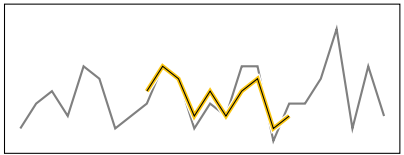

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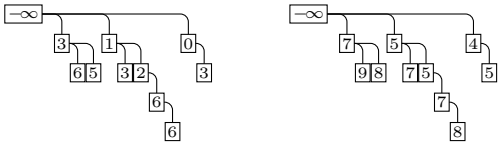

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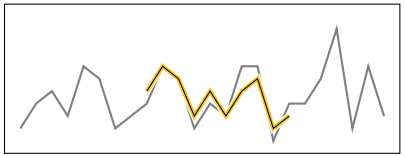

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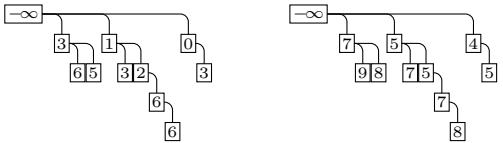

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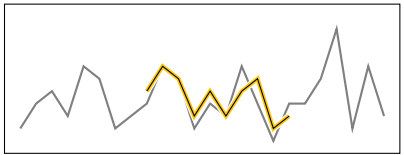

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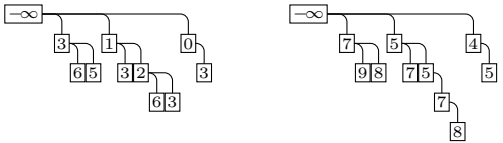

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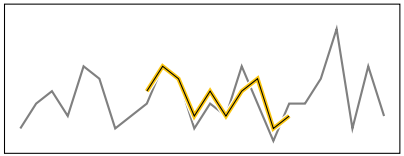

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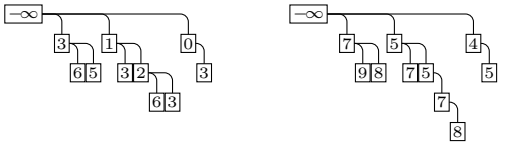

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solution: allow k substitutions in text *before* computing tree

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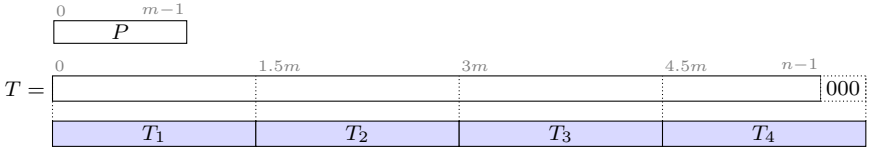

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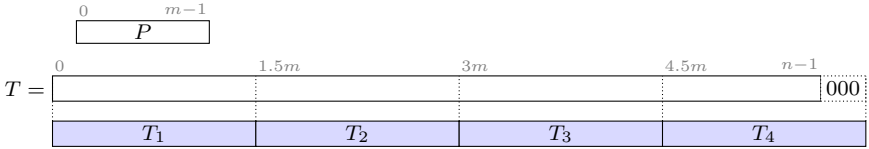

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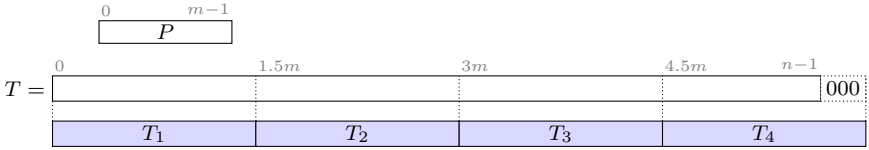

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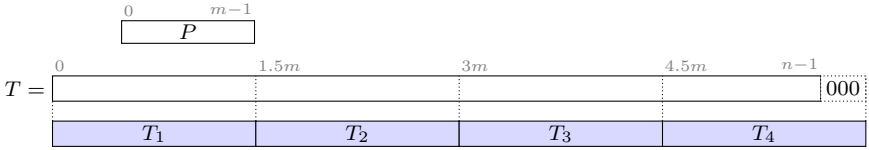

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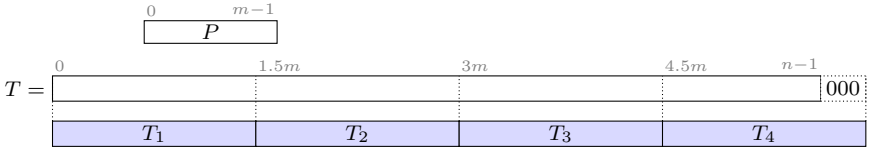

