Advancements in Online Edge Coloring Algorithms

Ola Svensson, based on joint works with Joakim Blikstad, Radu Vintan, and David Wajc

Outline

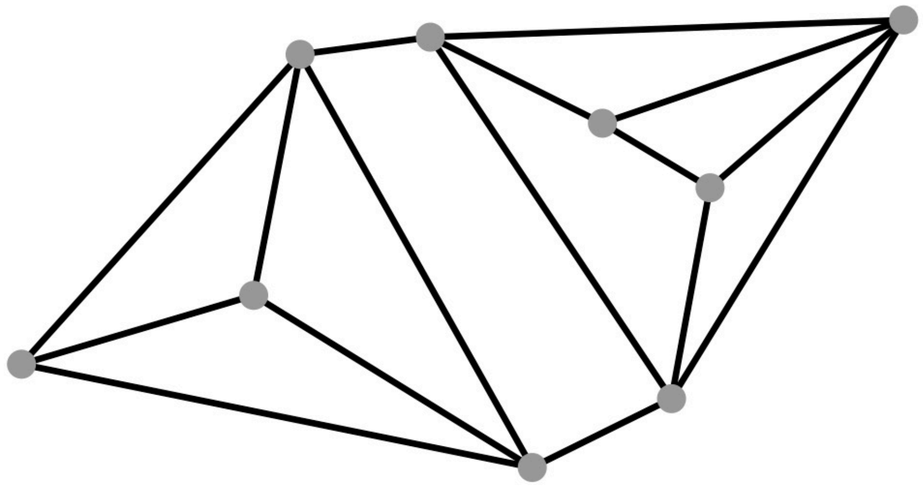
- Edge Coloring and Online Algorithms
- Greedy algorithm is optimal
- Improving upon Greedy
    - Fair Matchings
    - Embracing Correlations via Martingales

Edge Coloring

Given: Graph $G = (V, E)$

Goal: Color *edges* with few colors

Constraint: No two incident edges get the same color



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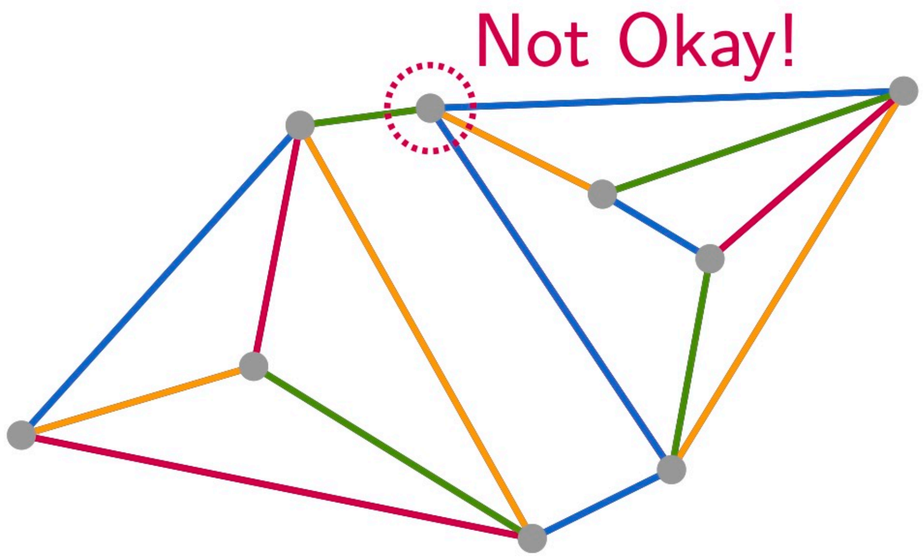

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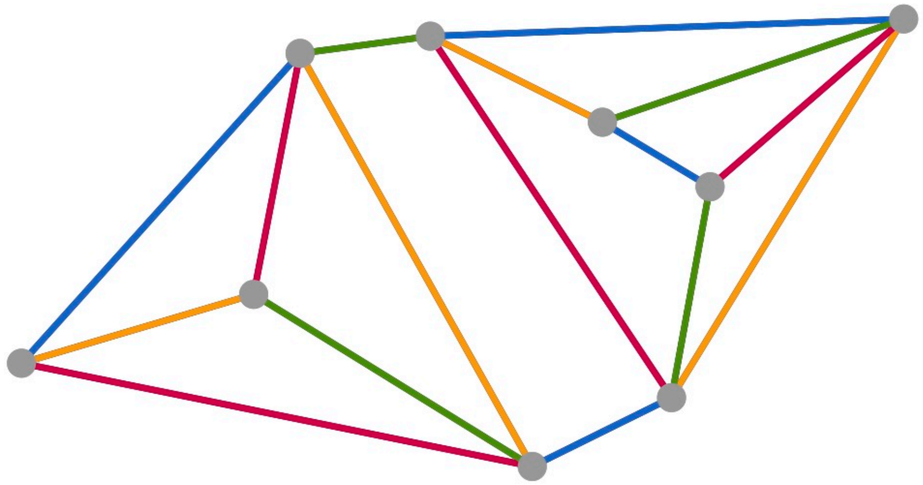

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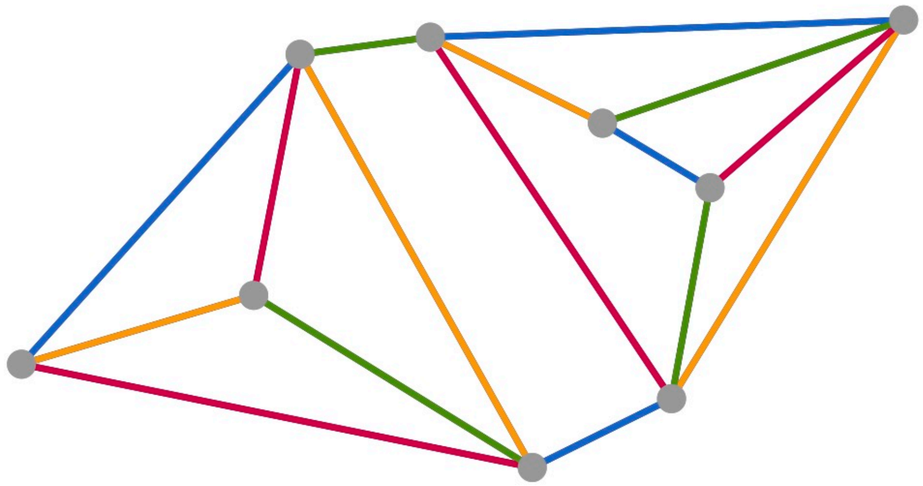

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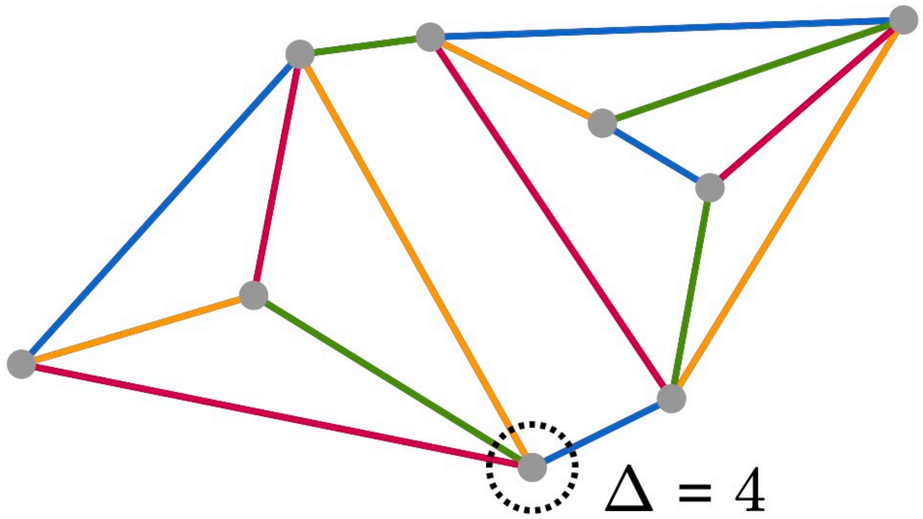

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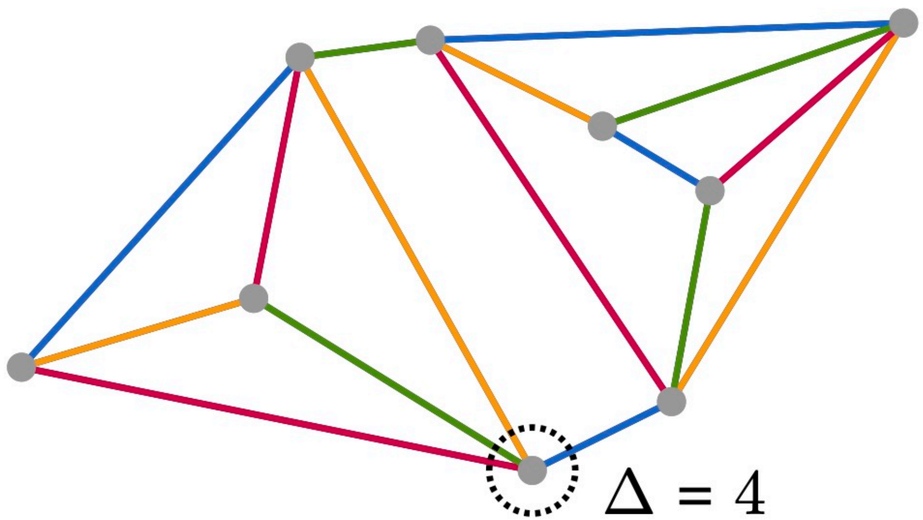

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Claim: #Colors ≥ Δ

Theorem: #Colors ≤ Δ + 1

[Vizing 1964]

Edge Coloring Algorithms

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- Many algorithms compute $(\Delta + 1)$-edge colorings, now near-linear time
  [Vizing'64, Gabow/Nishizeki/Kariv/Leven/Osmau'85,…, AssadiBehnezhadBhattacharyaCostaSolomonZhang'25]

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- Studied in various computational models

  Distributed [PanconesiSrinivasan'97,DubhashiGrablePanconessi'98,...]

  PRAM [LevPippengerValiant'81,...]

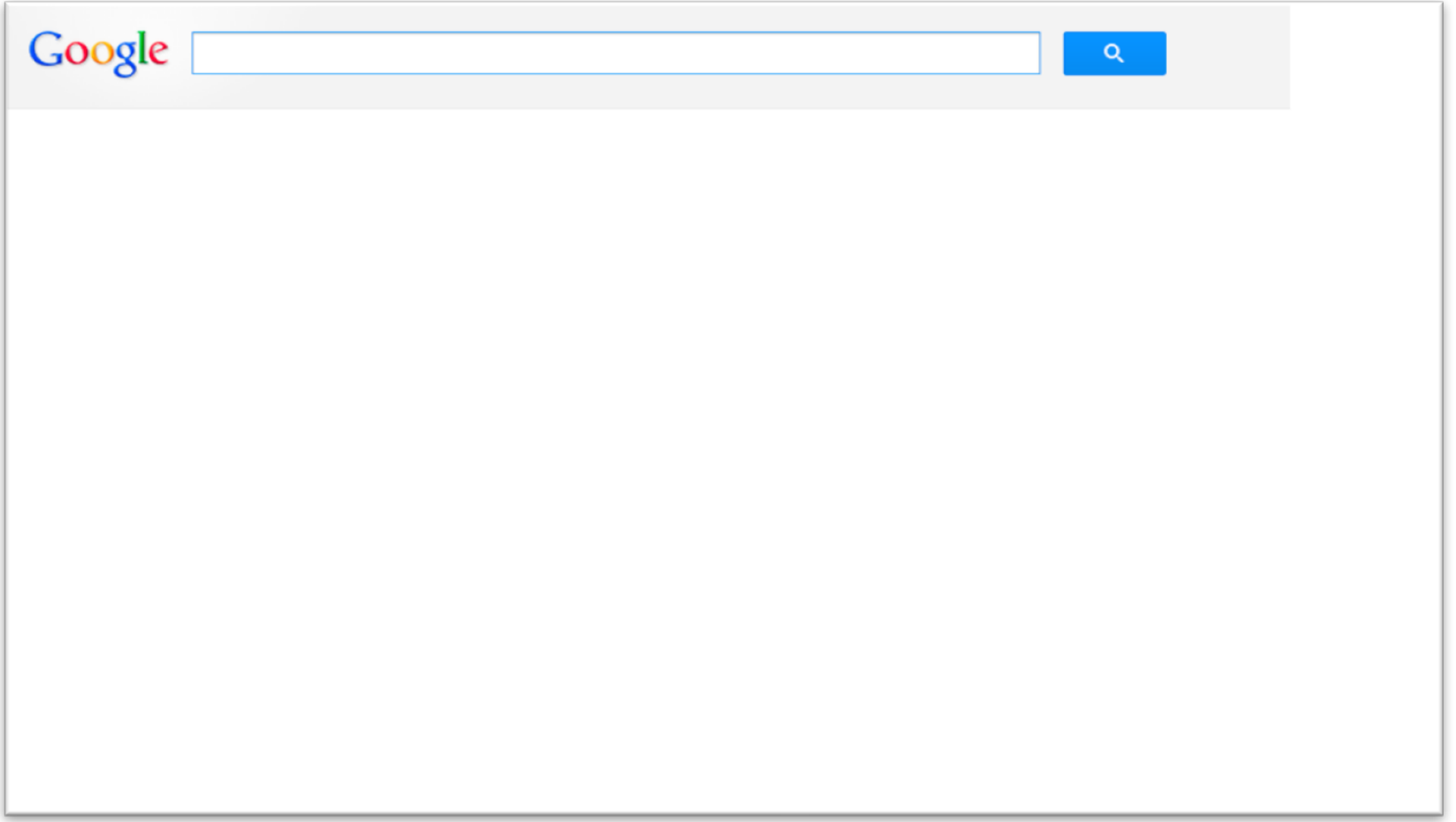
  NC & RNC [KarloffShmoys'87, MotwaniNaorNaor'94,...]

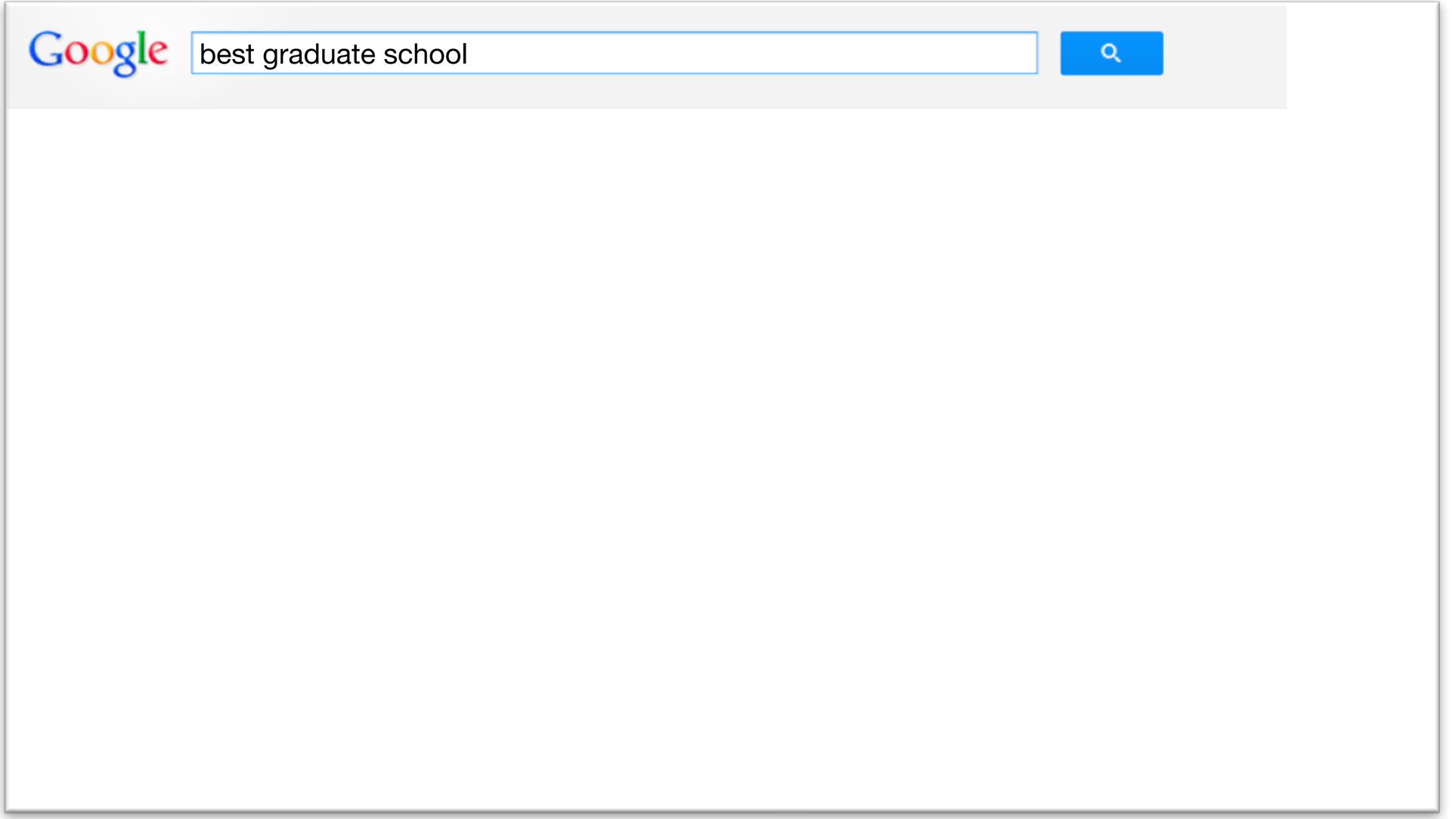
  Dynamic [Bhattacharya/Chakrabarty/Henzinger/Nanongkai'18,...]

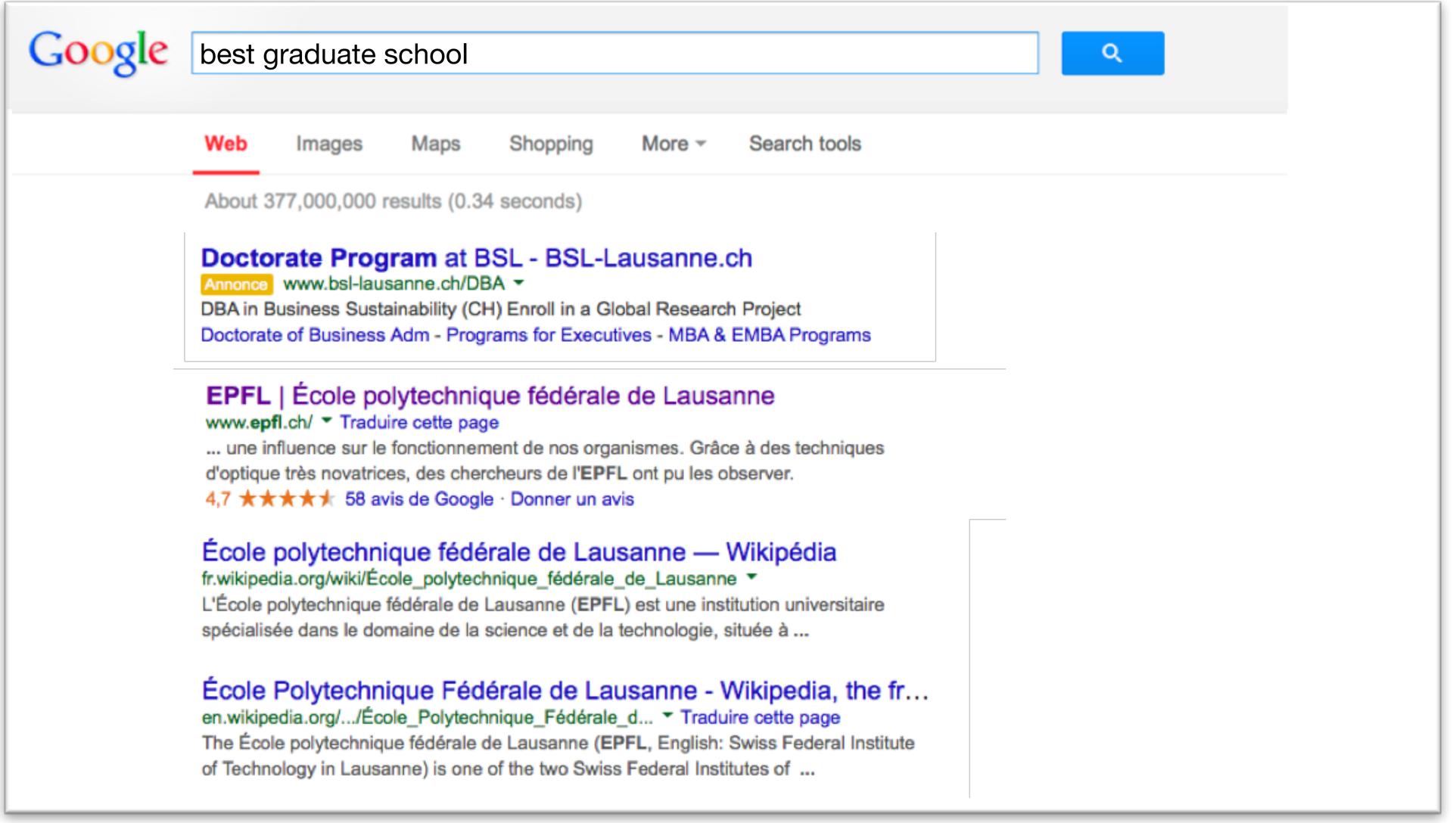
  …

Rent or buy?

