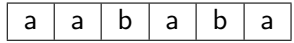
Algebraic Characterizations of Classes of Regular Languages in DynFO

Corentin Barloy, Felix Tschirbs, Nils Vortmeier, Thomas Zeume

Incremental maintenance and DynFO



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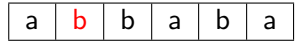

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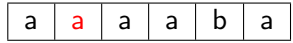

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→ We refine this with algebra!

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finite automaton $\approx$ finite monoid $(M, \cdot)$

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finite set → M

associative operation → $\cdot$

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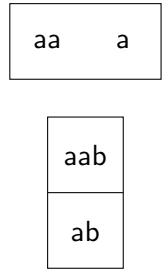
Previous proof of Reg $\subseteq$ DynFO: Maintain the evaluation of infixes in a monoid.

The regular languages of UDynΣ_2

Ideal structure

A division-based representation of the syntactic monoid of $(\mathtt{aa})^*\mathtt{b}(\mathtt{a}+\mathtt{b})^*$:



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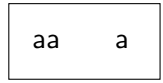

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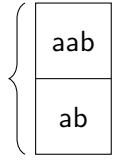

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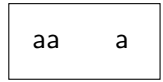

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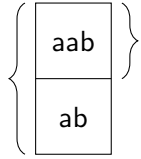

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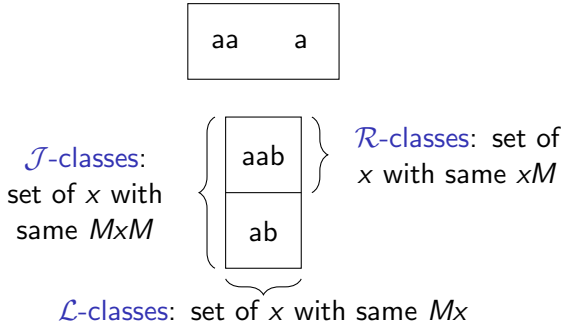

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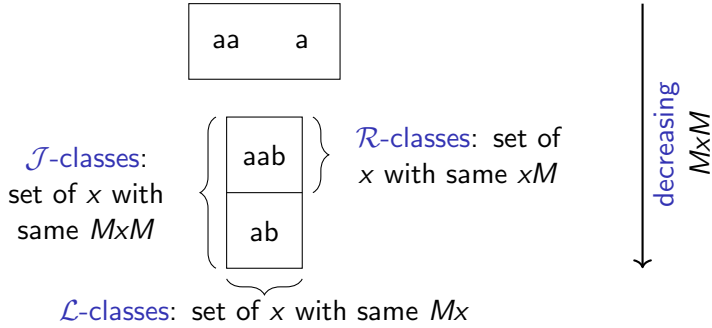

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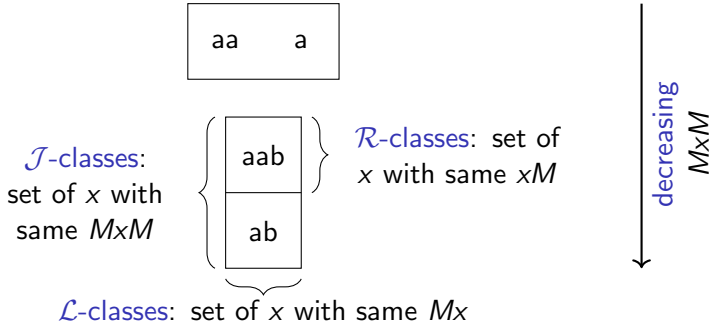

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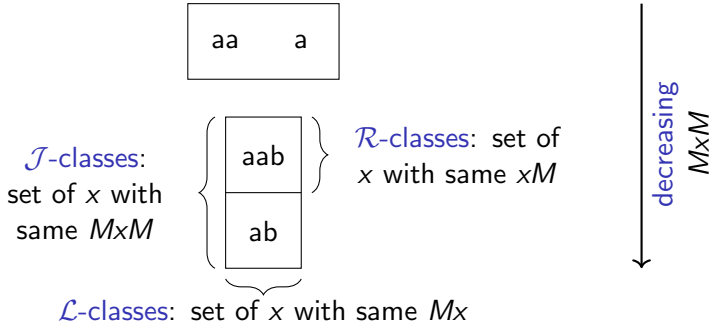

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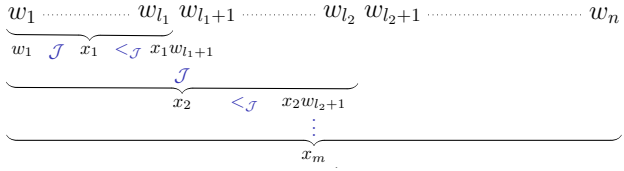

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$\exists i = l_1 < \cdots < l_m = j$,
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Updates of $R_{J,x}$ at i: there is an index j such that $w[i,j]$ evaluates to x and $w[i,j+1]$ is $> J$.

The regular languages of subclasses of UDynProp

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Usefull to define a class of languages by their syntactic monoids
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→ This language in not in UDynProp [Schwentick, Zeume 2015]

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$\rightarrow$ This language can be shown to not be in UDynΣ_1^+

Conclusion

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Recap:

Dynamic class	UDynProp	$\mathrm{UDyn}\Sigma_1^+$	$\mathrm{UDyn}\Sigma_2$
Regular languages	$\mathbf{G}$	$\mathbf{J}^+ * \mathbf{G}$	Reg