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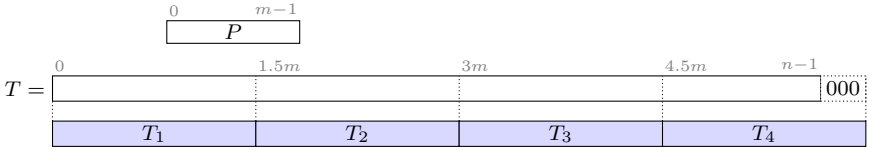

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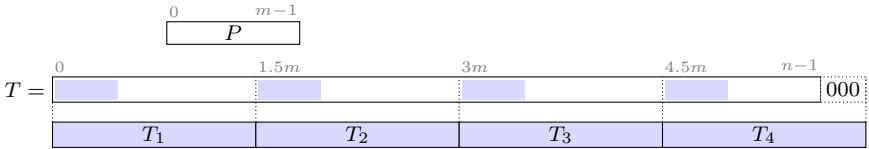

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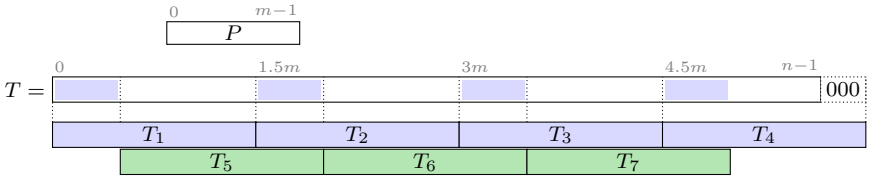

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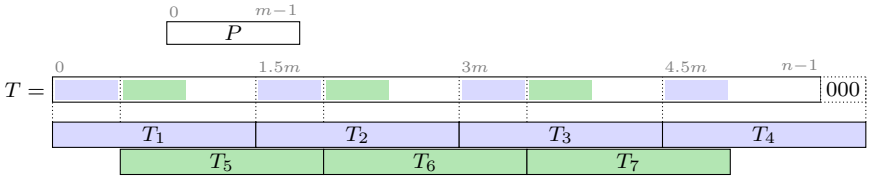

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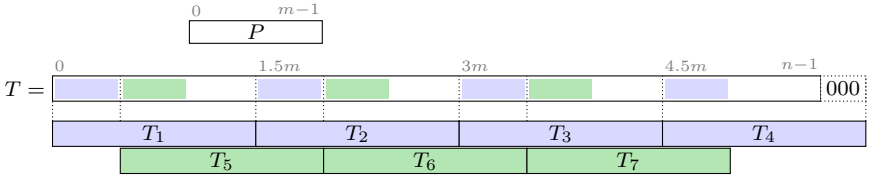

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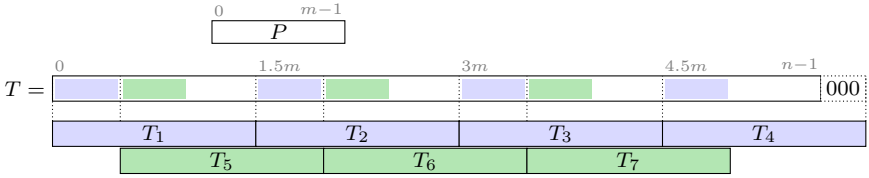

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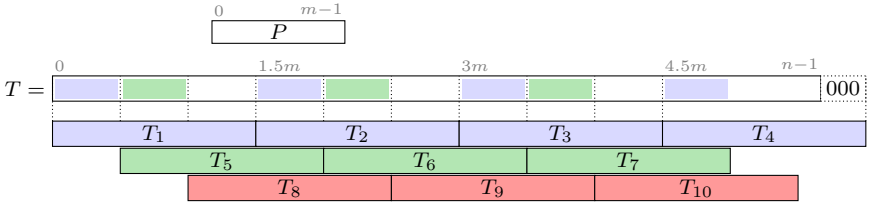