Today: Online Algorithms


Google


Google
best graduate school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best graduate school
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... une influence sur le fonctionnement de nos organismes. Grâce à des techniques d'optique très novatrices, des chercheurs de l'EPFL ont pu les observer.
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L'École polytechnique fédérale de Lausanne (EPFL) est une institution universitaire spécialisée dans le domaine de la science et de la technologie, située à ...
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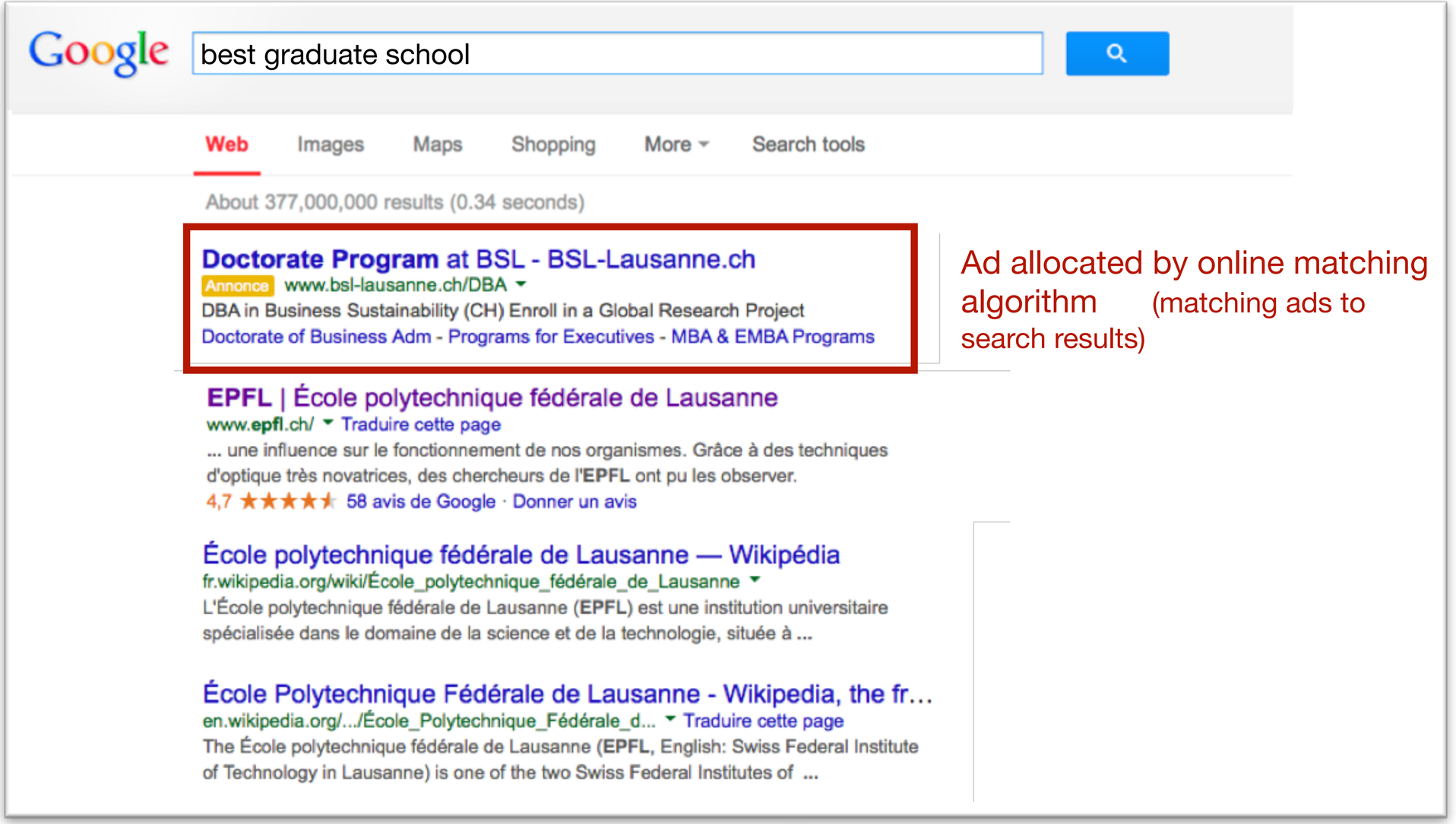

Ad allocated by online matching algorithm (matching ads to search results)

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Online: Graph revealed over time. Max degree Δ known

Task: Color edge irrevocably when revealed

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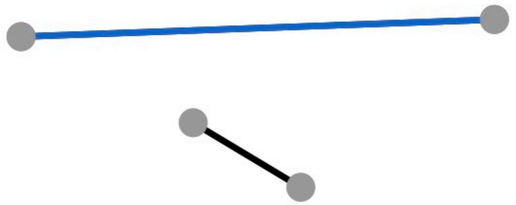

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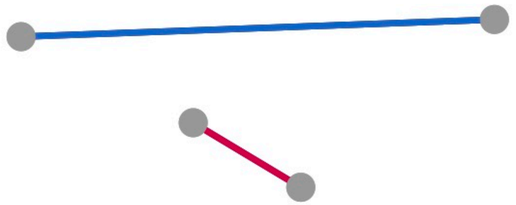

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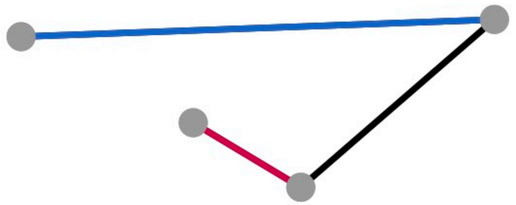

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Variant:

Edge arrivals

Adversarial order

General graphs

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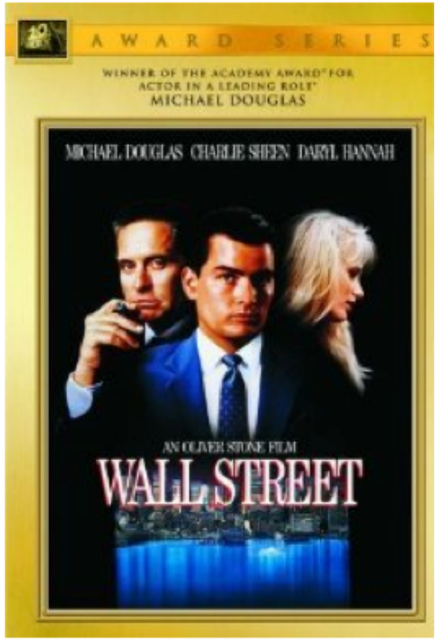
How many colors do we need? Still $\approx \Delta$?



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Warm-up: Greedy Algorithm

Greedy: Color edge with “lowest” available color

Colors = {1, 2, 3, …}

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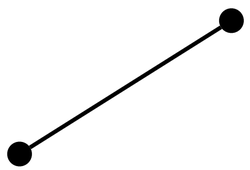

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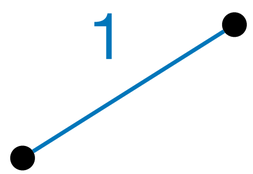

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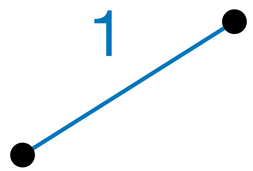

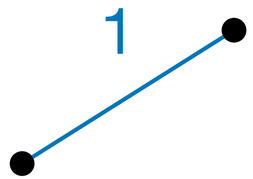

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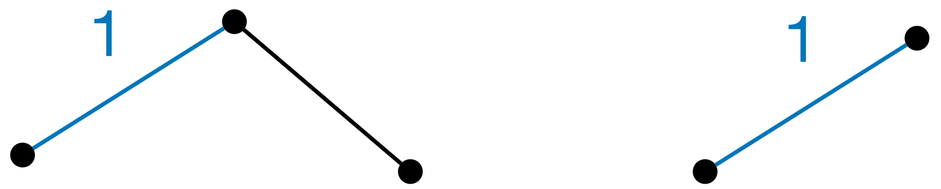

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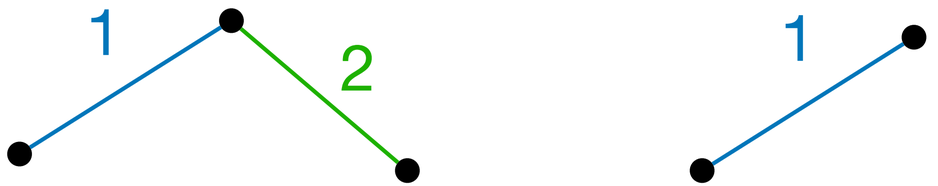

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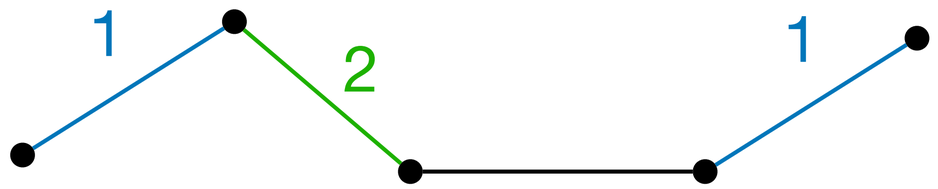

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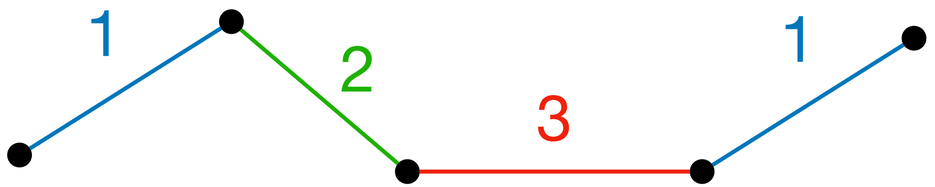

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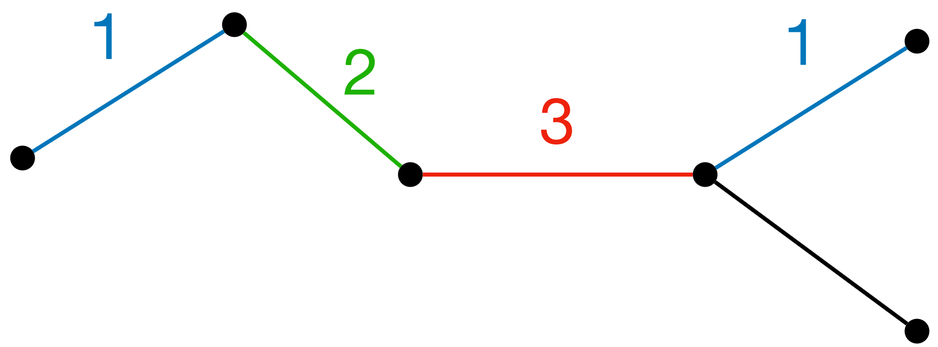

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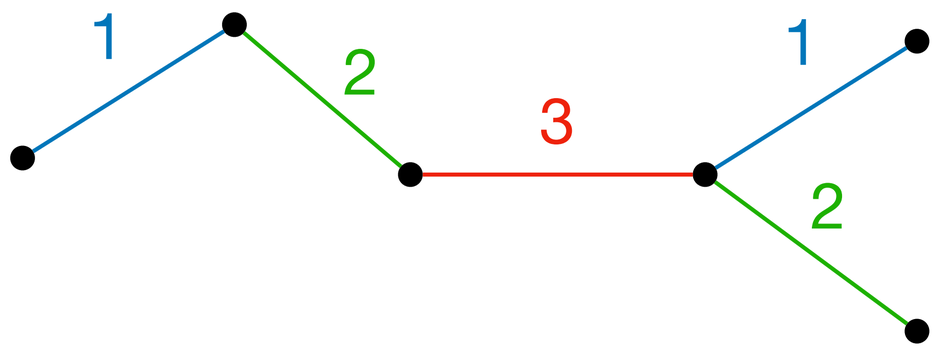

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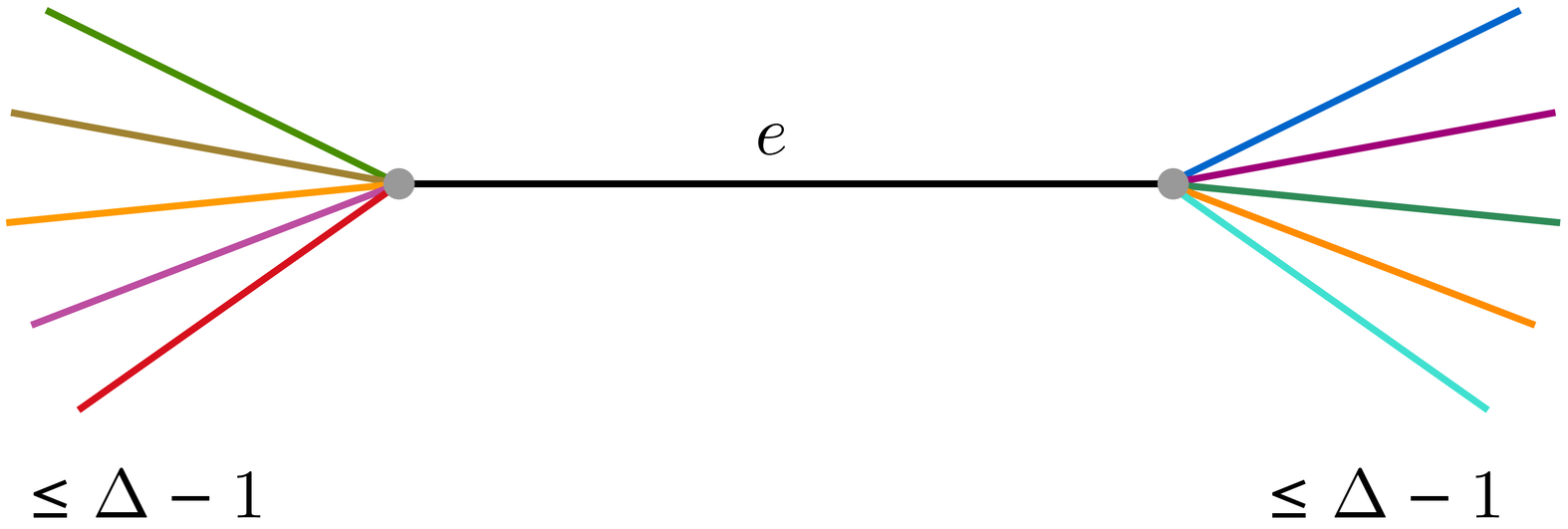

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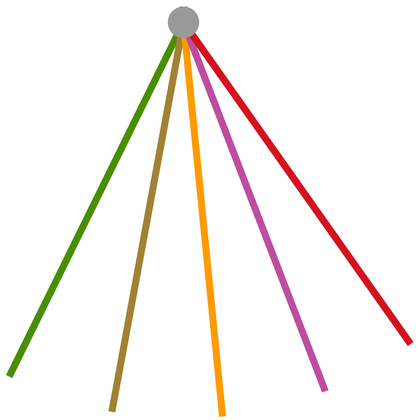

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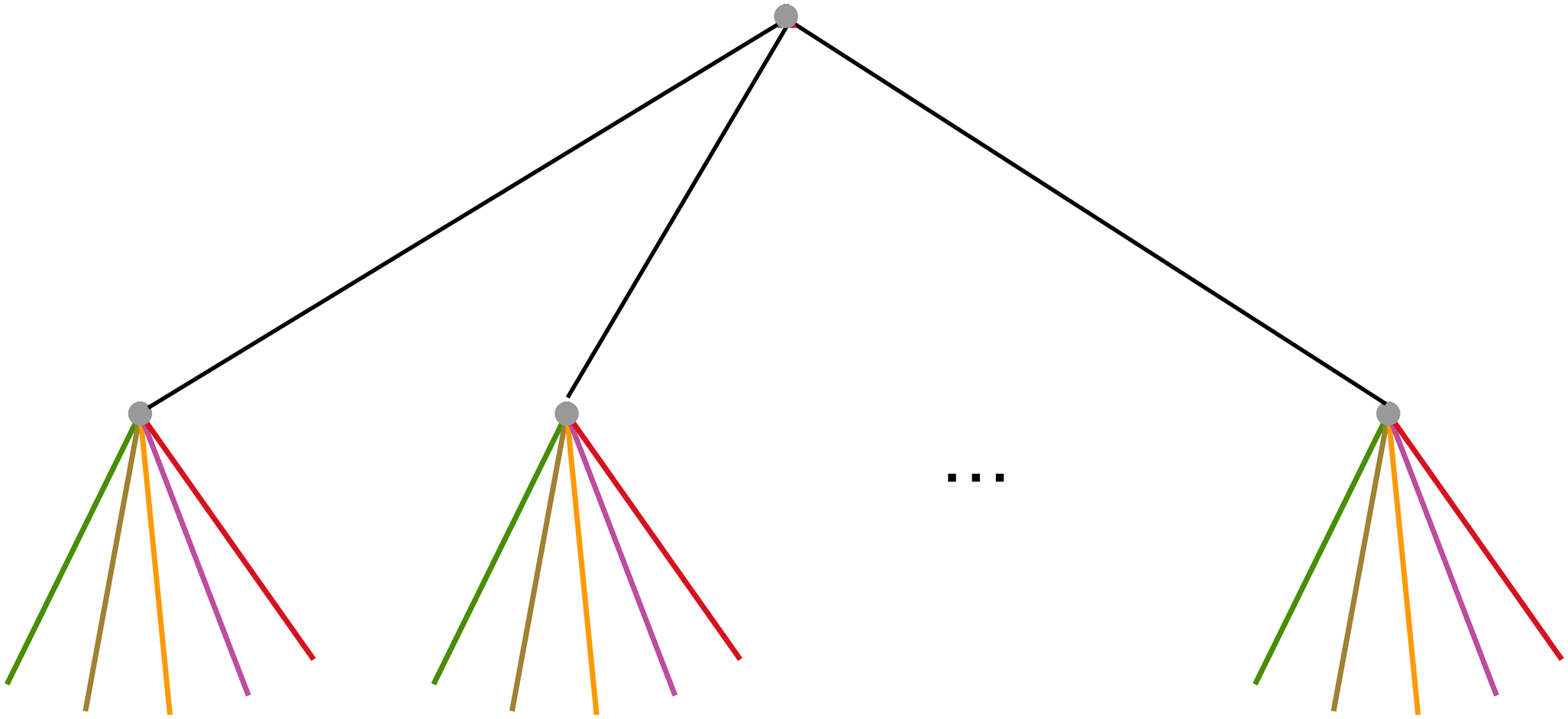

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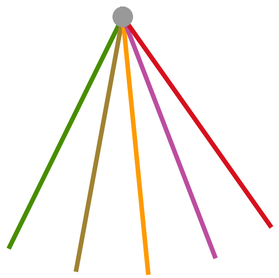

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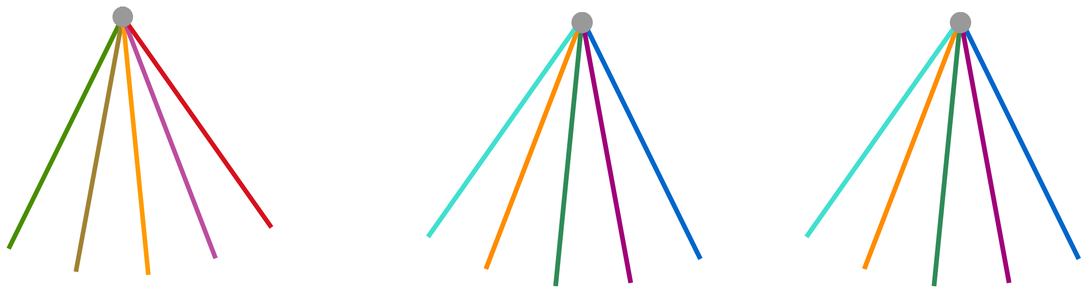