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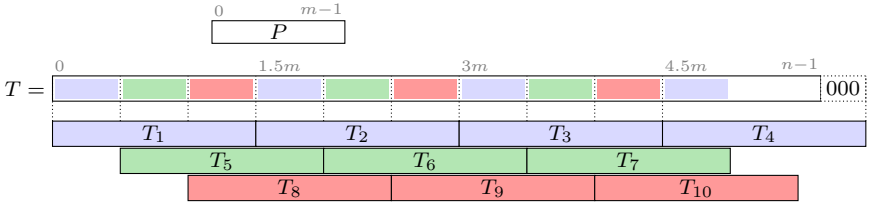

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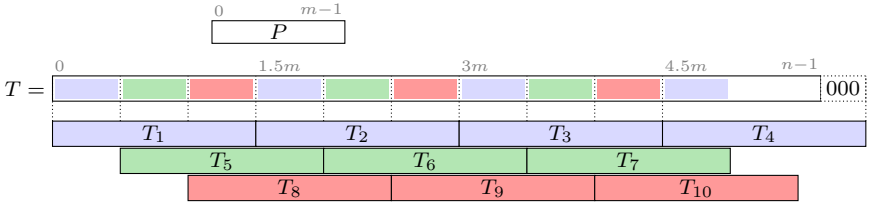

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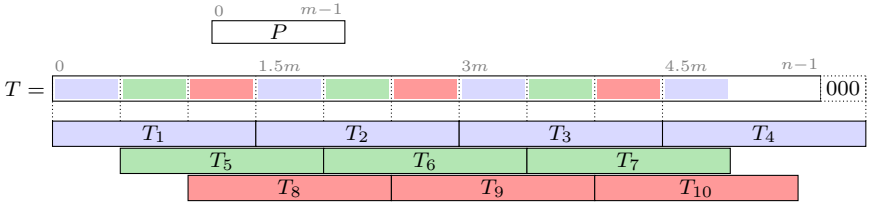

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- text $T[1\,..\,1.5m]$, pattern $P[1\,..\,m]$, threshold k
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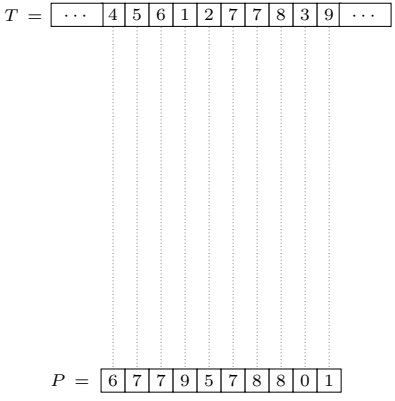

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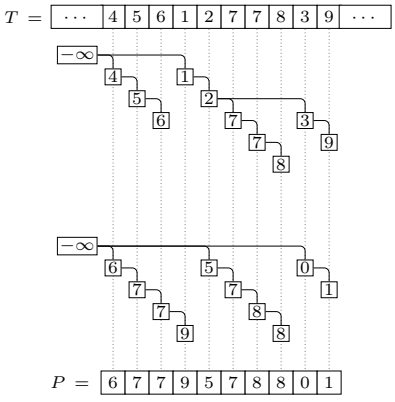

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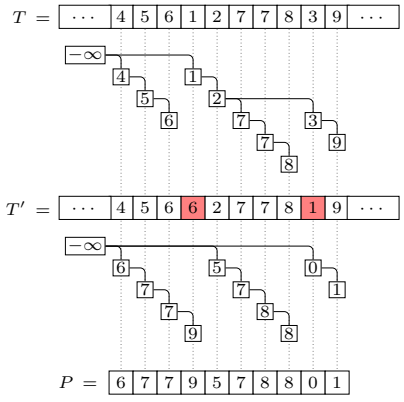

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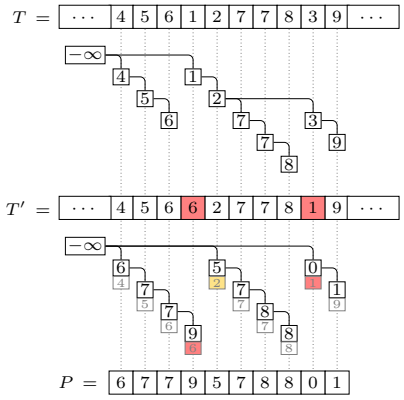

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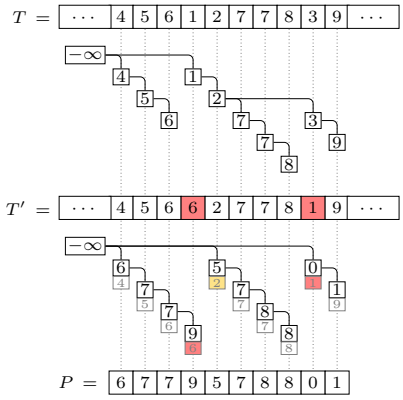

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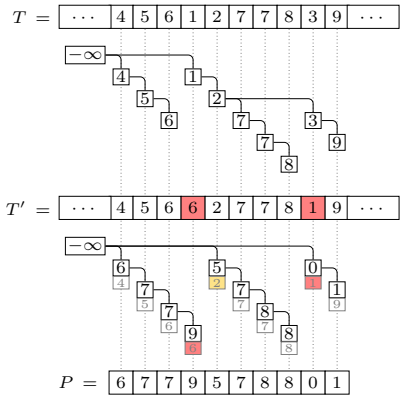

Some other flavors of this kind of pattern matching

Equality matching (with k mismatches)

- not very well suited for finding trends in series data

[Gawrychowski+Uznanski '18; Kim+Han '25; Gawrychowski+Uznanski '16; Labib et al. '19; Boneh et al. '24]

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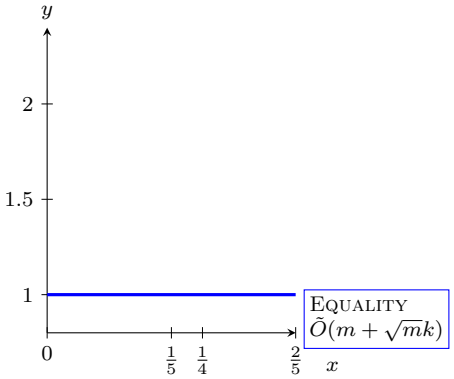

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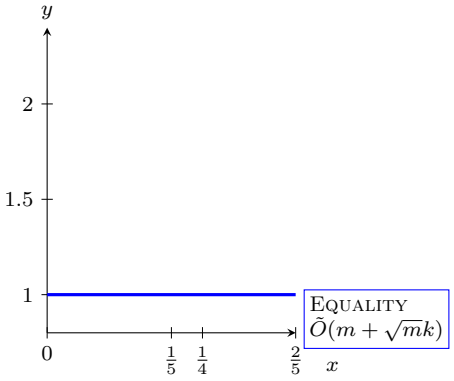

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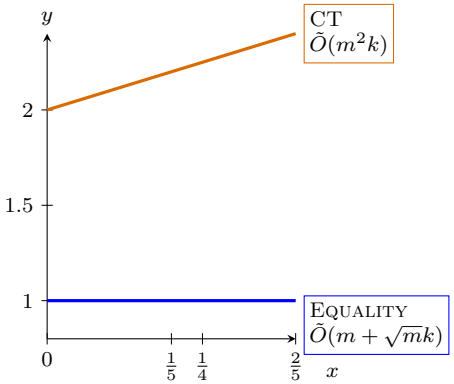

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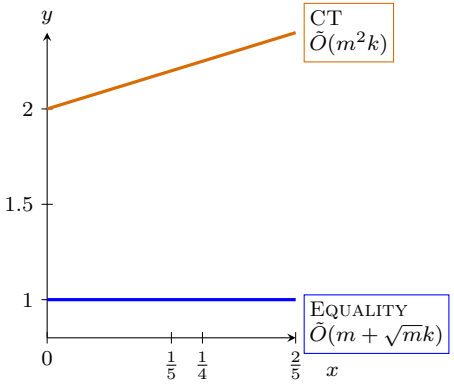

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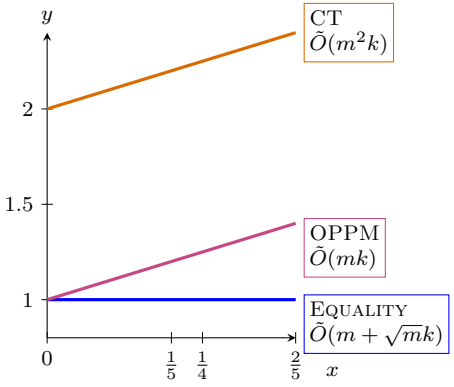