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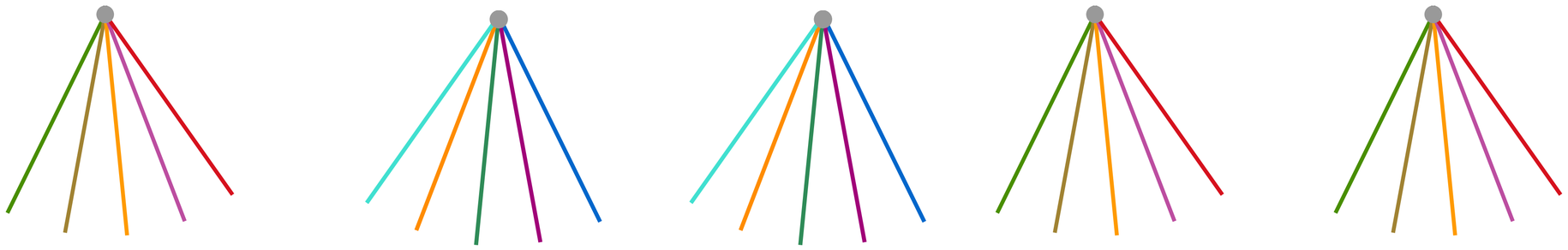

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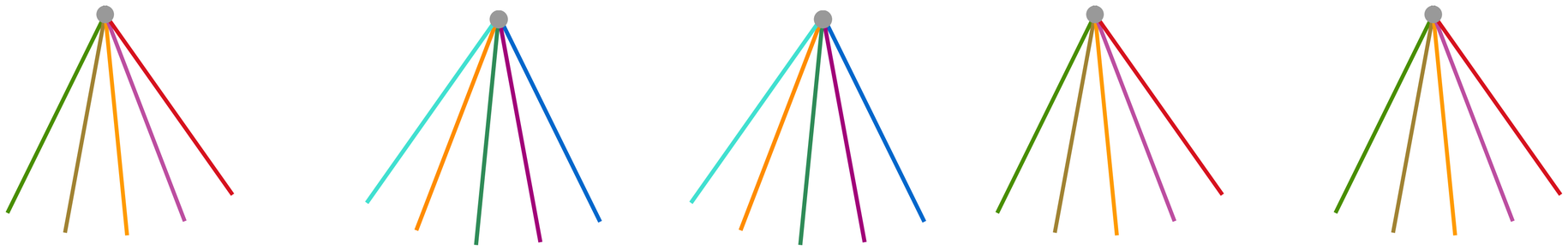

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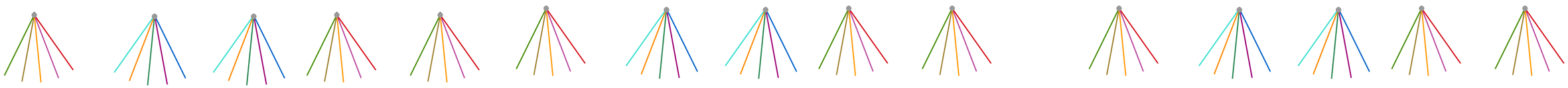

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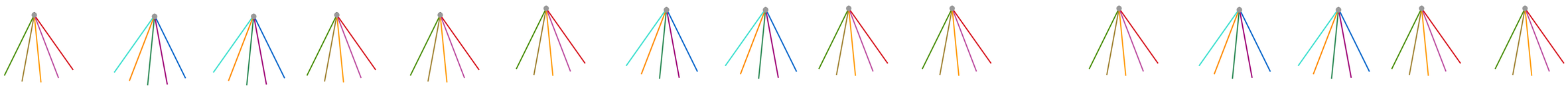
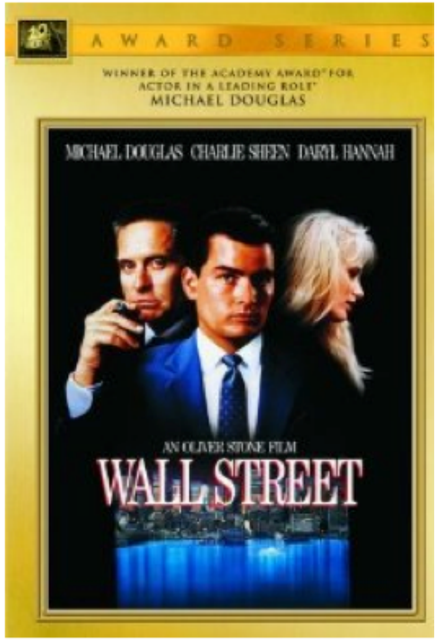
[Bar-Noy/Motwani/Naor 1992]

No algorithm can do better if $\Delta = \begin{cases} o(\log n) & \text{if deterministic} \\ o(\sqrt{\log n}) & \text{if randomized} \end{cases}$





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Conjecture: [Bar-Noy/Motwani/Naor 1992]

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- **Random order** edge arrivals:
  - [Aggarwal/Motwani/Shah/Zhu'03]: $\approx \Delta$ -coloring if $\boldsymbol{\Delta = \omega(n^2)}$ **(multigraphs)**
  - [Bahmani/Mehta/Motwani'10]: $\approx 1.27\Delta$ -coloring **general graphs**
  - [Bhattacharya/Grandoni/Wajc'21]: $\approx \Delta$ -coloring **general graphs**

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- Adversarial vertex arrivals:
  - [Cohen/Peng/Wajc'19] (simplified [Blikstad/**S.**/Vintan/Wajc '24]):
    $\approx \Delta$ -coloring **bipartite graphs**

    For unknown Δ: $\approx \frac{e}{e-1}\Delta$ -coloring **bipartite graphs (optimal)**
  - [Saberi/Wajc'21]: $\approx 1.9\Delta$ -coloring **general graphs**

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Our result

Conjecture:

$(1 + o(1))\Delta$-colors sufficient when $\Delta = \omega(\log n)$

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~~**Conjecture:**~~ **Theorem:**

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Fresh extension of techniques

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[BSVW'25]

Theorem:

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Outline of ideas

- Edge Coloring $\Longleftrightarrow$ Fair Matchings
- Online Fair Matching Algorithm
  - Previous attempts to control correlations
  - Embracing correlations via Martingales

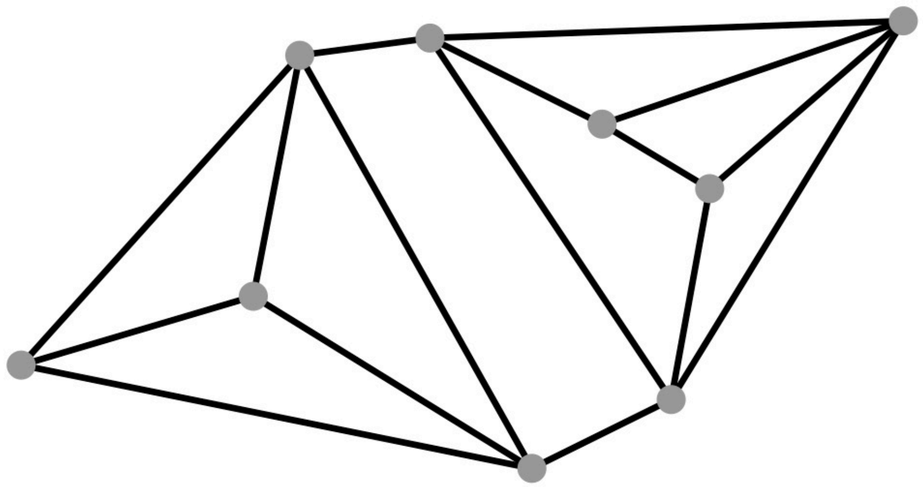
Edge Coloring $\Longleftrightarrow$ Fair Matchings

Fair Matching Problem

Given: Graph $G = (V, E)$

Goal: Find a matching M

α-Fairness: $\Pr[e \in M] \geq \frac{1}{\alpha\Delta}$ for every edge $e \in E$



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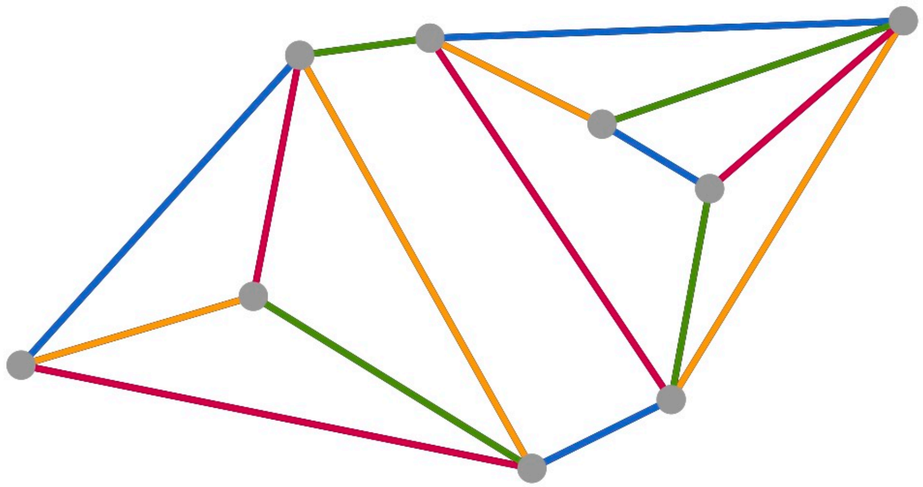

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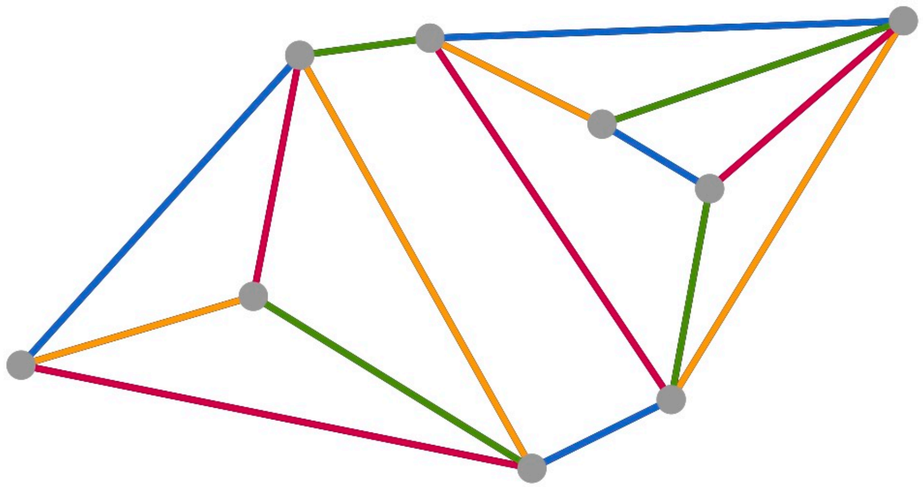

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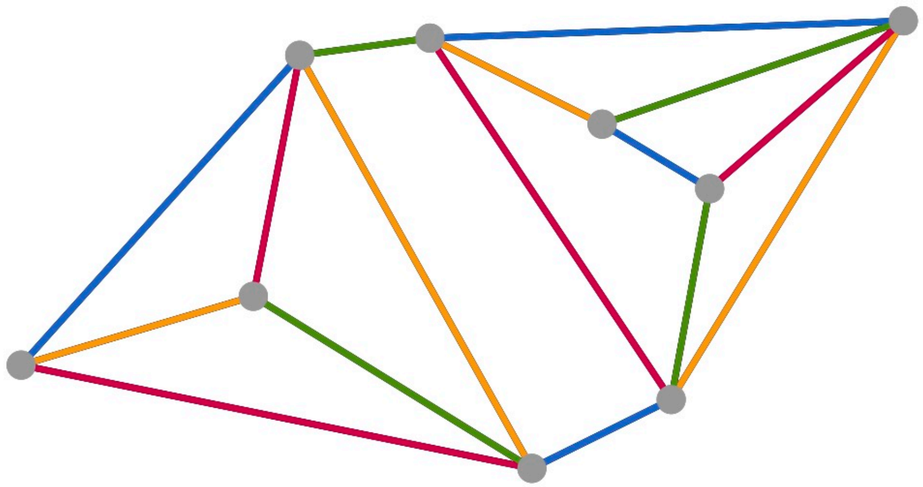

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[Cohen/Peng/Wajc'19]

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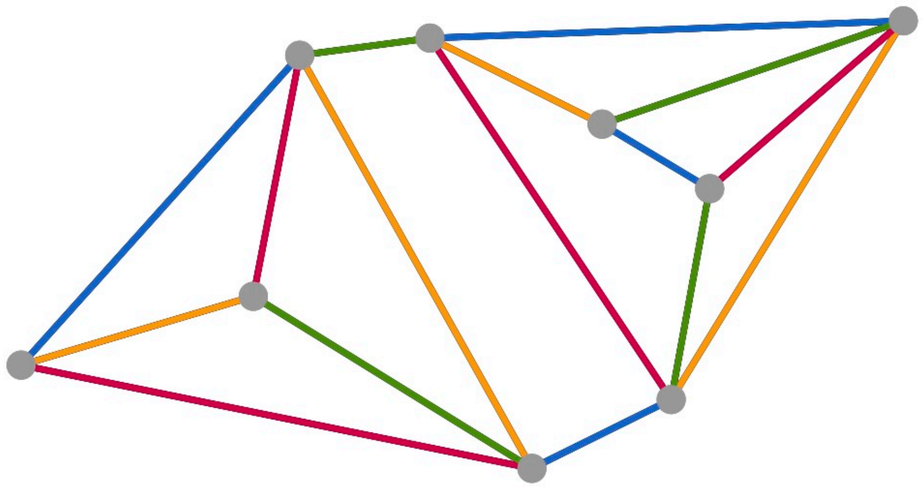

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Note: the above results are for online (similar reductions used by e.g. Kahn for list coloring)

New Objective

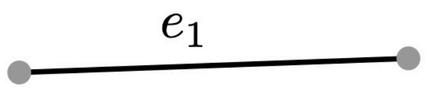
Find $(1 + o(1))$-fair matching, i.e., match each edge with probability

$$\Pr[e \in M] \approx \frac{1}{\Delta}$$

Fair Matching Algorithm

Goal: Match each edge with probability $\Pr[e \in M] \approx \frac{1}{\Delta}$

$$\Pr[e \in M] = \frac{1}{\Delta + q} \quad \text{with} \quad q = o(\Delta)$$



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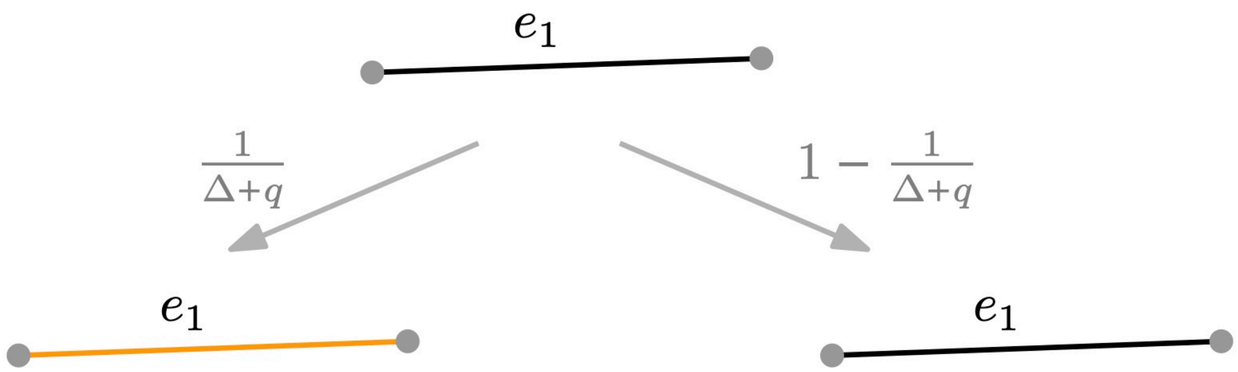

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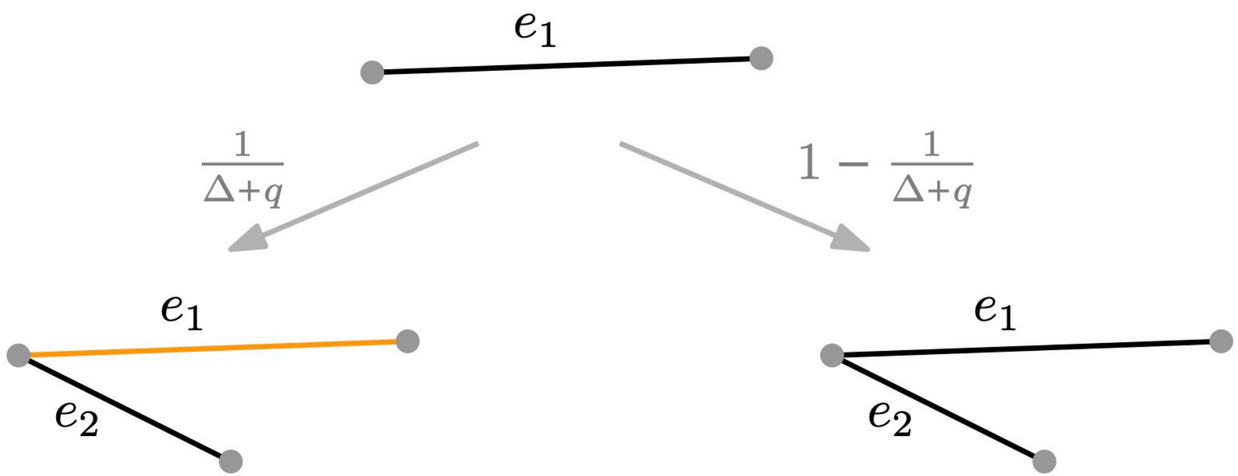

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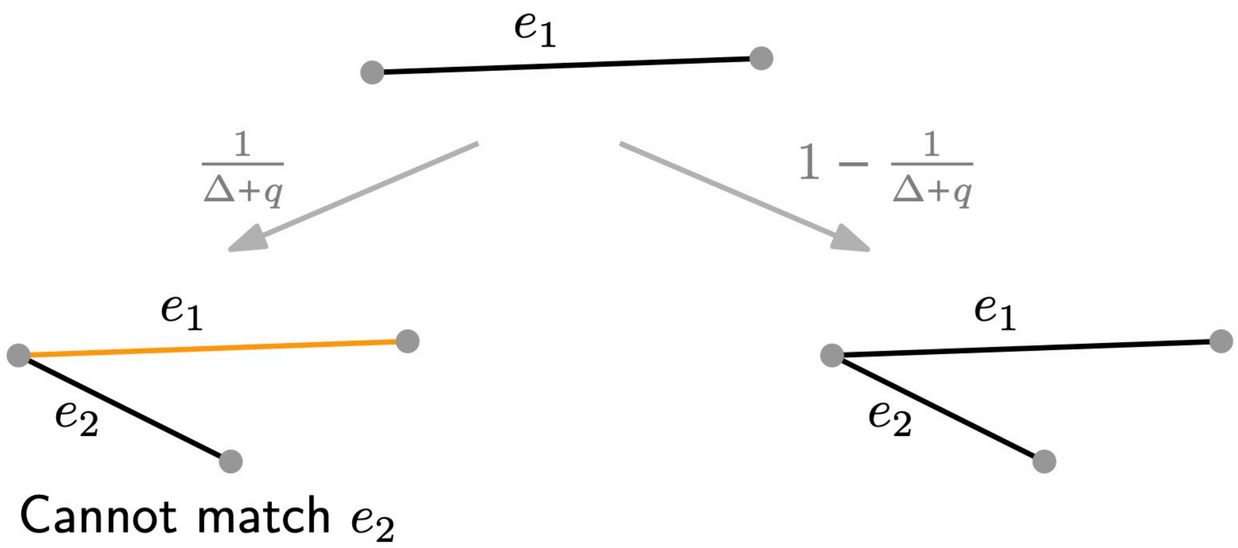