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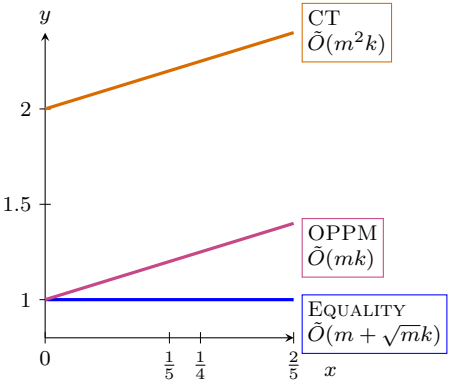

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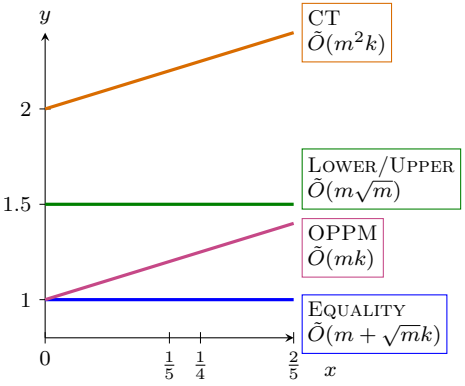

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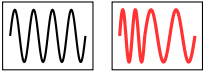
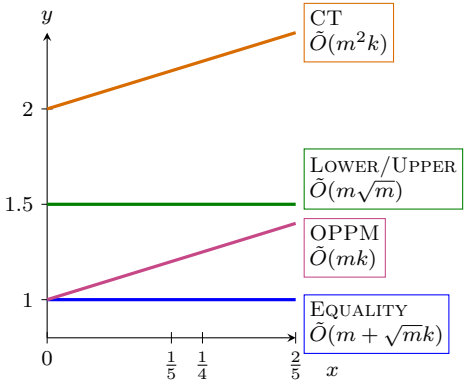
Dynamic time warping

- non-uniformly stretch and/or squeeze pattern to maximize similarity with text fragment





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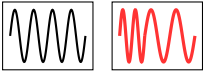

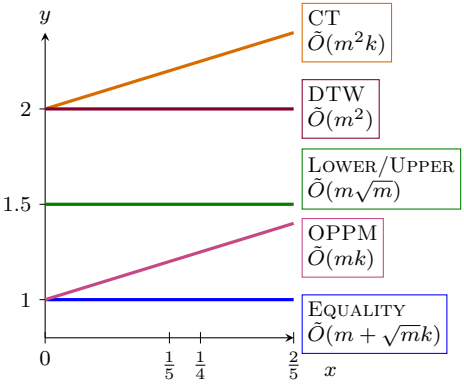

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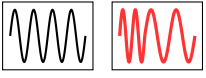

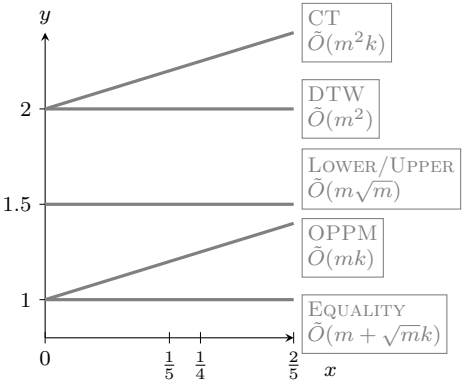

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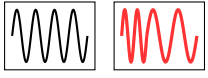

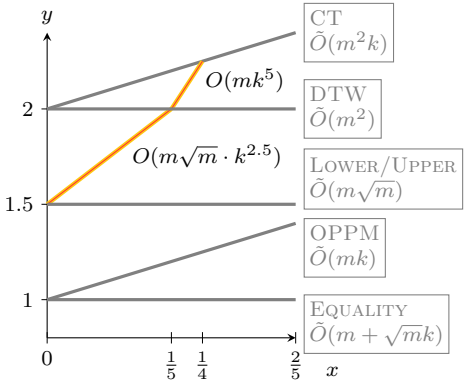

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... $\tilde{\mathcal{O}}(m + \sqrt{m}k)$ equality matching vs $\tilde{\mathcal{O}}(m\sqrt{m}k^{2.5})$ Cartesian tree matching. But why?