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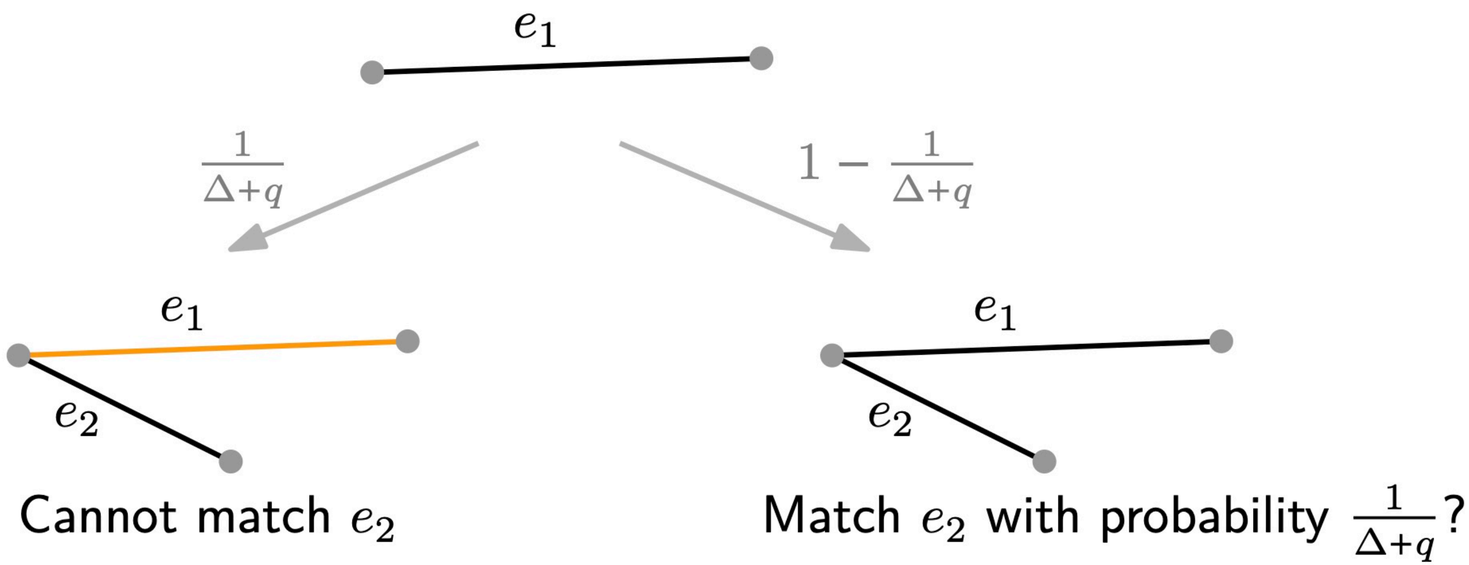

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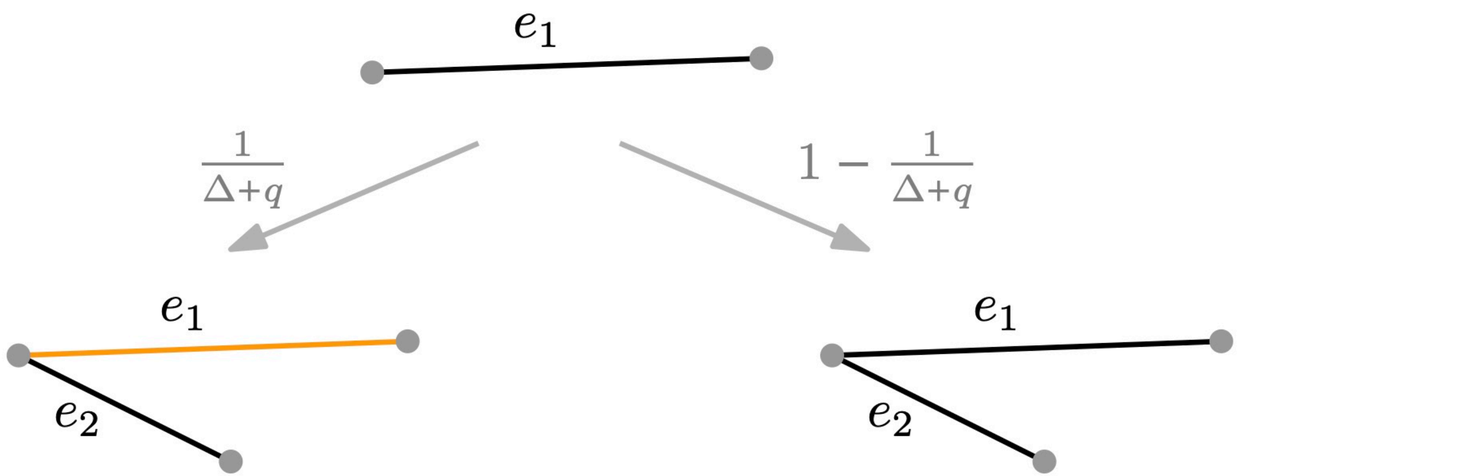

Cannot match e_2

Match e_2 with probability ~~$\frac{1}{\Delta+q}$~~ ?

Must scale up: $\frac{1}{\Delta+q} / (1 - \frac{1}{\Delta+q})$

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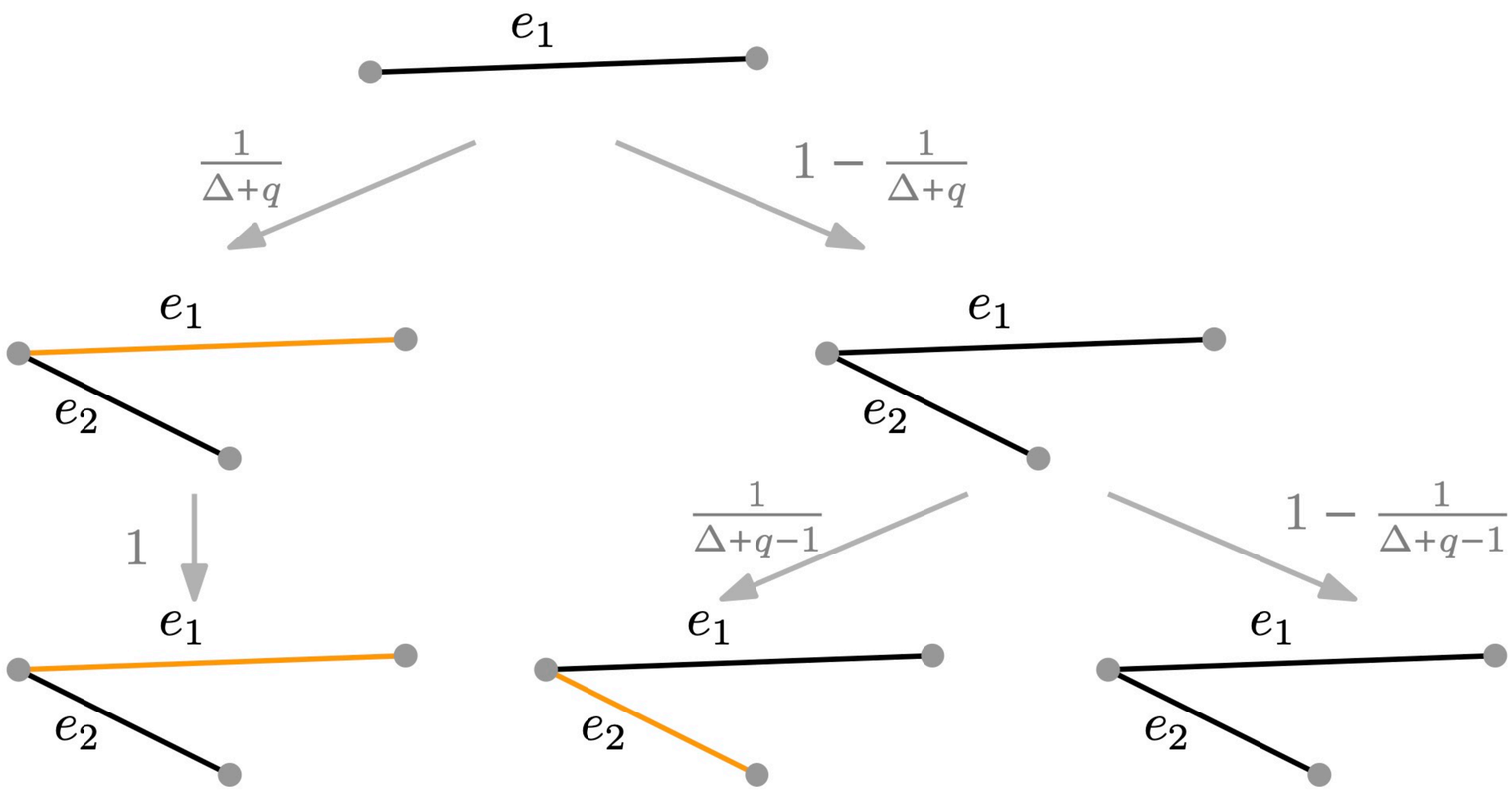

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Potential problem: $p_t > 1$

Easy to prove: works if G is tree

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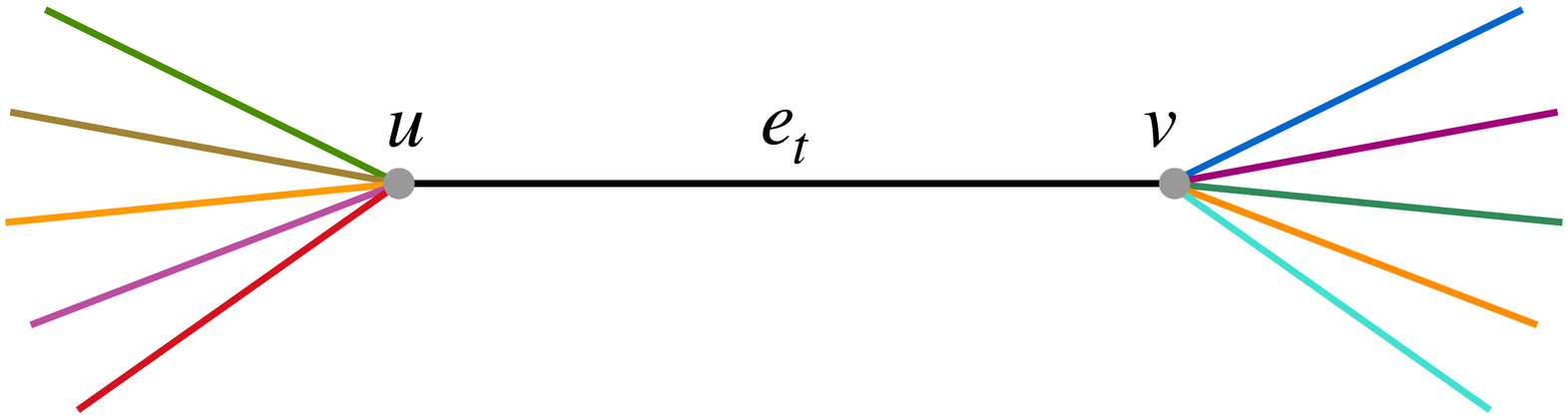

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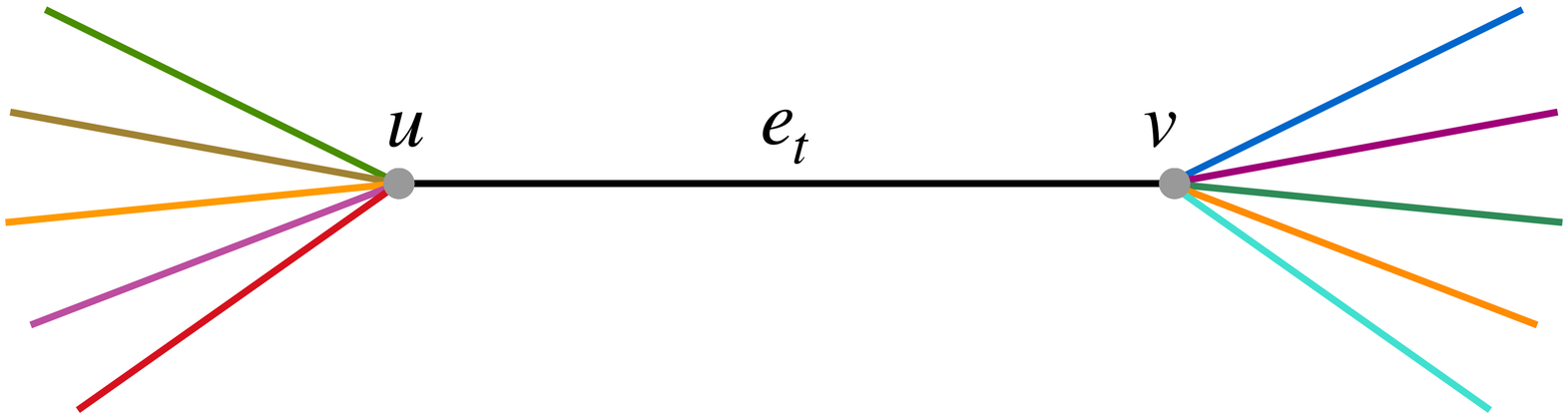

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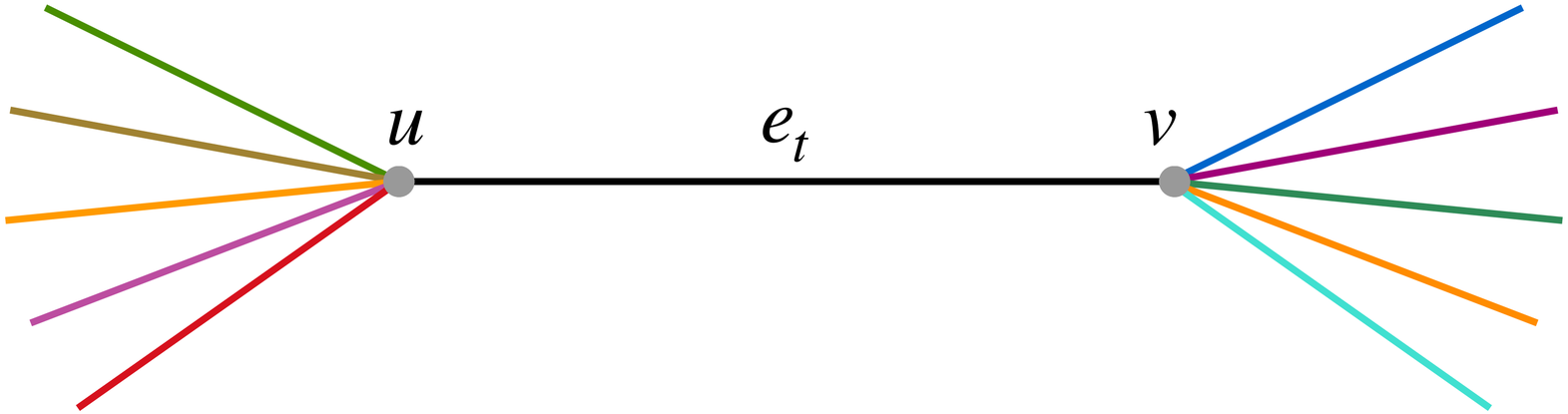


In this case "u is free" and "v is free" are **independent** so

Pr[u,v free] = Pr[u free] · Pr[v free] $\geq \left(\frac{q}{\Delta + q}\right)^2 \implies$ things work out if $q = \omega(\sqrt{\Delta})$

Problem with general graphs

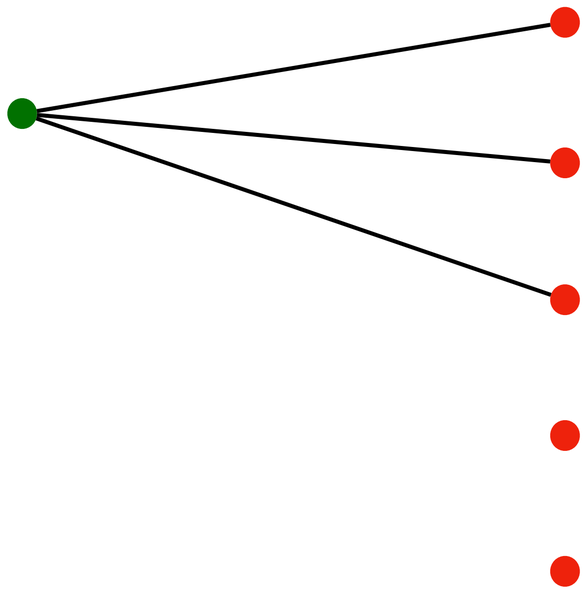
- Arbitrary (positive and negative) correlations

Prior work **controlled** these correlations

Vertex arrivals

Cohen, Peng, Wajc'19

Online vertices **Offline vertices**



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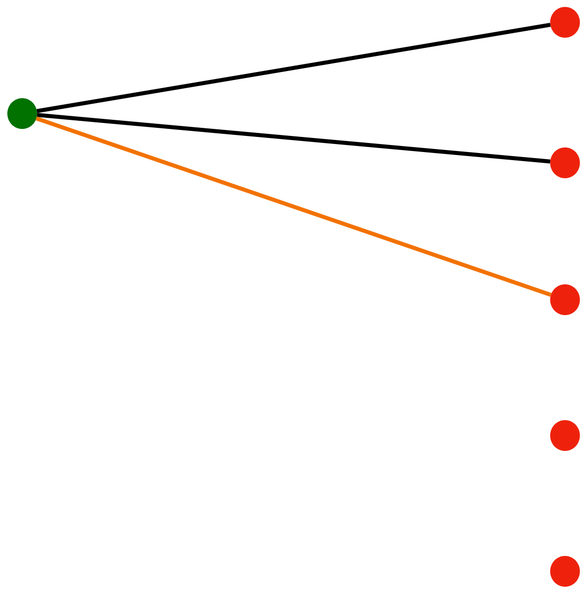

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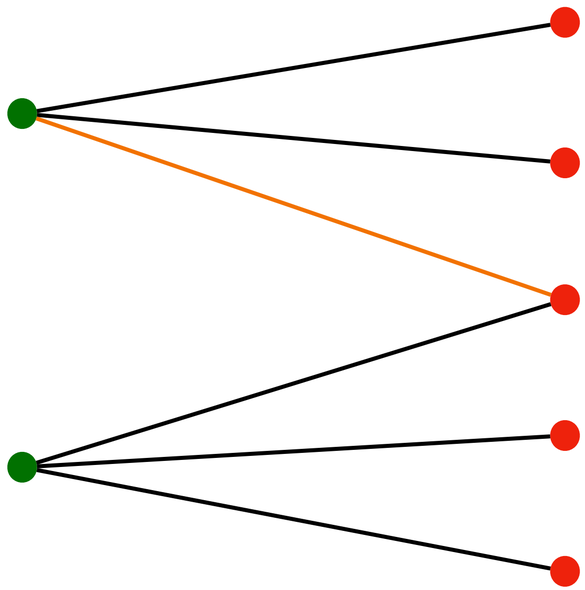

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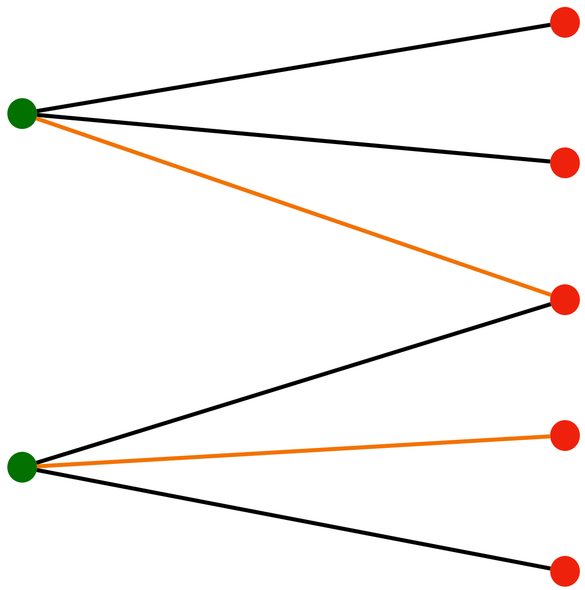

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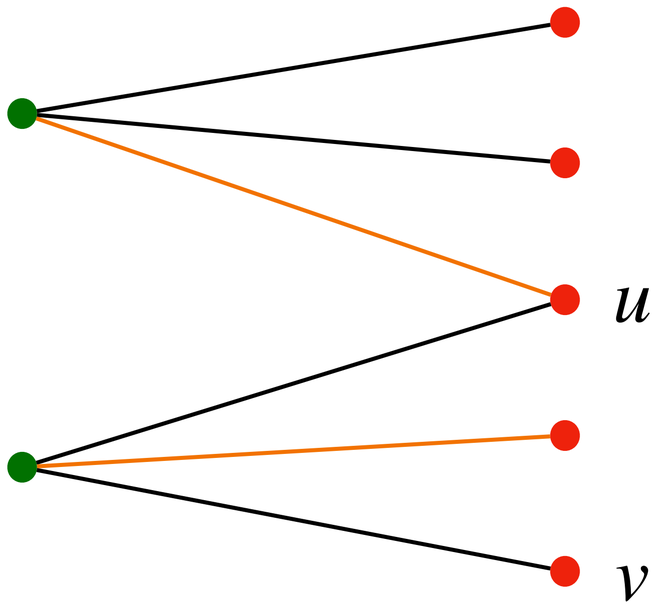

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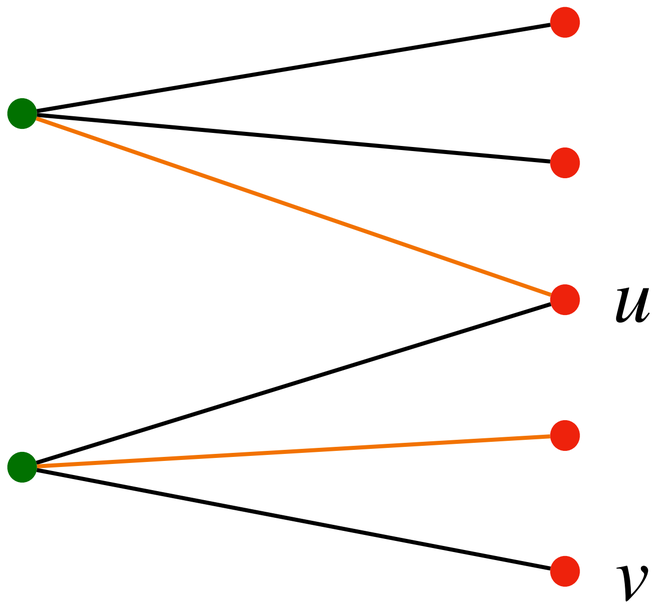

- "u free" and "v free" **negatively** correlated

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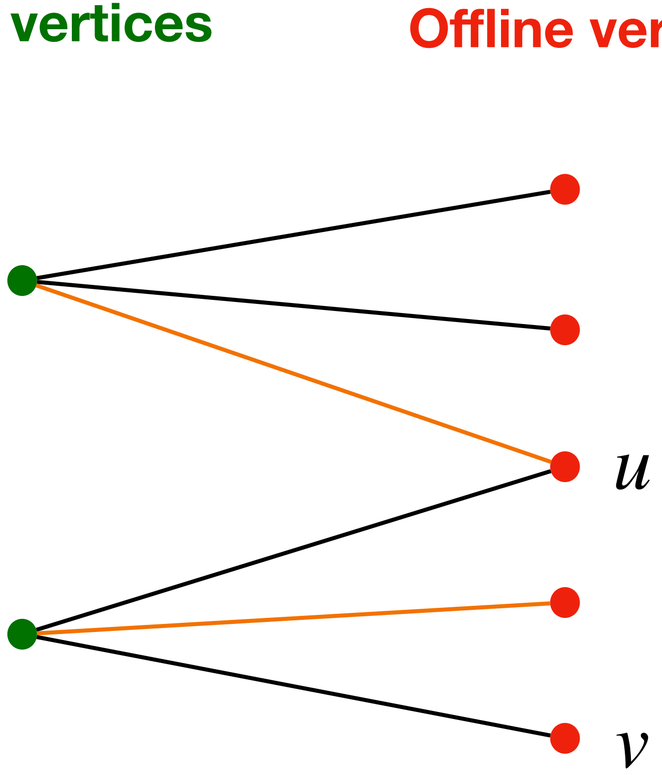

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- Chernoff bounds ensure concentration

General arrivals

Kulkarni, Liu, Sah, Sawhney, Tarnawski’22

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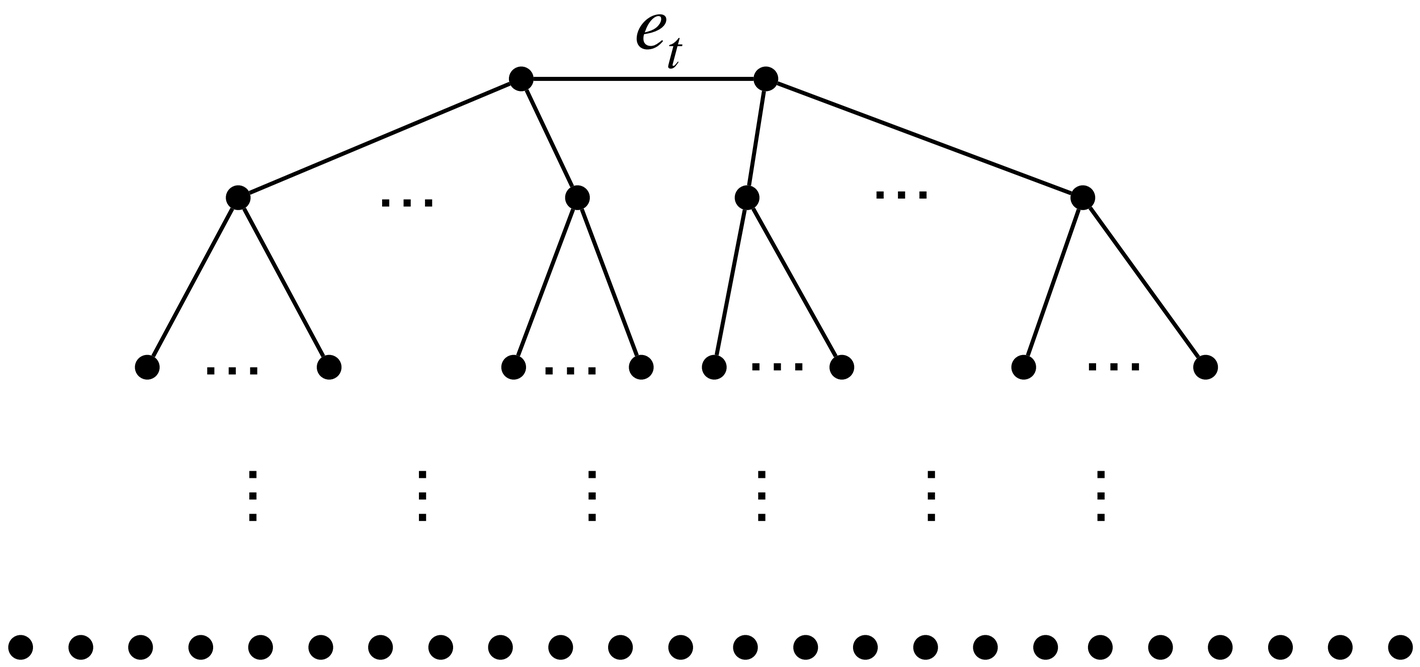

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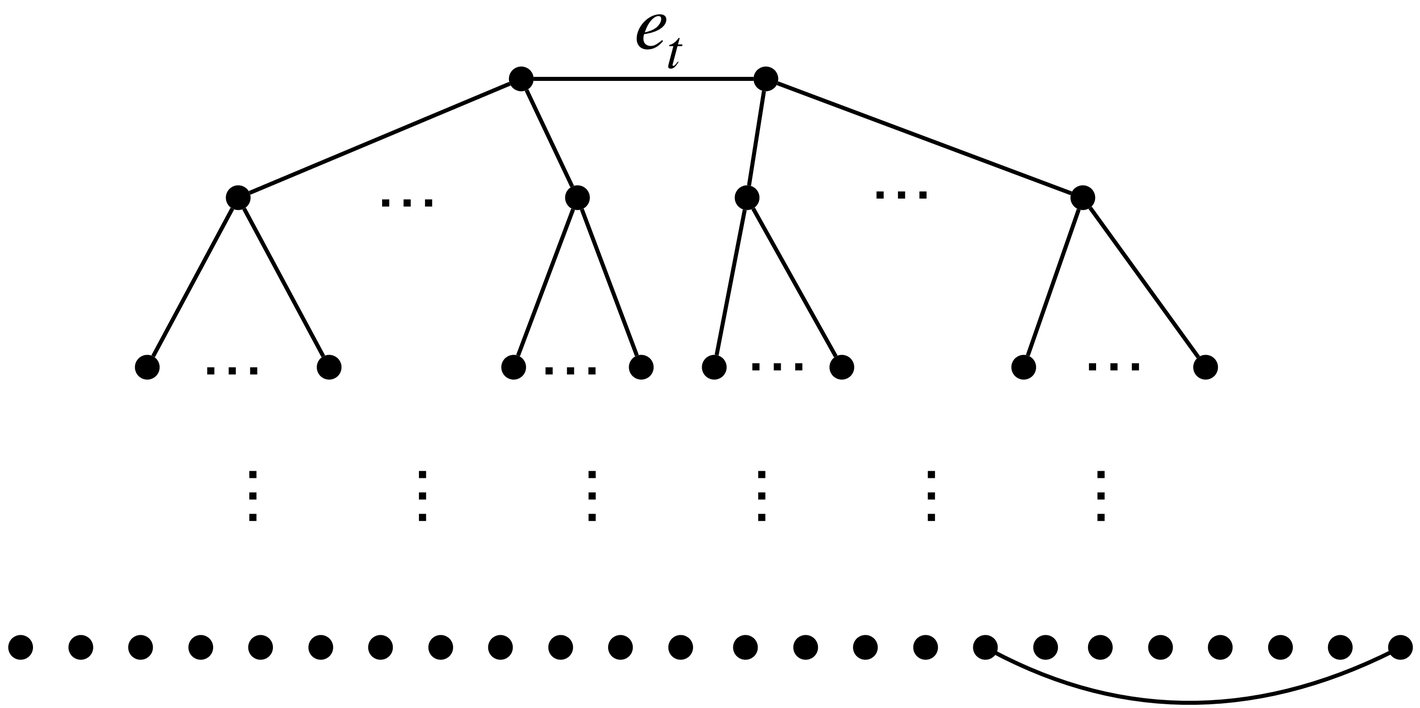

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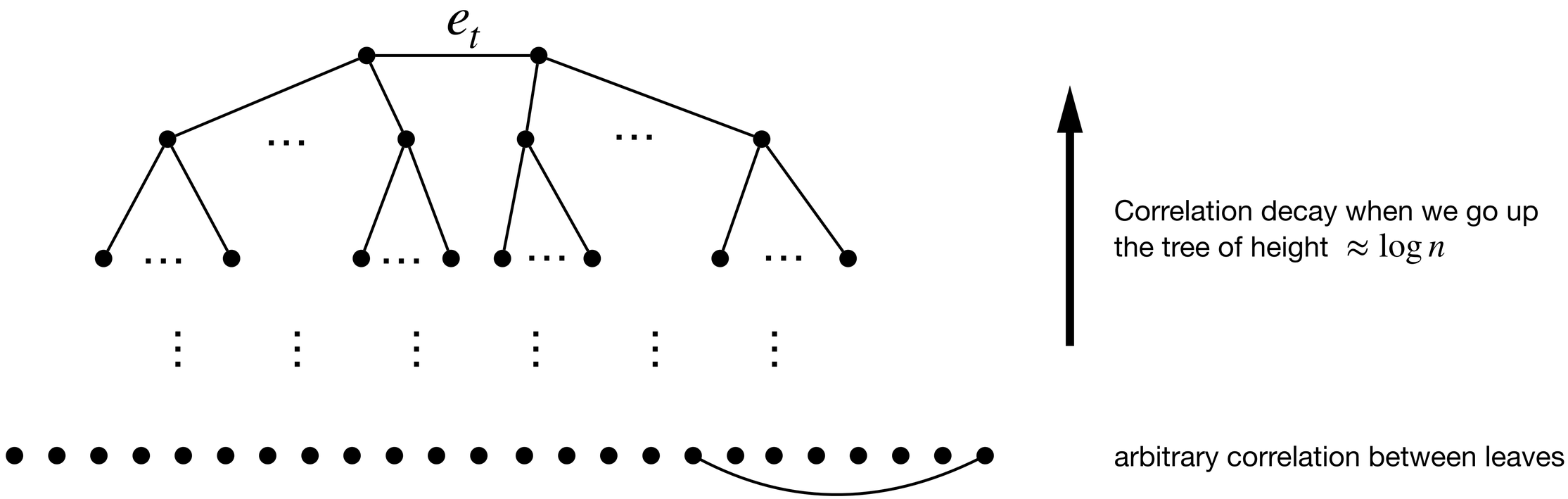

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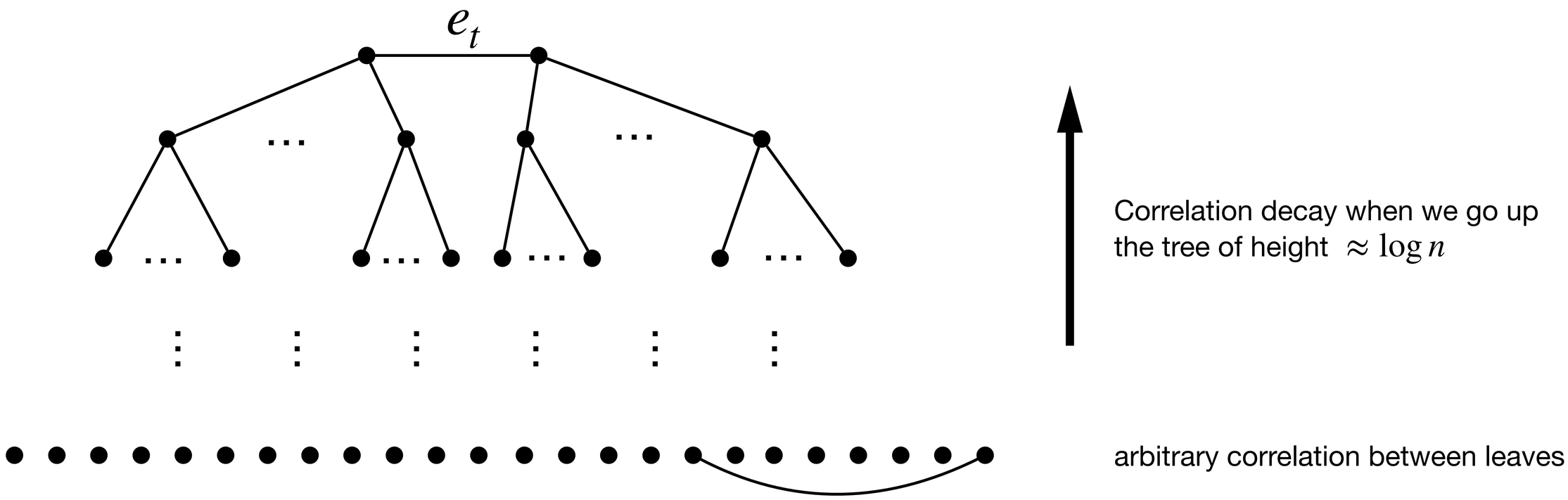
- Subsample graph G into sparse copies G_1, G_2 each copy sparse and locally tree like
- Correlation decay allow to treat as independent case but requires #colors $\approx e/(e-1)\Delta$



Our Work

Embrace correlations instead of controlling them

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Key takeaway:

Formulate **quantity as a martingale**, bound stepsize and observed variance

Chernoff-type concentration

Recall algorithm

When $e_t = (u, v)$ arrives and u, v free:

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Recall algorithm

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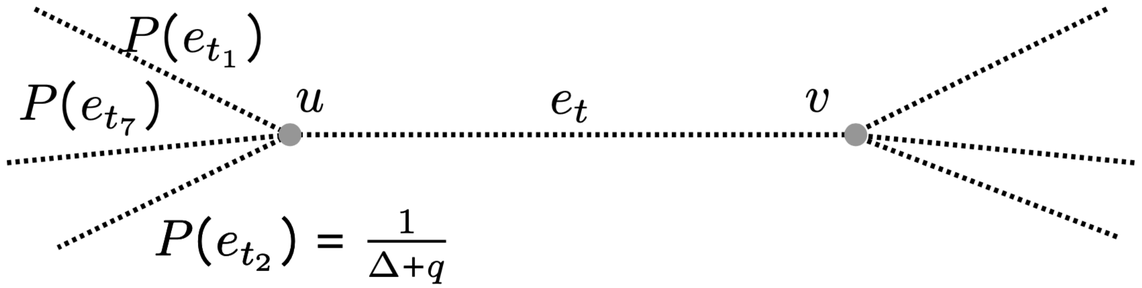
When $e_t = (u, v)$ arrives and u, v free:

Match e_t with probability $p_t = \dfrac{1/(\Delta + q)}{\text{execution-dependent scaling factor}}$

Our alternative algorithm

When $e_t = (u, v)$ arrives:

Match e_t with probability $P(e_t) = \begin{cases} 0 & \text{if } u \text{ or } v \text{ are matched} \\ \dfrac{1/(\Delta + q)}{\prod_{e_{t_j} \in \delta(u) \cup \delta(v)} (1 - P(e_{t_j}))} & \text{otherwise} \end{cases}$



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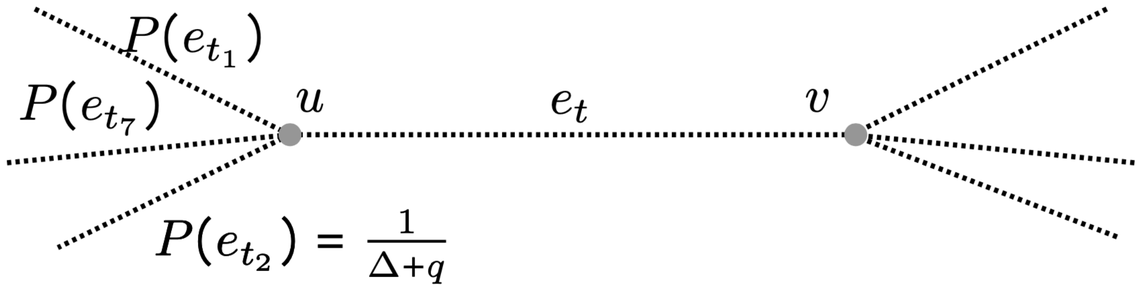
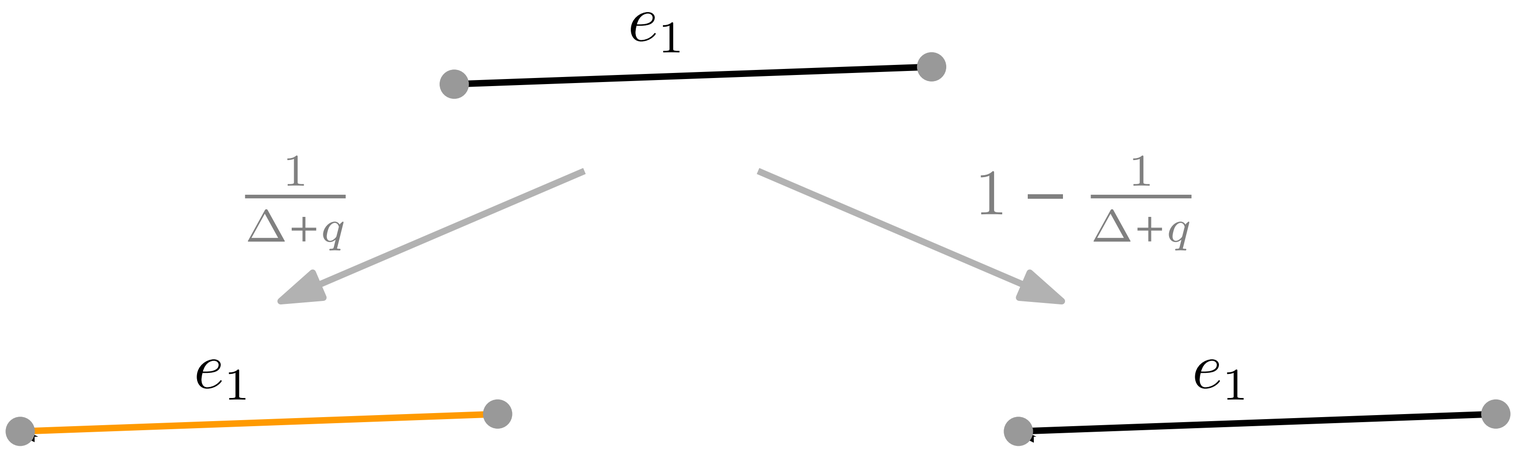
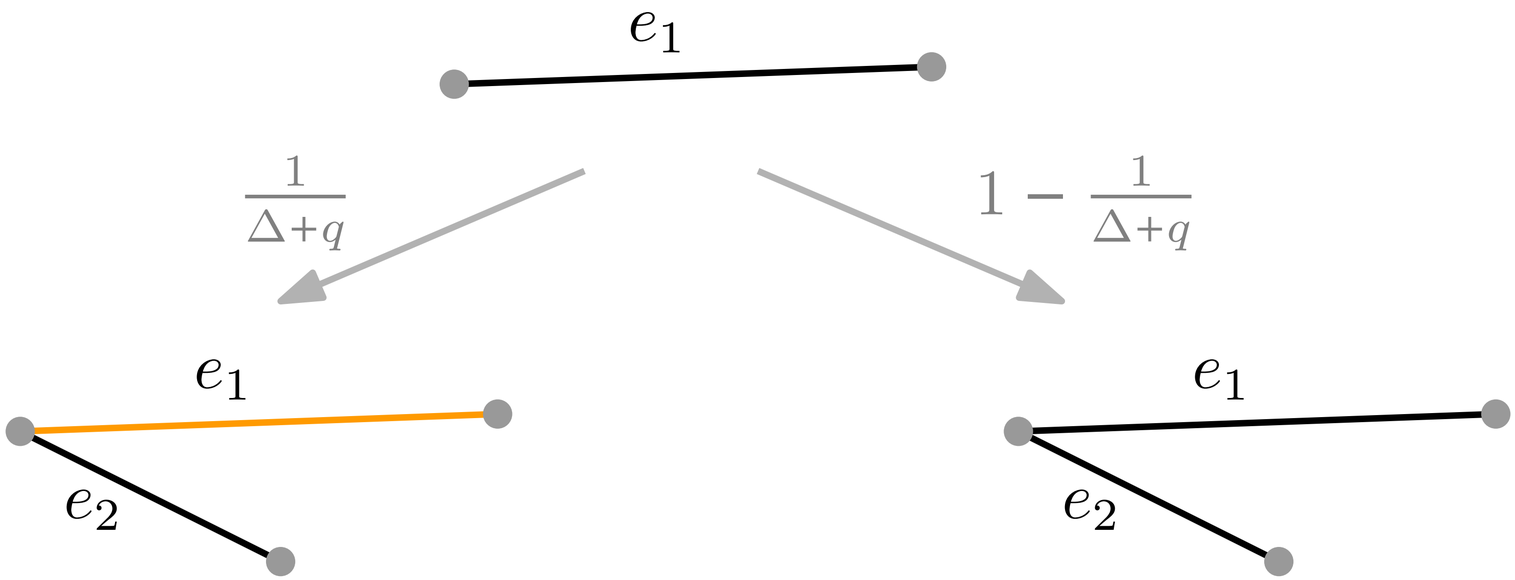