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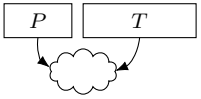

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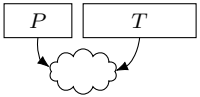

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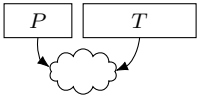

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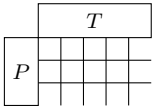

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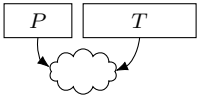

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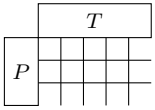

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- algorithms for equality matching heavily exploit periodicity

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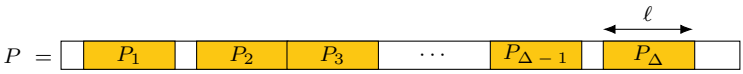

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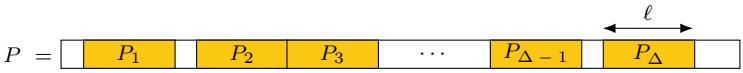

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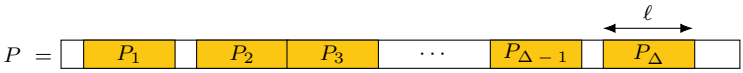

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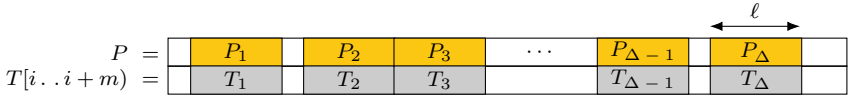

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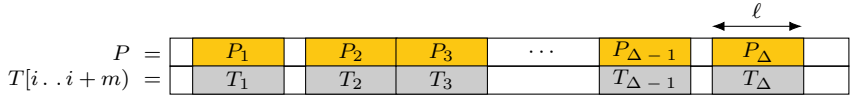

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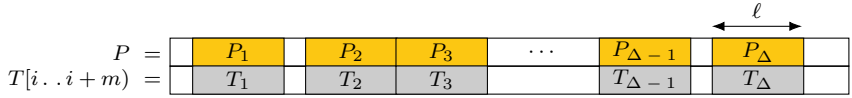

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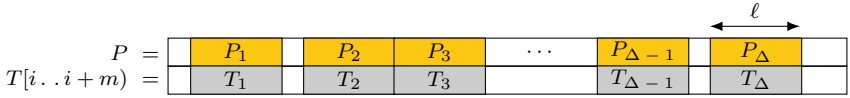

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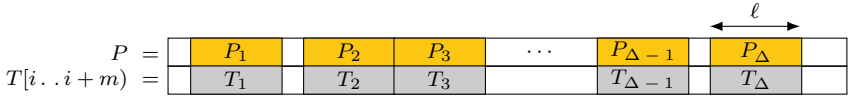

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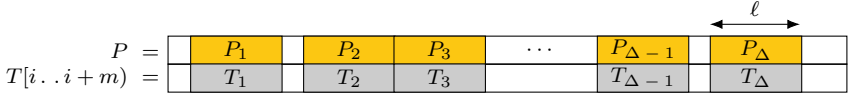

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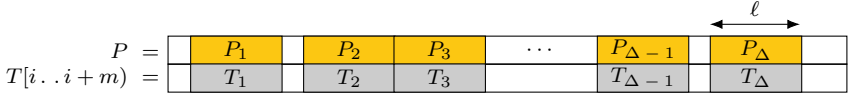

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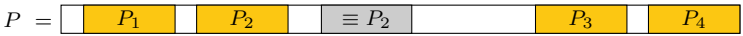

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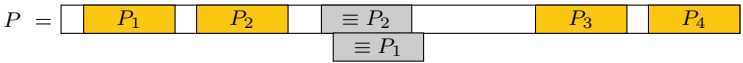

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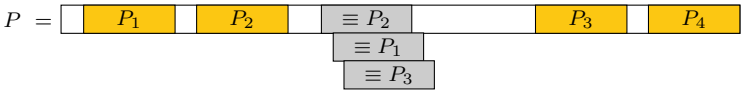

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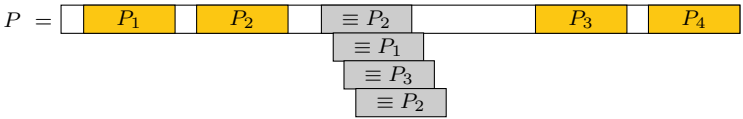

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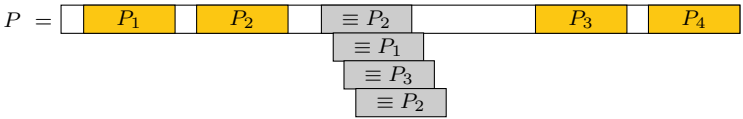