Potential problem: $P(e_t) > 1$


e_1
$\frac{1}{\Delta+q}$
$1-\frac{1}{\Delta+q}$
e_1
e_1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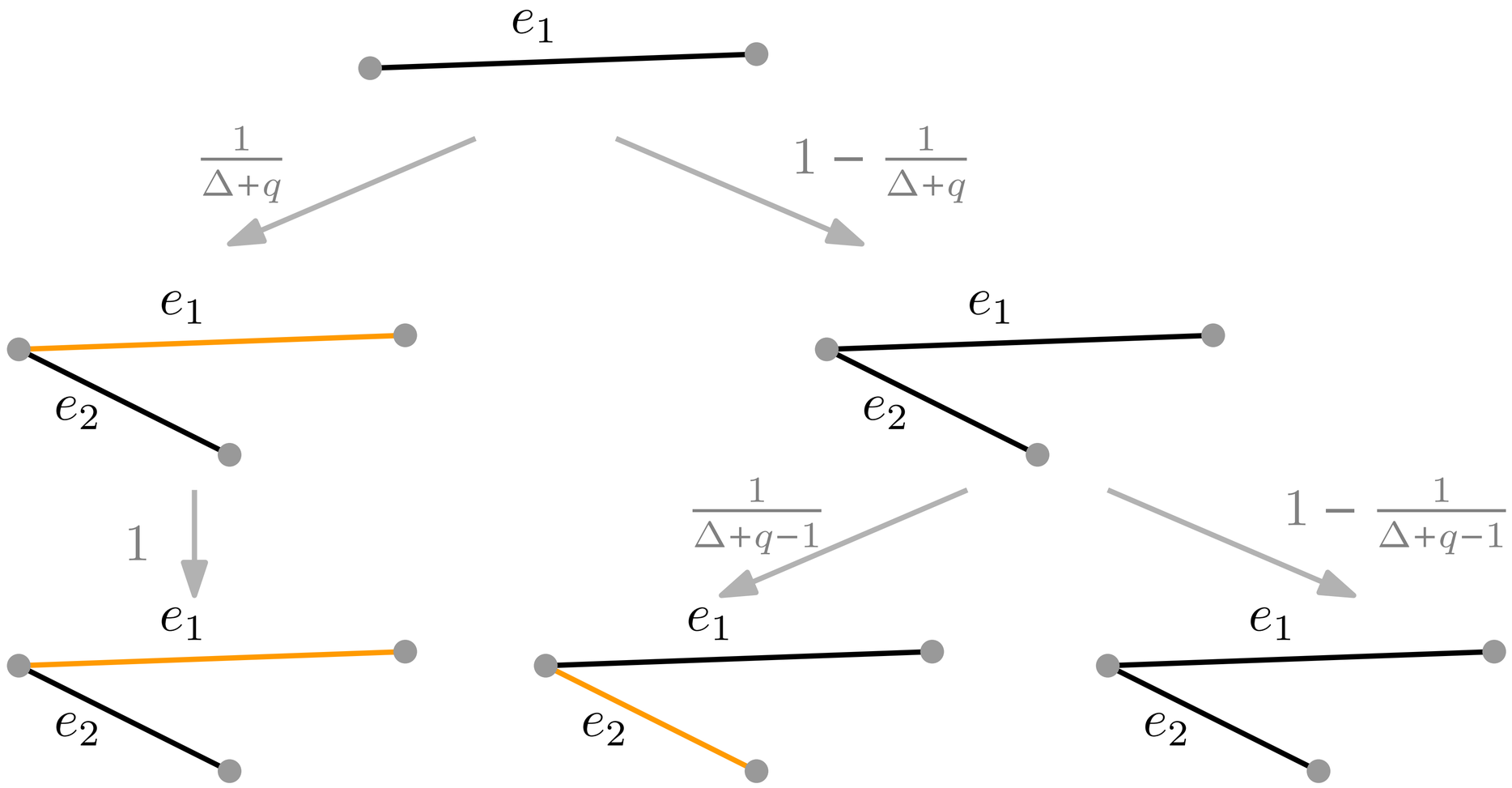
e_1
e_2
e_1
e_2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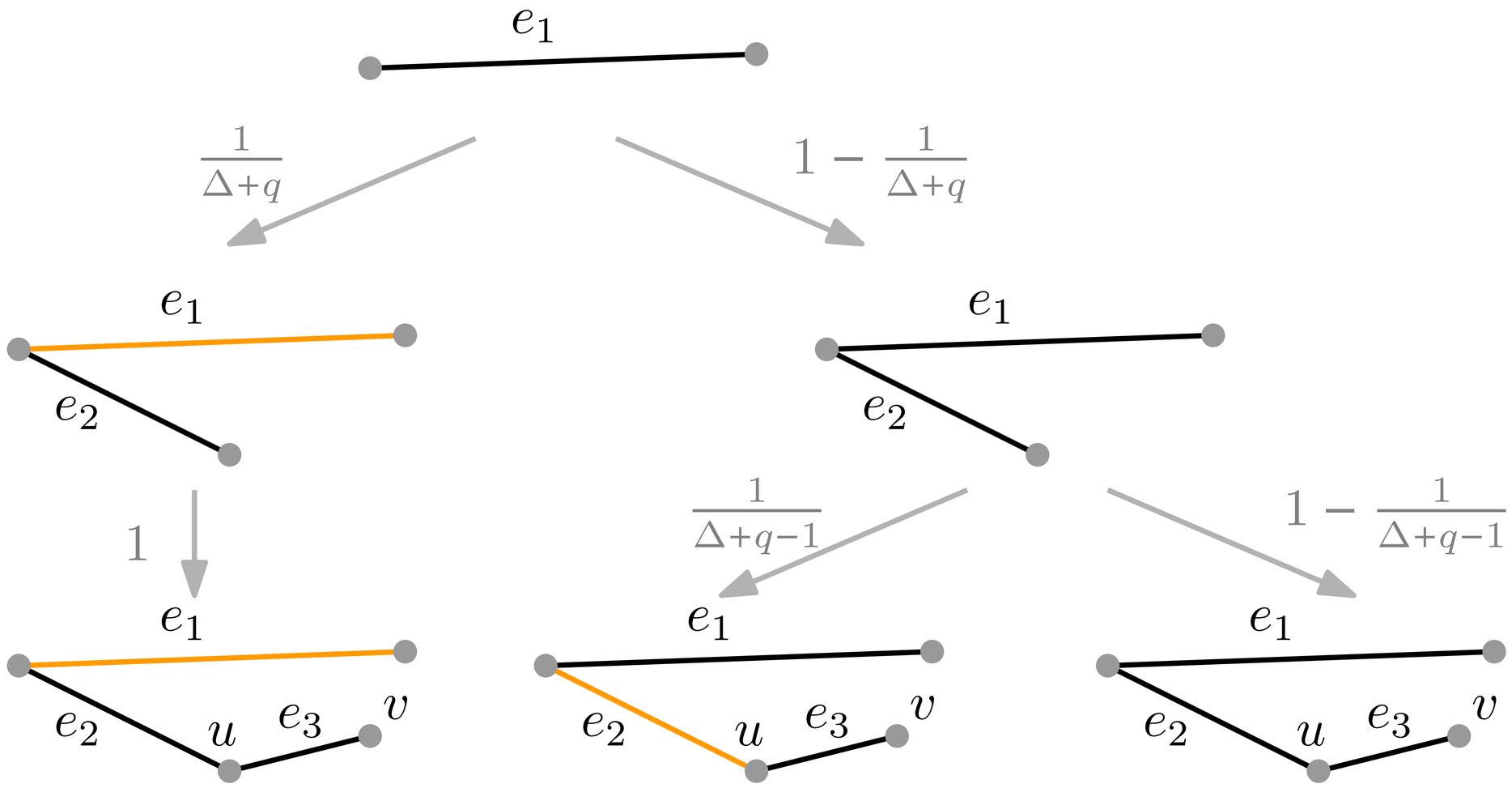
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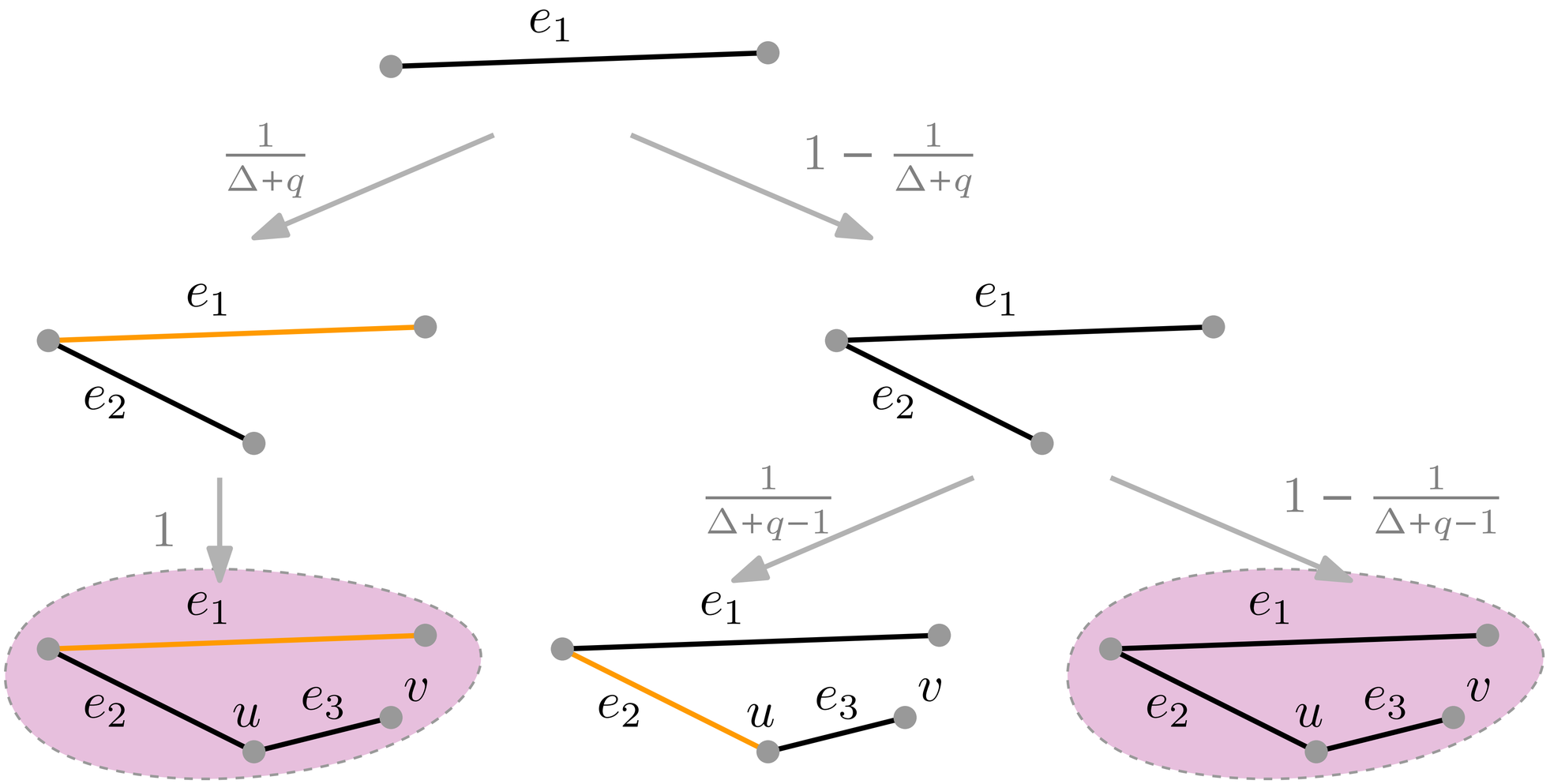
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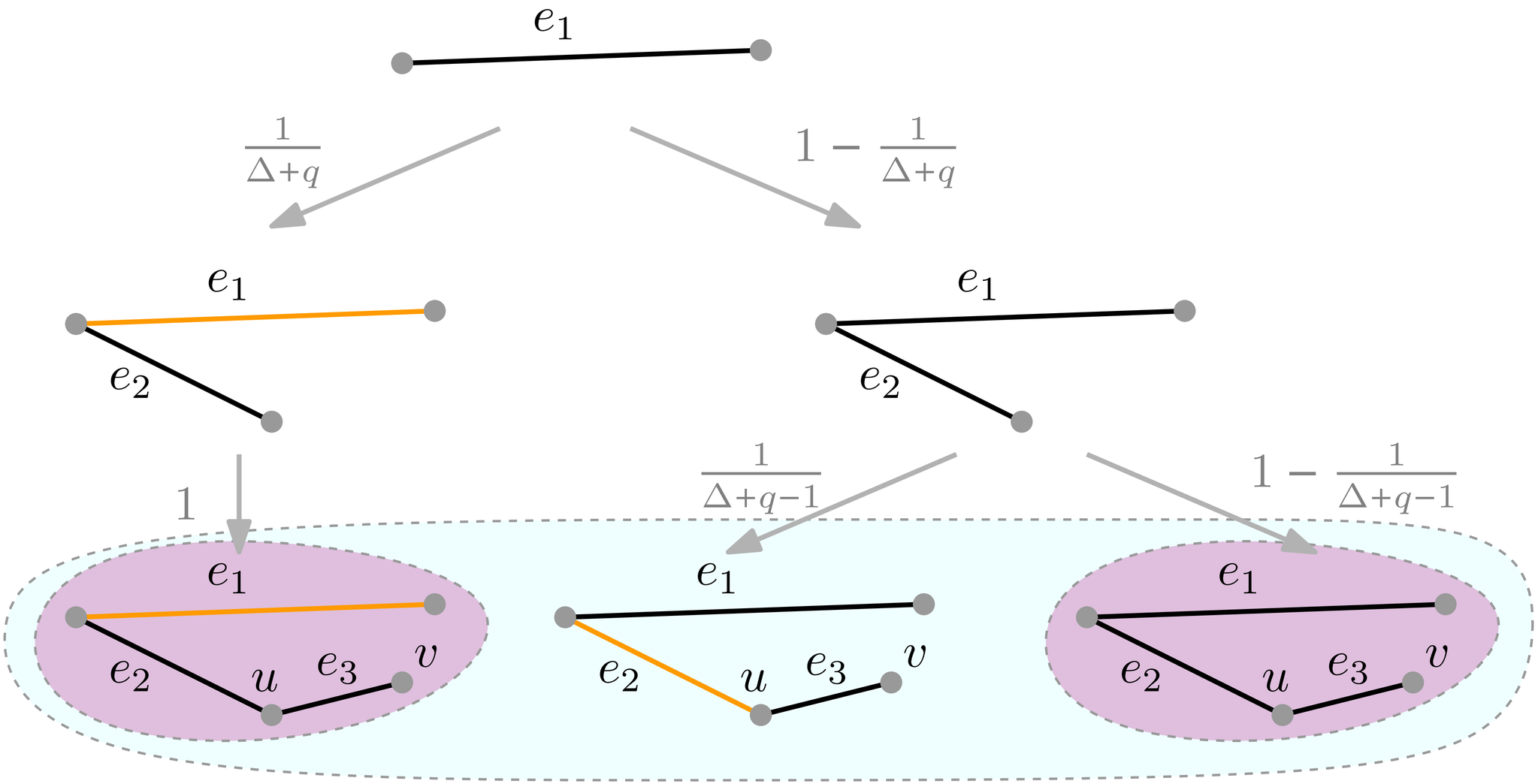
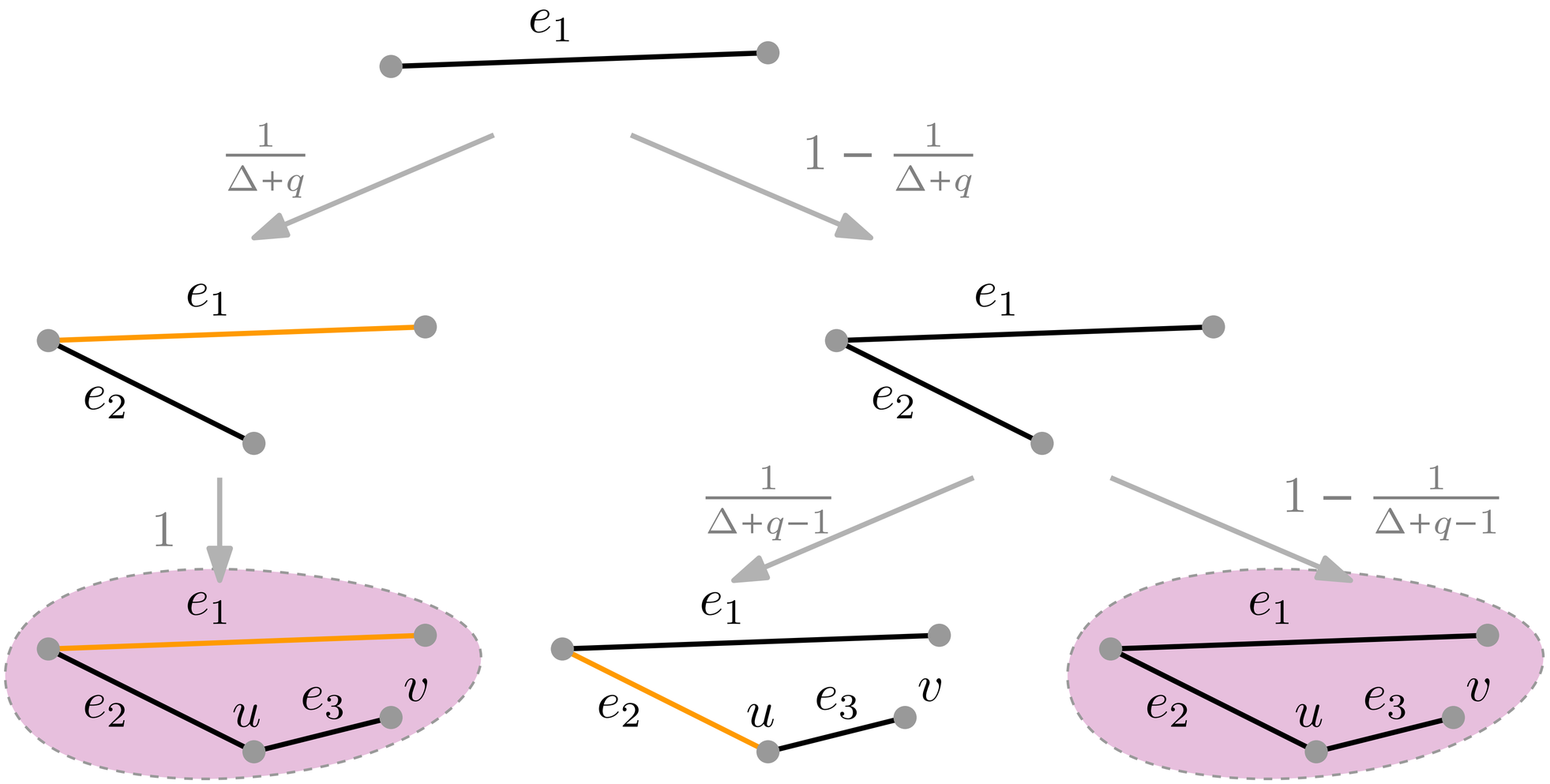
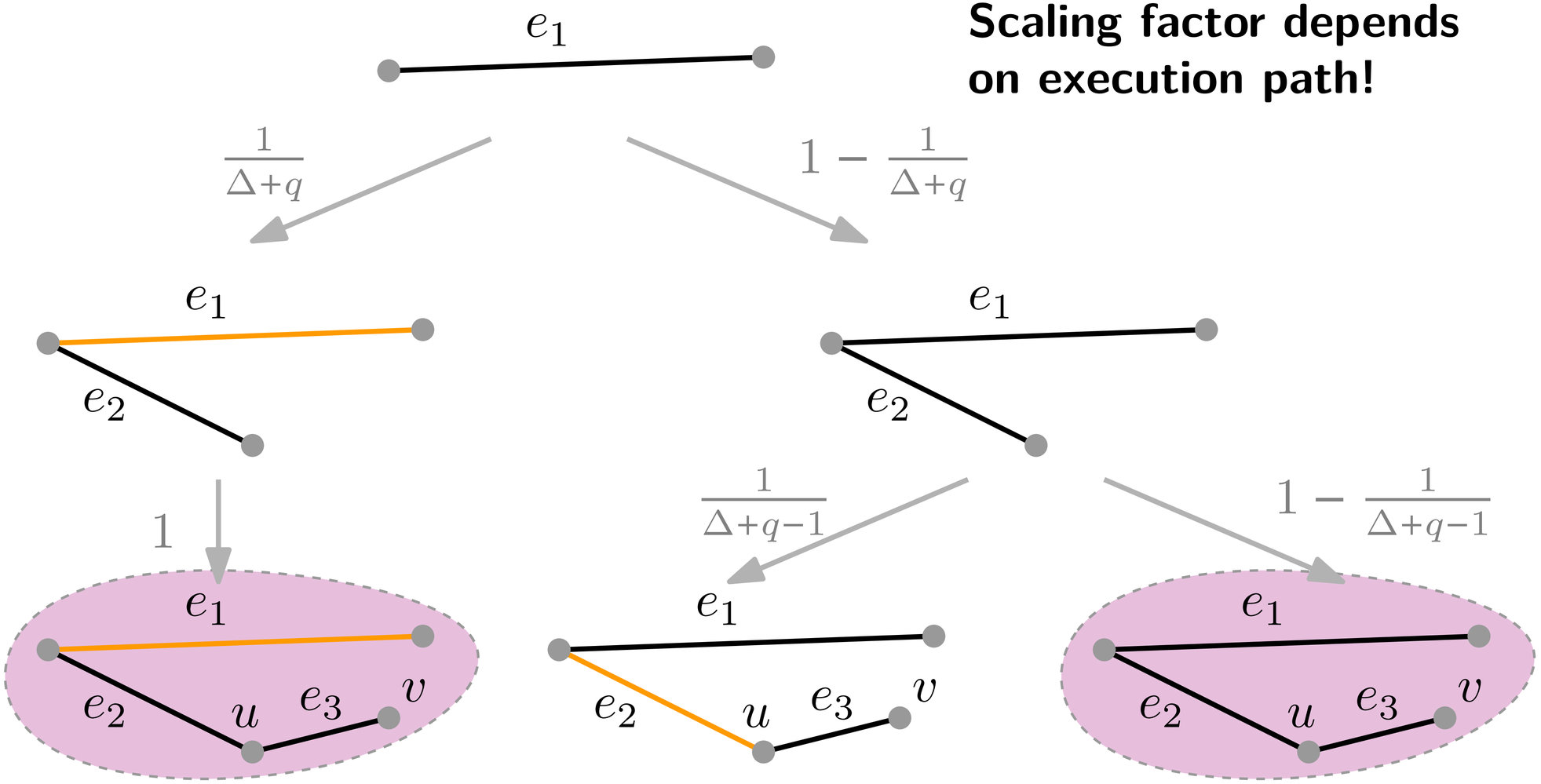


Old Algo: $\Pr[u, v \text{ free}] = (1 - \frac{1}{\Delta+q})$

Uses same scaling factor


e_1
$\frac{1}{\Delta+q}$
$1-\frac{1}{\Delta+q}$
e_1
e_2
e_1
e_2
1
$\frac{1}{\Delta+q-1}$
$1-\frac{1}{\Delta+q-1}$
e_1
e_2
u
e_3
v
e_1
e_2
u
e_3
v
e_1
e_2
u
e_3
v

Scaling factor depends on execution path!



scaling $= 1/(1 - P(e_2)) = 1$

scaling $= 1/(1 - P(e_2)) = 1/(1 - \frac{1}{\Delta+q-1})$

Analysis Idea — Random walk

Core of Analysis: Prove $P(e_t) \leq 1$

Our Algorithm: $P(e_t) := \frac{1/(\Delta+q)}{\prod_{e \in \delta(u) \cup \delta(v)} (1-P(e))}$

$$q := \approx \Delta^{3/4}$$

Analysis Idea — Random walk

Core of Analysis: Prove $P(e_t) \leq \frac{10}{\sqrt{\Delta}}$

Our Algorithm: $P(e_t) := \frac{1/(\Delta+q)}{\prod_{e \in \delta(u) \cup \delta(v)} (1-P(e))}$

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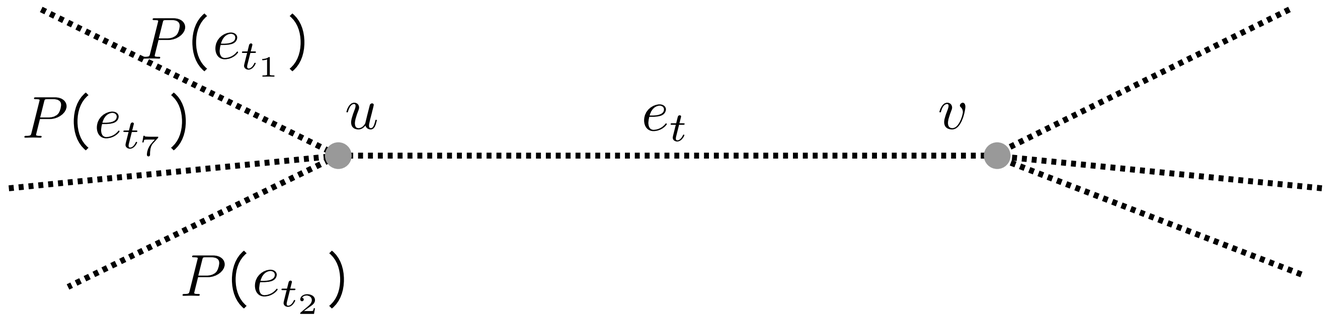
Analysis Idea — Random walk

Scaling factor $S_u := \prod_{e_{t_j} \in \delta(u)} (1 - P(e_{t_j}))$ **Goal:** Show $S_u \gtrapprox \sqrt[4]{\frac{1}{10\Delta}}$



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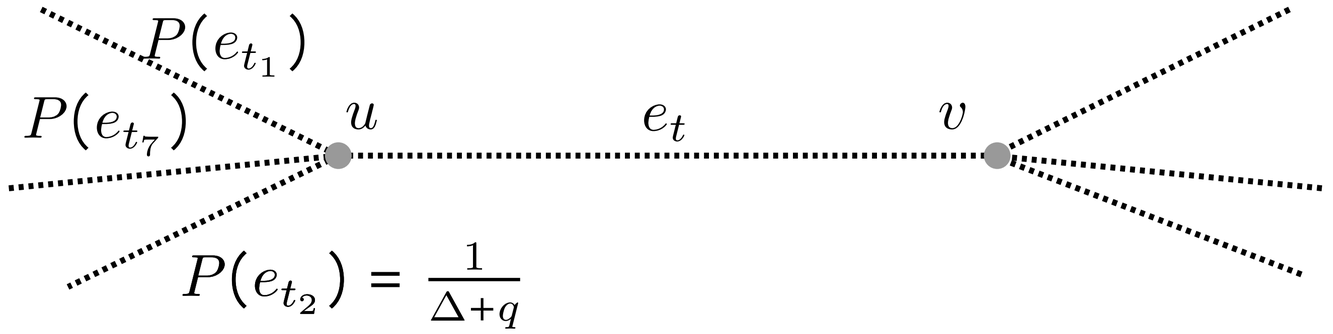

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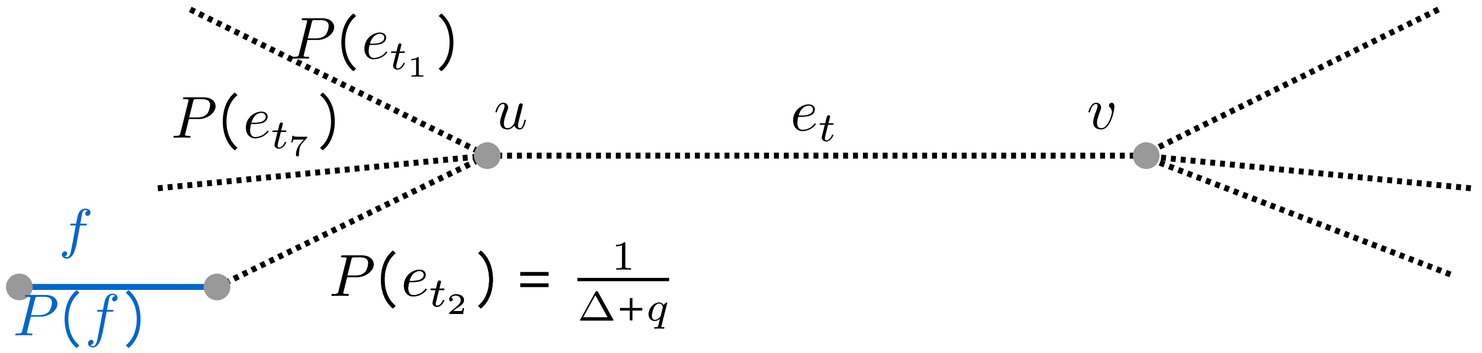

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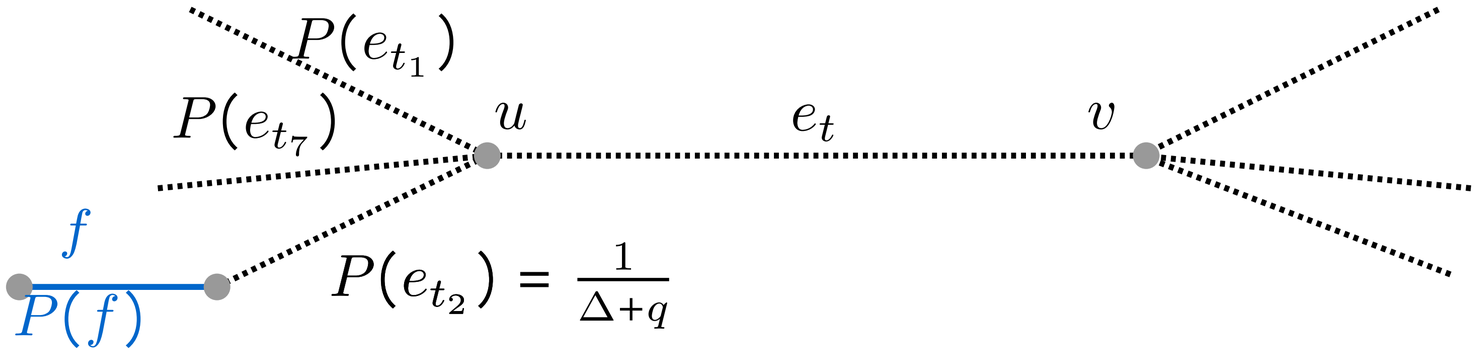

If f matched $\implies P^{new}(e_{t_2}) \leftarrow 0$

If f not matched $\implies P^{new}(e_{t_2}) \leftarrow P(e_{t_2})/(1 - P(f))$

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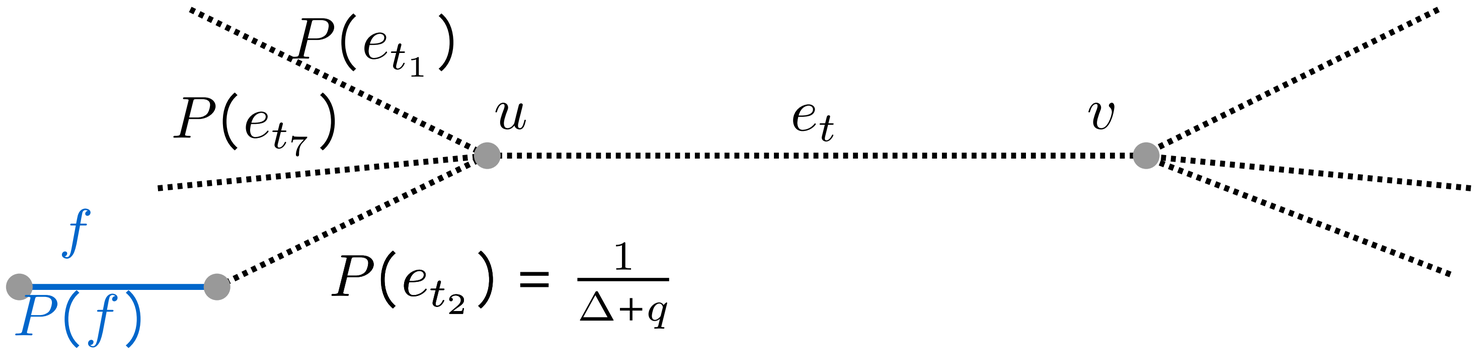

If f matched $\implies P^{new}(e_{t_2}) \leftarrow 0$ $\mathbb{E}[S_u^{new}] = S_u$

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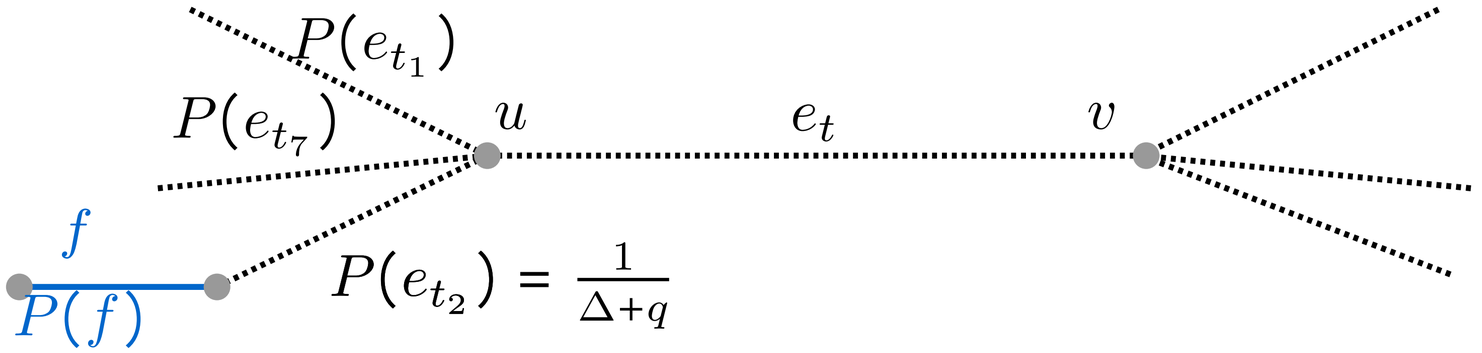

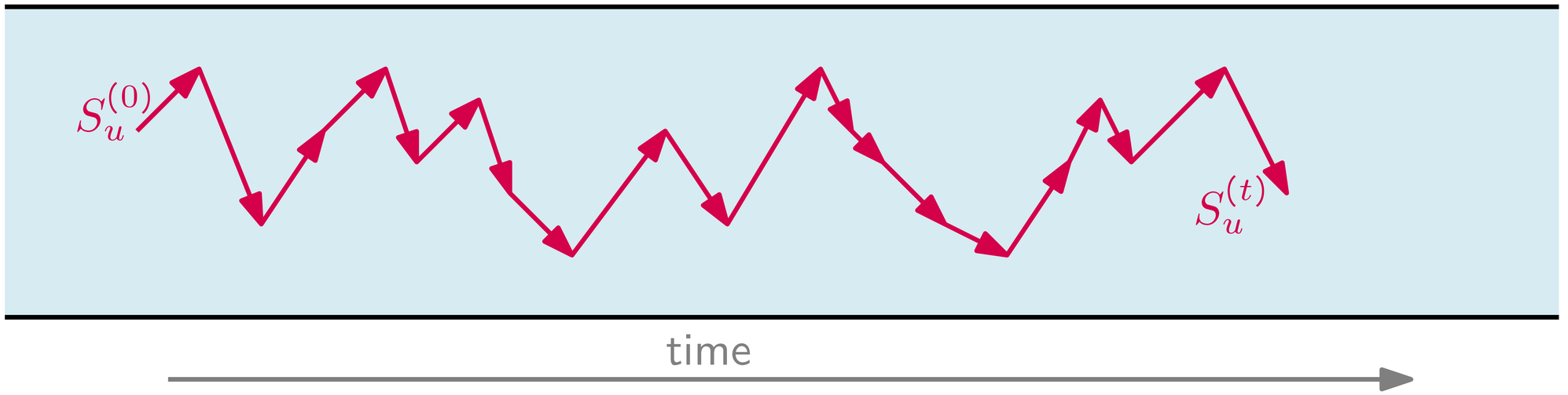

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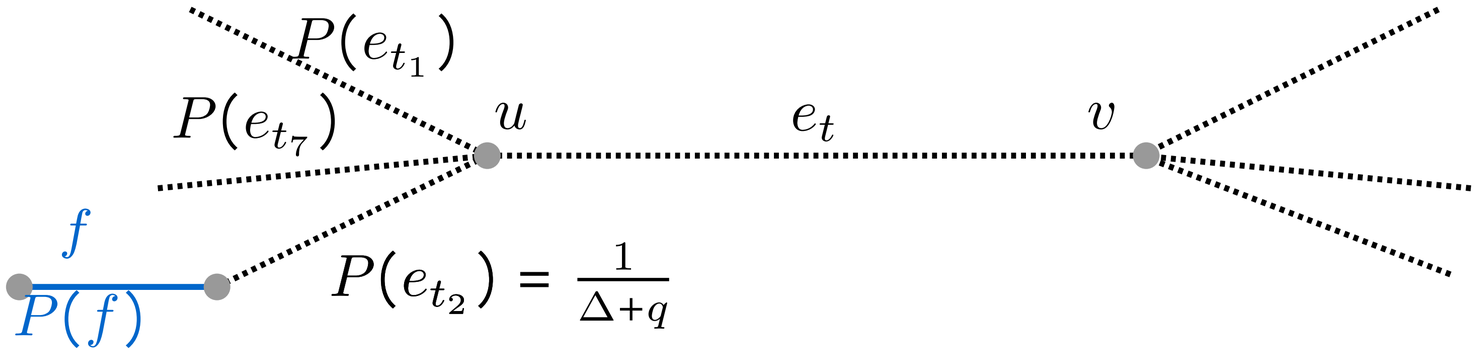