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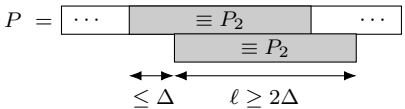

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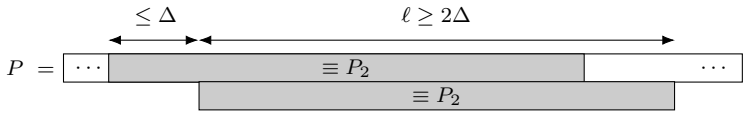

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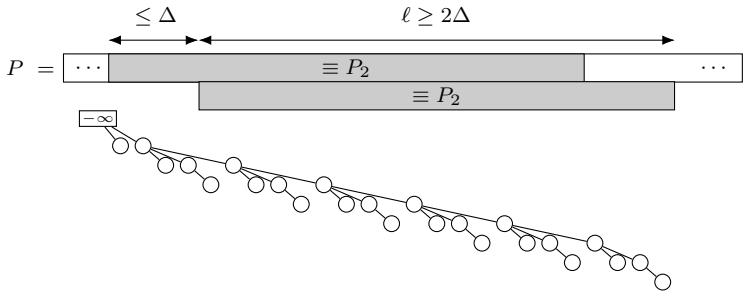

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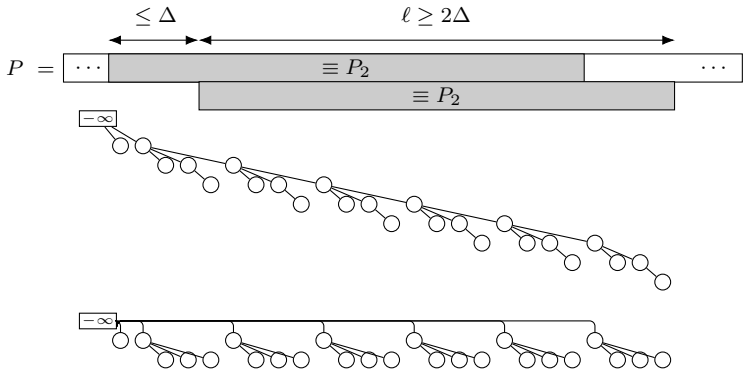

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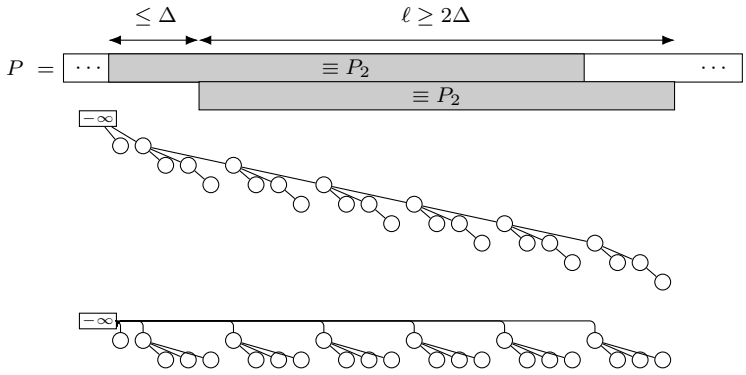

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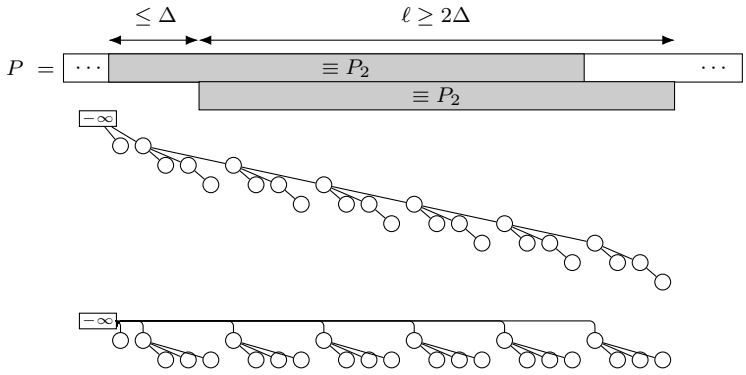

- highly periodic with respect to structure of CT
- adapt all the tricks for periodic fragments from equality pattern matching

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- CT/OPPM/DTW... how do they actually work on real data?

References


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