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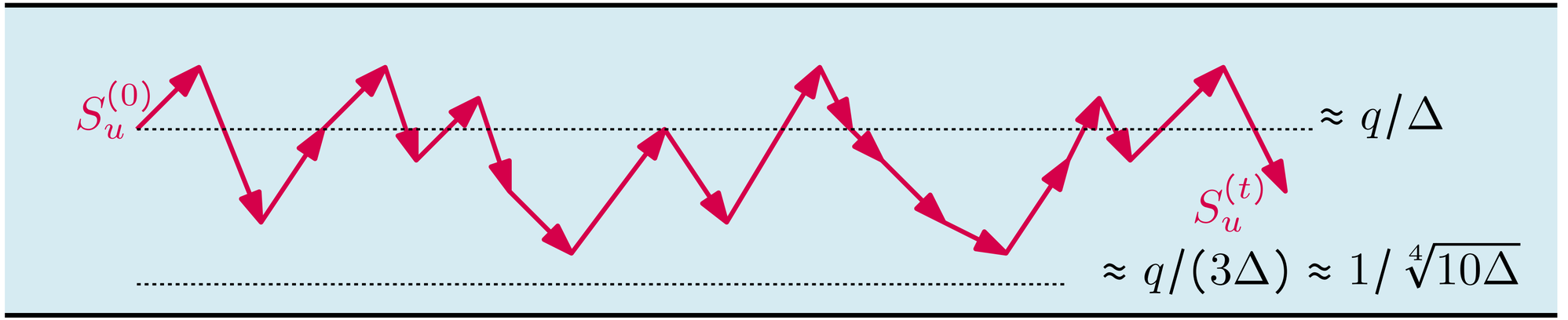
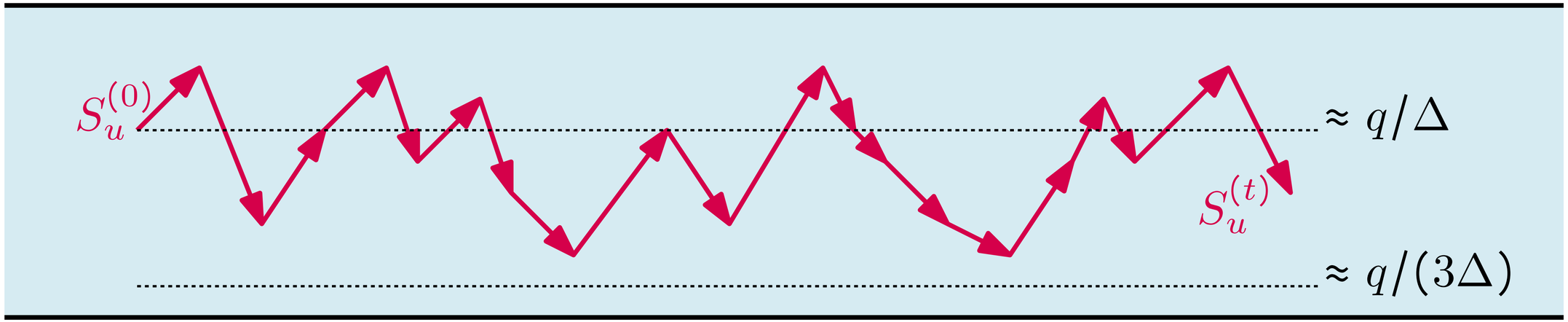


Martingale Process



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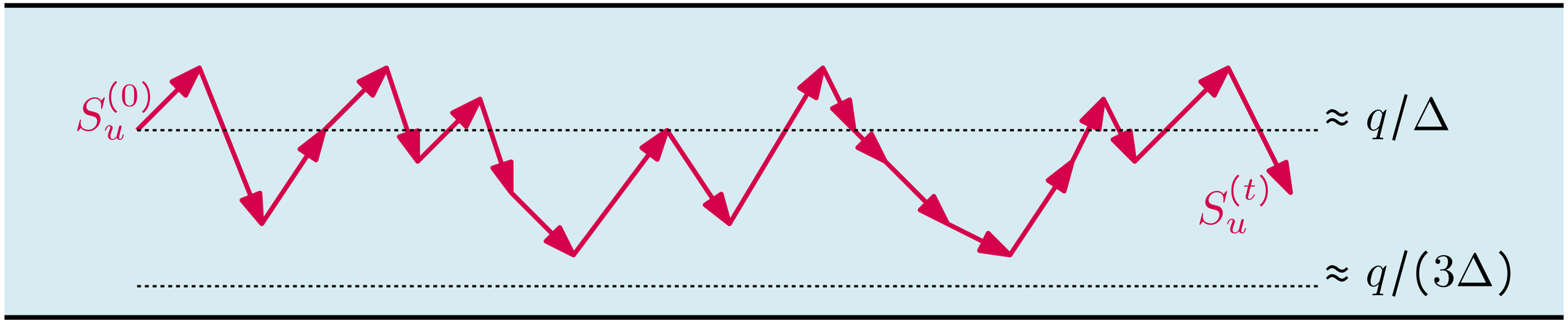

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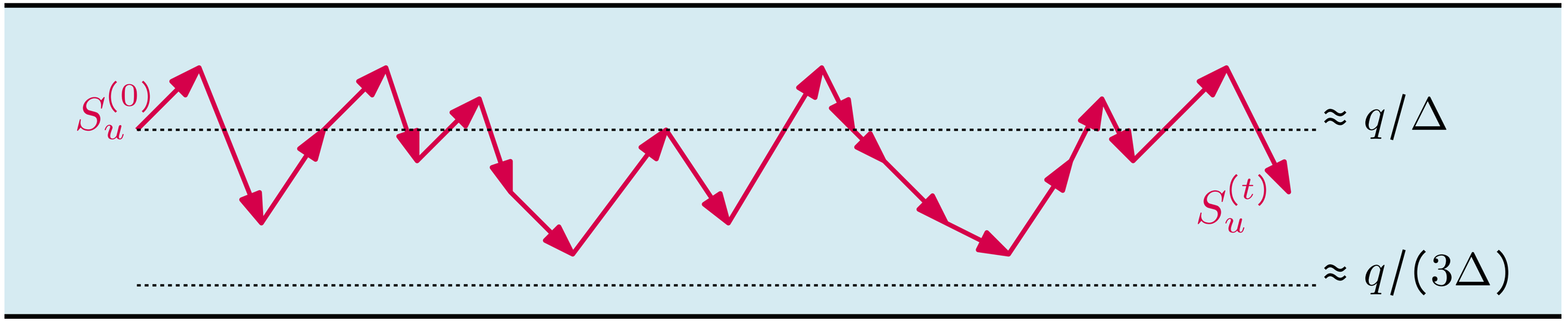

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Freedman's Inequality:

- Martingale $\mathbb{E}[S_u^{(t+1)} - S_u^{(t)} \mid S_u^{(1)}, S_u^{(2)}, \ldots, S_u^{(t)}] = 0$
- Step size $|S_u^{(t+1)} - S_u^{(t)}| \leq A$
- Observed variance $\sum_t \mathbb{E}\left[(S_u^{(t+1)} - S_u^{(t)})^2 \mid S_u^{(1)}, \ldots, S_u^{(t)}\right] \leq \sigma^2$

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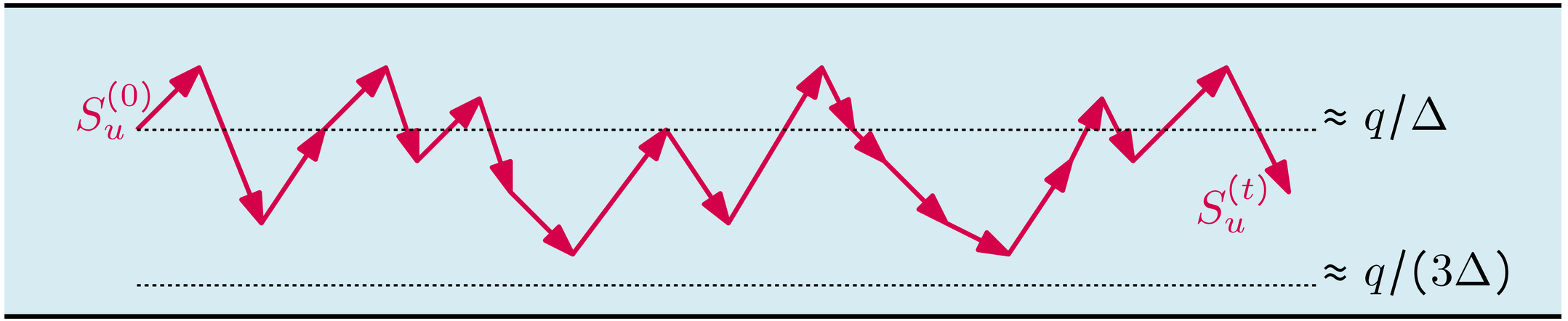

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$$\Longrightarrow \Pr\left[|S_u^{(t)} - S_u^{(0)}| \ge \varepsilon\right] \le 2\exp\left(-\frac{\varepsilon^2}{2(\sigma^2 + A\varepsilon/3)}\right)$$

Fair Matching Result

Main Technical Theorem:

There is an online algorithm which outputs a random matching M so that

$$\Pr[e \in M] \geq \frac{1}{\Delta + q} \ \forall e \in E, \qquad \text{where } q = O(\Delta^{3/4}\sqrt{\log \Delta})$$

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- This result has no requirement on Δ

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Corollary: online $(1 + o(1))\Delta$-edge-coloring algorithm when $\Delta = \omega(\log n)$

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Main Theorem: online $(1 + o(1))$-fair matching algorithm

Corollary: online $(1 + o(1))\Delta$-edge-coloring algorithm when $\Delta = \omega(\log n)$

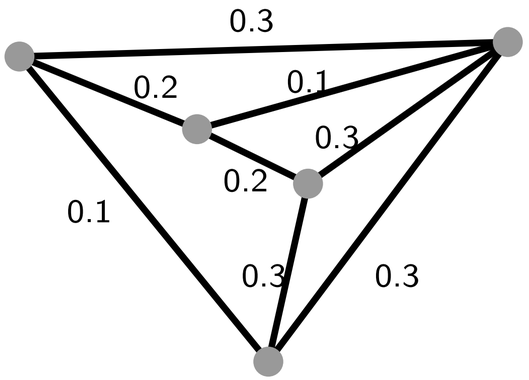
Key takeaway:

Formulate **quantity as a martingale**, bound stepsize and observed variance

Chernoff-type concentration

Extensions (1/3)
Fractional Matchings

Online Rounding of "Spread Out" Fractional Matchings:
Given (online) fractional matching $x \in \mathbb{R}^E$ satisfying $x_e \leq \varepsilon^5$,
output matching M so that $\Pr[e \in M] \geq (1-\varepsilon)x_e$.



$x_e := \frac{1}{\Delta}$ recovers fair matching theorem

Extensions (2/3)

List edge-coloring

Theorem:

There exists an online algorithm that computes an edge coloring which, with high probability, assigns each edge e a color from its list $L(e)$ (revealed online, with edge e), provided each list has sufficiently large size $(1 + o(1))\Delta$ and that $\Delta = \omega(\log n)$.

- Generalizes Kahn's seminal work from 96 to the online setting (albeit with worse bounds)

Extensions (3/3)

Local edge-coloring

Theorem:

There exists an online algorithm that computes an edge coloring assigning each edge $e = (u, v)$ a color from the set $\{1, 2, \ldots, d_{\max}(e) \cdot (1 + o(1))\}$ with high probability, where $d_{\max}(e) := \max\{\deg(u), \deg(v)\}$, provided that $d_{\max}(e) = \omega(\log n)$.

- Generalizes Christansen'23 work on local colorings (albeit with worse bounds)

Sharp Thresholds

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[BSVW'25]

Theorem:

$$(1+o(1))\Delta\text{-colors sufficient for when } \Delta = \begin{cases} \omega(\log n) & \text{if deterministic} \\ \omega(\sqrt{\log n}) & \text{if randomized} \end{cases}$$

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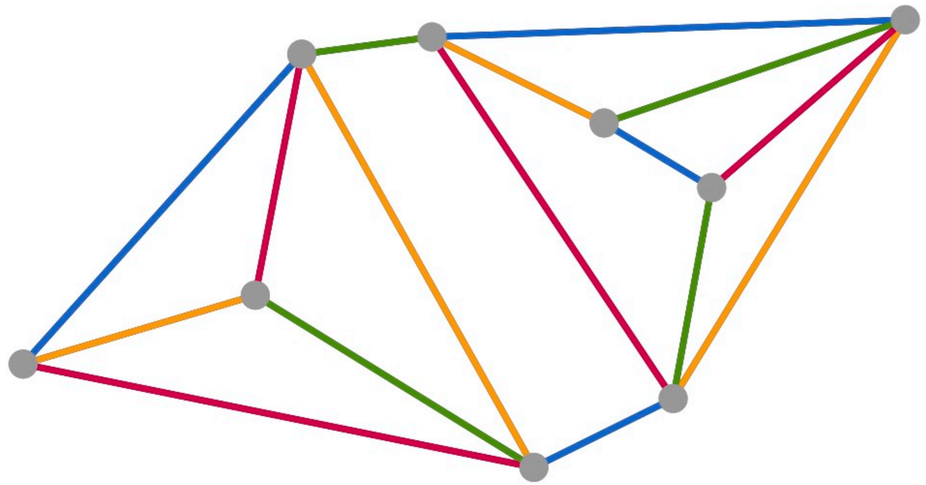
Theorem:

$(1+o(1))\Delta$-colors sufficient for when $\Delta = \begin{cases} \omega(\log n) & \text{if deterministic} \\ \omega(\sqrt{\log n}) & \text{if randomized} \end{cases}$

- Cannot go via fair matchings
- Interestingly, this stronger result cannot generalize to list edge coloring

One slide intuition



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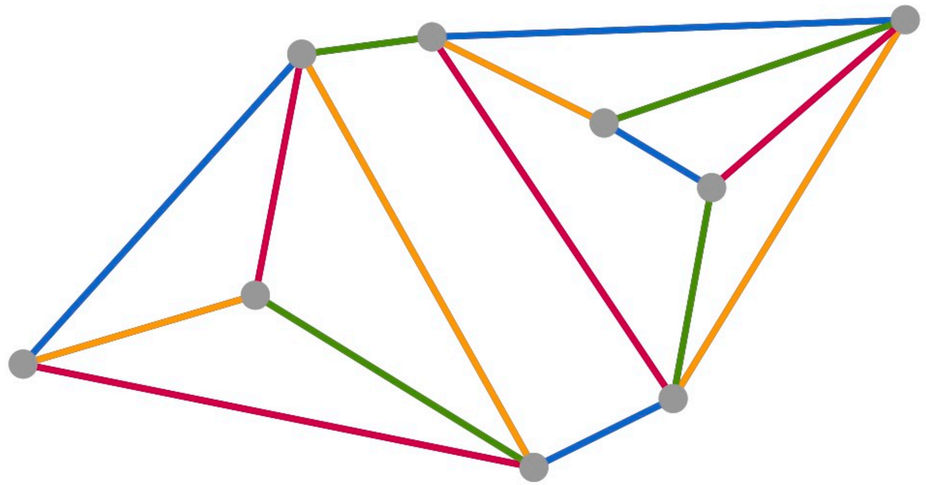
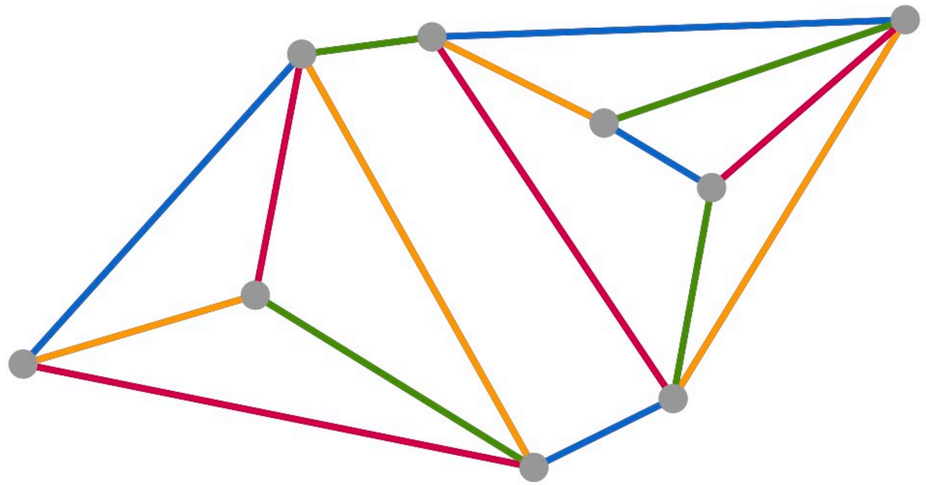
- First result shows that every matching <=> every color is good with probability $e^{-\Delta}$ so we need to select $\Delta = \Omega(\log n)$



One slide intuition

- First result shows that every matching <=> every color is good with probability $e^{-\Delta}$ so we need to select $\Delta = \Omega(\log n)$
- Instead show that the probability that more than a vanishing number of colors $\epsilon\Delta$ fail is at most $\left(e^{-\Delta}\right)^{\epsilon\Delta}$ so we can select $\Delta = \Omega(\sqrt{\log n})$



Open Problems (1/3)

Randomized Greedy

When $e_t = (u, v)$ arrives:

Color e_t with a **uniformly at random** color in $\mathscr{C}$ available to both u and v

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- Very recent positive results for special cases by Dudeja, Goswami, Saks'24

Open Problems (2/3)

Efficient Deterministic algorithms

> *... An interesting open problem is whether better bounds [than greedy's] can be achieved for graphs whose maximal degree is larger [than* $\log n$*]. This seems less plausible in the deterministic case, but perhaps one can devise a randomized algorithm that would edge color a graph better than the greedy algorithm for high degree graphs.*

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  - Randomized against adaptive => deterministic

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Tight guarantees on Δ

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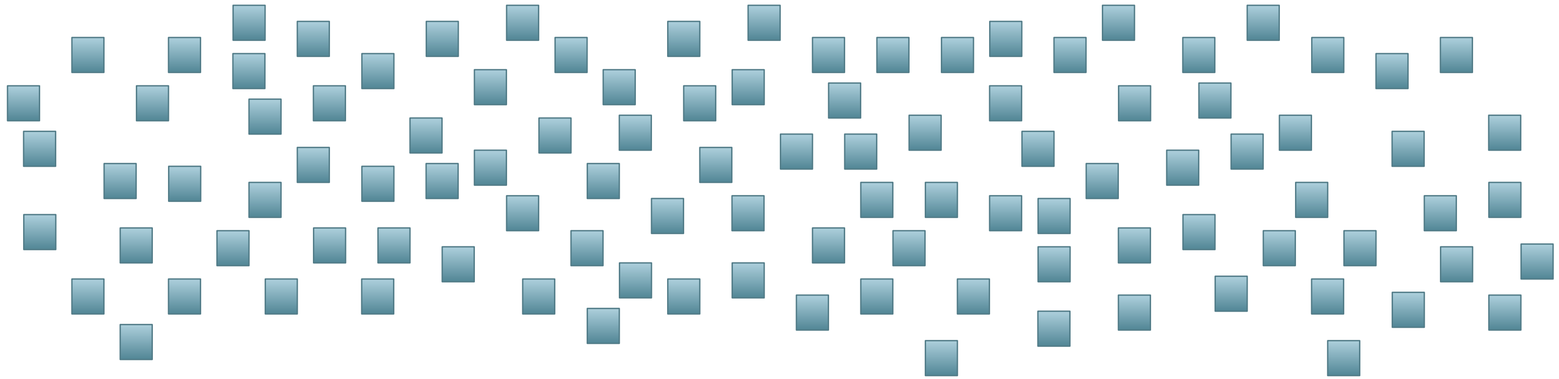
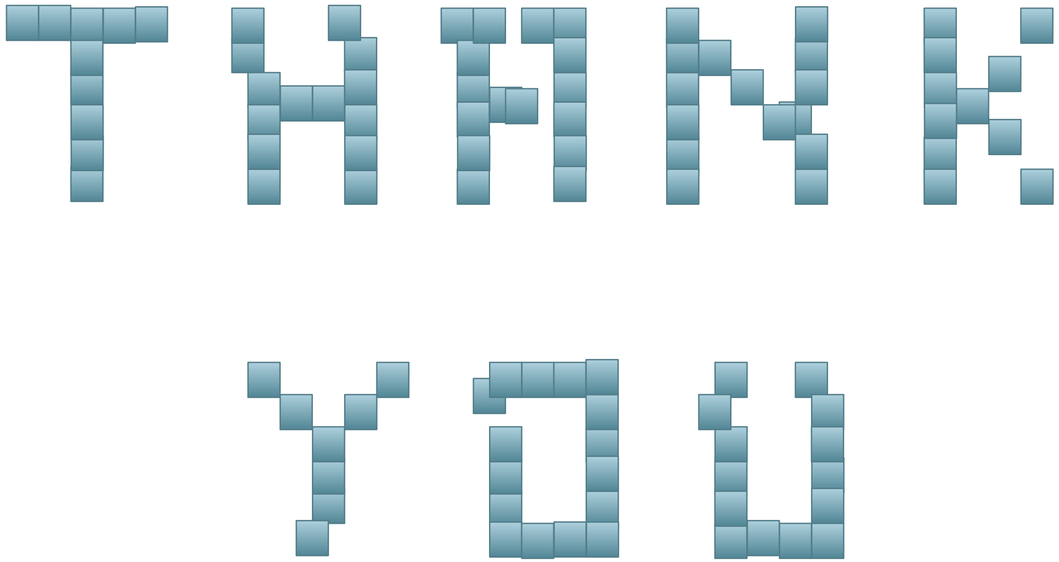
- Lower bound is $\Omega(\sqrt{\Delta})$
- Our proof yields $O(\Delta^{3/4} \log \Delta)$




THANK
YOU